SPACES OF MULTIPLIERS AND THEIR PREDUALS
FOR THE ORDER MULTIPLICATION ON [0, 1]. II

BY

SAVITA BHNATNAGAR (Chandigarh)

Abstract. Consider $I = [0, 1]$ as a compact topological semigroup with max multiplication and usual topology, and let $C(I), L^p(I), 1 \leq p \leq \infty$, be the associated algebras. The aim of this paper is to study the spaces $\text{Hom}_{C(I)}(L^r(I), L^p(I)), r > p$, and their preduals.

1. Introduction. The multipliers from $L^r(I)$ to $L^p(I), 1 \leq r, p \leq \infty$, where $I$ denotes the topological semigroup $[0, 1]$ with max multiplication and the usual topology, have been studied by Baker, Pym and Vasudeva [1]. The identification of multiplier spaces and their preduals from $L^r(I)$ to $L^p(I), 1 \leq r \leq p \leq \infty$, has been carried out by Bhatnagar and Vasudeva [2]. The case when $r > p$ has evaded the authors. The present study does not close this gap completely; however, it provides a set of necessary conditions and another set of sufficient conditions for a linear operator to be a multiplier from $L^r(I)$ to $L^p(I), r > p$. As a natural outcome of the methods employed, we find that a multiplier from $L^r(I)$ to $L^p(I), r > p$, need not be bounded as was assumed in [1]. As there is substantial overlap between the arguments presented in [2] and the present note, we have tried to be as brief as possible. For details, the reader may refer to [2].

2. Preliminaries. The set $I = [0, 1]$ equipped with max multiplication and usual topology is a compact topological semigroup. Let $C(I)$ and $L^p(I), 1 \leq p \leq \infty$, have their usual meanings. $L^p(I)$ is a left Banach $C(I)$-module under convolution $*$ defined by

$$\varphi * g(t) = \varphi(t) \int_0^t g(s) \, ds + g(t) \int_0^t \varphi(s) \, ds$$

for almost all $t \in I$, where $\varphi \in C(I)$ and $g \in L^p(I)$. We denote this left Banach $C(I)$-module by $L^p_I$. The adjoint action $\circ$ of an element $\varphi \in C(I)$ on $L^{p'}(I), 1/p + 1/p' = 1$, under which $L^{p'}(I)$ becomes a right Banach

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$C(I)$-module is defined by
\[ g \circ \varphi(s) = g(s) \int_0^s \varphi(t) \, dt + \int_s^1 \varphi(t) g(t) \, dt, \quad g \in L^{p'}(I). \]

The right Banach $C(I)$-module $L^{p'}(I)$ with the adjoint action defined above is denoted by $L^p_\circ$. The above definitions can be extended to $\varphi \in L^r(I)$, $r > p$, by denseness of $C(I)$ in $L^r(I)$.

Let $M^{r,p}_o$ denote $\text{Hom}_{C(I)}(L^r_*, L^p_0)$ and $M^{*,p}_s$ denote $\text{Hom}_{C(I)}(L^*_{s'}, L^p_0)$ for $1 \leq r, p \leq \infty$. If
\[ A^{r,p}_s = L^r_* \hat{\otimes}_{C(I)} L^{p'}_* , \quad A^{r,p}_o = L^r_* \hat{\otimes}_{C(I)} L^{p'}_0, \]
where the tensor product is the projective tensor product of Banach modules, then it follows, using a theorem of Rieffel [5], that $(A^{r,p}_s)^* = M^{r,p}_s$ and $(A^{r,p}_o)^* = M^{r,p}_o$.

3. Description of preduals. We define an operator
\[ B : L^r(I) \hat{\otimes} L^{p'}(I) \to L^{p'}(I), \]
where $1 \leq r, p' < \infty$, by
\[ B(f \otimes g)(s) = g(s) \int_0^s f(t) \, dt. \]

The image of $B$ in $L^{p'}(I)$ will be called $B^{r,p}$ if $1 < r, p < \infty$, and $B^\infty$ in $L^\infty$ if $r = p' = \infty$. Let $I_n = [0, 2^{-n}]$ and $J_n = [2^{-n}, 2^{-n+1}]$, $n = 1, 2, \ldots$. For a measurable function $\varphi$ on $I$, let $P_n\varphi$ denote the function $\chi_{J_n} \varphi$, $n = 1, 2, \ldots$. Define $e_n = 2^n \chi_{I_n}$, $n = 1, 2, \ldots$. Since $\int_0^1 e_n(s) \, ds = 1$ for each $n$, we can easily see that if $f \equiv 0$ on $I_n$ then $B(e_n \otimes f) = f$. As $f = \sum_{n=1}^\infty P_n f$, we obtain
\[ f = \sum_{n=1}^\infty B(e_n \otimes P_n f) = \sum_{n=1}^\infty e_n \circ P_n f = \sum_{n=1}^\infty e_n * P_n f. \]

Let
\[ C^{r,p} = \left\{ \varphi : \varphi \text{ is measurable and } \sum_{n=1}^\infty 2^{n/r'} \|P_n \varphi\|_{p'} < \infty \right\}, \]
\[ C^{r,p}_u = \left\{ \varphi : \varphi \text{ is measurable and } \sum_{n=1}^\infty (2^{n/r'} \|P_n \varphi\|_{p'})^u < \infty \right\}, \]
where $1/u = 1/r + 1/p'$. It may be noted that for $r = p$, $C^{p,p} = C^{p,p}_u$. We first characterize $B^\infty$.

**Proposition 1.** $B^\infty = \{ \varphi : \varphi \text{ is measurable and } \varphi(s)/s \text{ is essentially bounded} \}$. 
Proof. If \( \varphi \) is measurable and \( \varphi(s)/s \) is essentially bounded then \( \varphi \) can be written as \( \varphi(s) = (\varphi(s)/s) \int_0^1 f dt \). Consequently, \( \varphi = B(\psi) \), where \( \psi(s) = 1 \otimes \varphi(s)/s \). On the other hand, if \( \varphi = B(f \otimes g) \), where \( f, g \in L^\infty \), then \( \varphi(s) = g(s) \int_0^s f(t) dt, s \in I \), is measurable. Moreover,

\[
\left| \frac{\varphi(s)}{s} \right| = \left| \frac{g(s)}{s} \int_0^s f(t) dt \right| \leq \|g\|_\infty \|f\|_\infty.
\]

Remark. It is not difficult to see that the requirement that \( \varphi(s)/s \) be essentially bounded is equivalent to \( \sup_n 2^n \|P_n \varphi\|_\infty < \infty \).

**Theorem 1.** For \( r > p \), \( C^{r,p} \subseteq B^{r,p} \subseteq C_u^{r,p} \), where \( 1/u = 1/r + 1/p' \).

Proof. If \( \varphi \in C^{r,p} \), then \( \sum_{n=1}^\infty 2^{n/r'} \|P_n \varphi\|_{p'} < \infty \), so that \( \varphi = \sum_{n=1}^\infty B(e_n \otimes P_n \varphi) \) and \( \sum_{n=1}^\infty \|e_n\|_r \|P_n \varphi\|_{p'} < \infty \). Thus \( \psi = \sum_{n=1}^\infty e_n \otimes P_n \varphi \in L^r \otimes L^{p'} \) and \( B(\psi) = \varphi \).

To prove the other inclusion, fix \( r \) and \( p \) and let \( q = 1 + r/p' \), so that \( q' = 1 + p'/r \). Let \( \alpha = q/r \). Then

\[
\alpha \cdot \frac{1}{q} + (1 - \alpha) \cdot \frac{1}{\infty} = \frac{1}{r}, \quad \alpha \cdot \frac{1}{q'} + (1 - \alpha) \cdot \frac{1}{\infty} = \frac{1}{p'}.
\]

Since \( B(L^q \otimes L^{q'}) \subset B^{q,q} \) and \( B(L^\infty \otimes L^\infty) \subset B^\infty \), it follows by interpolation, using Calderón [3], that \( B \) maps \( L^r \otimes L^{p'} \) into a suitable intermediate space between \( B^{q,q} \) and \( B^\infty \). (It may be observed that \( B^{q,q} = C^{q,q} \).) To see this, note that \( B^{q,q} \) may be regarded as a mixed \( L^p \) space, viz., \( L^1(\mathbb{N}, \nu, L^q(I)) \), where \( \nu \) is the measure on \( \mathbb{N} \) assigning mass \( 2^{-n} \) to the element \( \{n\} \), and \( B^\infty \) may be regarded as \( L^\infty(\mathbb{N}, \nu, L^\infty(I)) \). These identifications are obtained by associating with \( \varphi \in B^{q,q} \) (or \( B^\infty \)) the function \( f : \mathbb{N} \times I \rightarrow \mathbb{C} \) given by

\[
f(n, t) = 2^n \varphi \left( \frac{t + 1/2^n}{n} \right), \quad n \in \mathbb{N}, t \in I.
\]

By Calderón [3], the intermediate space with index \( \alpha \) between \( L^1(\mathbb{N}, \nu, L^q(I)) \) and \( L^\infty(\mathbb{N}, \nu, L^\infty(I)) \) is contained in \( L^u(\mathbb{N}, \nu, L^v(I)) \), where

\[
\alpha \cdot \frac{1}{1} + (1 - \alpha) \cdot \frac{1}{\infty} = \frac{1}{u}, \quad \alpha \cdot \frac{1}{q'} + (1 - \alpha) \cdot \frac{1}{\infty} = \frac{1}{v}.
\]

Thus \( 1/u = 1/r + 1/p' \), and \( v = p' \). It follows that \( B \) maps \( L^r \otimes L^{p'} \) into \( L^u(\mathbb{N}, \nu, L^v(I)) \) and by identification (1) this corresponds to measurable functions \( \varphi \) on \( I \) such that \( \sum_{n=1}^\infty (2^{n/r'} \|P_n \varphi\|_{p'})^u < \infty \). Thus \( \varphi \in C^{r,p}_u \). This completes the proof.

We next characterize the predual \( A^{r,p}_0 \) of the multiplier space \( M^{r,p}_0 \). Let \( AC_u^0 (u \geq 1) \) be the space of absolutely continuous functions on \([0, 1]\) whose derivative belongs to \( L^u(I) \) and which vanish at 1.
Theorem 2. For \( r \geq p \), \( C^{r,p} \subseteq A^{r,p}_0 \subseteq B^{r,p} + AC^o_u \), where \( 1/r + 1/p' = 1/u \).

Proof. If \( \varphi \in C^{r,p} \), we can write \( \varphi = \sum_{n=1}^{\infty} e_n \circ P_n \varphi \) and by definition of \( C^{r,p} \), \( \sum_{n=1}^{\infty} ||e_n||_r ||P_n \varphi||_{p'} < \infty \) so that \( \varphi \in A^{r,p}_0 \).

Clearly, every element of \( A^{r,p}_0 \) is a sum of the form \( \varphi + \psi \), where \( \varphi \in B^{r,p} \) and \( \psi \in AC^o_u \), so that we have the required result.

Theorem 3. (a) \( A^{r,p}_* \) is a semisimple commutative Banach algebra under convolution. It has an approximate identity. The maximal ideal space of \( A^{r,p}_* \) is the interval \((0,1]\) with the interval topology.

(b) \( C^{r,p} + C^{p',r'} \subseteq A^{r,p}_* \subseteq B^{r,p} + B^{p',r'} \) for \( 1 \leq r, p \leq \infty \).

(c) \( C^{p',r'} \subseteq C^{r,p} \) if \( r \geq p' \), and \( C^{r,p} \subseteq C^{p',r'} \) if \( r \leq p' \).

(d) \( A^{r,p}_* \subseteq C^{r,p} \) if \( r \geq p' \), and \( C^{p',r'} \subseteq A^{r,p}_* \subseteq C^{p',r'} \) if \( r \leq p' \).

Proof. (a) \( A^{r,p}_* = L^r \hat{\otimes}_{C(I)} L^{p'} \), being the projective tensor product of two Banach algebras, is a Banach algebra. For the detailed proof, consult Theorem 7 of [2].

(b) If \( \varphi \in C^{r,p} \), then \( \sum_{n=1}^{\infty} 2^n/r ||P_n \varphi||_{p'} < \infty \) and \( \varphi = \sum_{n=1}^{\infty} e_n \ast P_n \varphi \in A^{r,p}_* \). Similarly for \( C^{p',r'} \). It is clear that every element of \( A^{r,p}_* \) is a sum \( \varphi + \psi \), where \( \varphi \in B^{r,p} \) and \( \psi \in B^{p',r'} \), so that \( A^{r,p}_* \subseteq B^{r,p} + B^{p',r'} \).

(c) We show that \( C^{p',r'} \subseteq C^{r,p} \) if \( r \geq p' \), the other proofs are similar. If \( \varphi \in C^{p',r'} \), then \( \sum_{n=1}^{\infty} 2^n/p ||P_n \varphi||_r < \infty \) and

\[
\sum_{n=1}^{\infty} 2^n/r ||P_n \varphi||_{p'} \leq \sum_{n=1}^{\infty} 2^n/r ||P_n \varphi||_r (2^{-n})^{1/p' - 1/r} = \sum_{n=1}^{\infty} 2^n/p ||P_n \varphi||_r < \infty
\]

so that \( \varphi \in C^{r,p} \).

(d) is clear in view of (c).

4. Multipliers. In this section we study the multipliers, namely, \( M^{r,p}_* = \text{Hom}_{C(I)}(L^*_r, L^p_0) \) and \( M^{r,p}_o = \text{Hom}_{C(I)}(L^*_r, L^p_o) \). We have \( M^{r,p}_* = (A^{r,p}_*)^* \), and we deal with the case \( r \geq p' \); the other case can be obtained by identifying \( A^{r,p}_* \) and \( A^{p',r'}_o \). The following theorem gives us a necessary condition for a multiplier to be in \( M^{r,p}_* \).

Theorem 4. Let \( r \geq p' \). If \( t \in M^{r,p}_* \) then \( t \) is measurable and

\[
\sup_n 2^{-n/r'} ||P_n t||_p < \infty.
\]

Proof. \( M^{r,p}_* = (A^{r,p}_*)^* \subseteq (C^{r,p})^* \) by Theorem 3. Therefore, if \( t \in M^{r,p}_* \) then \( t \) is measurable and \( \sup_n 2^{-n/r'} ||P_n t||_p < \infty \).

The following theorem gives us a sufficient condition for a multiplier to be in \( M^{r,p}_* \).
THEOREM 5. Let \( r \geq p' \). If \( t \) is measurable and satisfies
\[
\sum_{n=1}^{\infty} (2^{-n/r'} \|P_n t\|_p)^{u'} < \infty,
\]
then \( t \in \mathcal{M}_{r}^{p} \).

Proof. \( \mathcal{M}_{r}^{p'} = (A_r^{r,p})^* \supseteq (C_{u}^{r,p})^* \) by Theorem 3. Therefore if \( t \) is measurable and satisfies \( \sum_{n=1}^{\infty} (2^{-n/r'} \|P_n t\|_p)^{u'} < \infty \) then \( t \in \mathcal{M}_{r}^{p} \).

Next, we look at Hom\(_{C(I)}(L_r, \mathbb{L}) \), i.e., \( (L_r \hat{\otimes}_{C(I)} \mathbb{L})^* \), where the action of \( C(I) \) on \( L_r(I) \) is by \( * \) and on \( \mathbb{L}^r(I) \) by \( \circ \). The natural map \( w : L_r \hat{\otimes} \mathbb{L}^r \rightarrow L_r \circ \mathbb{L}^r \) factors through \( L_r \hat{\otimes}_{C(I)} \mathbb{L}^r \) (note that \( C(I) \) can be replaced by another dense subalgebra contained in \( L^r \cap \mathbb{L}^r \)). Theorem 1 of [2] shows that the map \( L_r \hat{\otimes}_{C(I)} \mathbb{L}^r \rightarrow L_r \circ \mathbb{L}^r \) induced by \( w \) is one-to-one. Then \( L_r \circ \mathbb{L}^r \) can be identified with \( \{\sum_{i=1}^{\infty} f_i \circ g_i : f_i \in L_r, g_i \in \mathbb{L}^r \text{ and } \sum_{i=1}^{\infty} \|f_i\|_r \|g_i\|_{p'} < \infty\} \), which we can identify with \( L_r \hat{\otimes}_{C(I)} \mathbb{L}^r \), i.e., the predual of Hom\(_{C(I)}(L_r, \mathbb{L}) \).

THEOREM 6. Let \( r \geq p \). If \( \beta \in \mathcal{M}_{0}^{r,p} \) then \( \beta \) is measurable and satisfies \( \sup_n 2^{-n/r'} \|P_n \beta\|_p < \infty \).

Proof. \( \mathcal{M}_{0}^{r,p} = (A_0^{r,p})^* \subseteq (C_{u}^{r,p})^* \) by Theorem 2. Thus if \( \beta \in \mathcal{M}_{0}^{r,p} \) then \( \beta \in (C_{r,p})^* \) and so \( \beta \) is measurable and satisfies \( \sup_n 2^{-n/r'} \|P_n \beta\|_p < \infty \).

THEOREM 7. Let \( r \geq p \). If \( \beta \in L^{u'}(I) \) has an a.e. derivative \( h \) which satisfies \( \sum_{n=1}^{\infty} (2^{-n/r'} \|P_n h\|_p)^{u'} < \infty \), then \( \beta \in \mathcal{M}_{0}^{r,p} \).

Proof. \( \mathcal{M}_{0}^{r,p} = (A_0^{r,p})^* \supseteq (C_{u}^{r,p} + AC_{u}^{r})^* \) by Theorems 1 and 2. Suppose \( \mu \in (C_{u}^{r,p} + AC_{u}^{r})^* \). Then \( \mu|_{AC_{u}^{r}} \in (AC_{u}^{r})^* \) and \( \mu|_{C_{u}^{r,p}} \in (C_{u}^{r,p})^* \). Note that \( (AC_{u}^{r})^* = L^{u'}(I) \), via the pairing
\[
\langle \mu, \varphi \rangle = \int_{0}^{1} \beta(s) f(s) \, ds,
\]
where \( \varphi(s) = \int_{s}^{1} f(t) \, dt \) is in \( AC_{u}^{r} \) and \( \mu \) corresponds to \( \beta \in L^{u'}(I) \). Since \( \mu|_{C_{u}^{r,p}} \in (C_{u}^{r,p})^* \) it follows that \( \beta \in (C_{u}^{r,p})^* \). For any \( \varepsilon > 0 \), \( L^{p'}(I_{\varepsilon}) \subseteq C_{u}^{r,p} (I_{\varepsilon} = [\varepsilon, 1]) \) and so \( \mu|_{L^{p'}(I_{\varepsilon})} \) corresponds to an \( L^{p}(I_{\varepsilon}) \) function \( h_{\varepsilon} \) such that for \( \varphi \in L^{p'}(I_{\varepsilon}) \),
\[
\langle \mu, \varphi \rangle = \int_{\varepsilon}^{1} h_{\varepsilon}(s) \varphi(s) \, ds = \int_{0}^{1} h_{\varepsilon}(s) \varphi(s) \, ds,
\]
where \( \varphi \) is taken to be zero on \([0, \varepsilon) \). It is clear that the \( h_{\varepsilon}'s \) are compatible,
Comparing this with (2) we get

\[ \langle \mu, \varphi \rangle = \int_0^t h_\varepsilon(s) ds dt = \int_0^{\varepsilon} f(t) h_\varepsilon(s) ds dt. \]

Comparing this with (2) we get

\[ \beta(t) = \int_{\varepsilon}^t h_\varepsilon(s) ds \quad \text{for } t > \varepsilon. \]

Thus \( \beta'(t) = h_\varepsilon(t) \) a.e. on \((\varepsilon, 1]\) so we have proved that there exists \( h \) measurable on \((0, 1]\) such that \( h \in L^p(I_\varepsilon) \) for every \( \varepsilon > 0 \) and \( \beta'(t) = h(t) \) a.e. on \((0, 1] \). If we take \( \varphi \in B^{r,p} \subseteq C^{r,p}_u \), then

\[ \langle \mu, \varphi \rangle = \int_0^1 h(t) \varphi(t) dt \]

exists and is finite because \( \sum_{n=1}^{\infty} (2^{-n/r'}\|P_n h\|_p)^u' < \infty. \) The multiplier \( M_\beta \) corresponding to \( \beta \) is given by

\[ M_\beta(f)(t) = \left[ \beta(1) - \int_0^t h(s) ds \right] f(s) + h(t) \int_0^t f(s) ds \quad \text{for } f \in L^r(I). \]

Indeed,

\[ \langle M_\beta(f), g \rangle = \langle \beta, f \circ g \rangle \]

\[ = \int_0^1 \left\{ h(t) g(t) \int_0^t f(s) ds + g(t) f(t) \beta(t) \right\} dt, \]

\[ = \int_0^1 g(t) \left\{ \left[ \beta(1) - \int_0^t h(s) ds \right] f(t) + h(t) \int_0^t f(s) ds \right\} dt. \]

Remarks. (i) For \( r = p \) (\( u = 1 \)), \( \beta \in L^\infty(I) \), and consequently \( x \mapsto \int_0^t h(t) dt \) is bounded.

(ii) The condition that \( h(s) \int_0^t f(t) dt \) is in \( L^p(I) \) for every \( f \in L^r(I) \) is equivalent to \( \sum_{n=1}^{\infty} (2^{-n/r'}\|P_n h\|_p)^u' < \infty. \) To see this, let \( g \in L^p \). If \( \hat{f}(x) \) denotes \( \int_0^x f(y) dy \) for \( f \in L^r \), then

\[ |\langle h \hat{f}, g \rangle| = \int_0^1 (h \hat{f})(x) g(x) dx \leq \sum_{n} \|P_n h\|_p \|P_n(g \hat{f})\|_{p'} \]

\[ \leq \left[ \sum_{n=1}^{\infty} (2^{-n/r'}\|P_n h\|_p)^u' \right]^{1/u'} \left[ \sum_{n=1}^{\infty} (2^{n/r'}\|P_n(g \hat{f})\|_{p'})^u \right]^{1/u} < \infty, \]

since \( g \hat{f} \in B^{r,p} \subseteq C^{r,p}_u \). Thus \( h \hat{f} \in L^p \) for all \( f \in L^r \).
Conversely, suppose \( h \hat{f} \in L^p \) for all \( f \in L^r \). Since
\[
\left[ \sum_{n=1}^{\infty} (2^{-n/r'} \| P_n h \|_p)^{u'} \right]^{1/u'} = \sup_{\varphi \in C_u^{r,p}} \frac{\| h \varphi \|_1}{\| \varphi \|},
\]
and
\[
\| \varphi \|_{p'} \leq \| \varphi \| \quad \text{for} \quad \varphi \in C_u^{r,p},
\]
we get
\[
\| h \varphi \|_1 \leq \| (x + 1)^{1/r'} h \|_p \| (x + 1)^{-1/r'} \varphi \|_{p'} \leq \| (x + 1)^{1/r'} h \|_p \| \varphi \| < \infty,
\]
since \( f(x) = (x + 1)^{-1/r} \in L^r \). Thus \([\sum_{n=1}^{\infty} (2^{-n/r'} \| P_n h \|_p)^{u'}]^{1/u'} < \infty\). Here \( \| \varphi \| \) stands for \([\sum_{n=1}^{\infty} (2^{n/r'} \| P_n \varphi \|_{p'})^{u'}]^{1/u'}\).

(iii) In [1] the fact that a multiplier \( T \) from a Banach algebra \( A \) to itself gives rise to a bounded continuous function was used heavily. It is natural to expect that if \( T \) is a multiplier from a Banach algebra \( A \) to a Banach algebra \( B \), where \( A \subset B \), \( A \neq B \), then \( T \) need not give rise to a bounded continuous function. See Larsen [4, Theorem 1.2.2]. This is corroborated by the fact that \( \beta(x) = \log x, x \in (0, 1] \), gives a multiplier from \( L^r(I) \) to \( L^p(I) \), \( r > p \ (u > 1) \), whereas \( t \mapsto \int_0^t (1/s) \, ds \) is not bounded.

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Department of Mathematics
Panjab University
Chandigarh 160014, India
E-mail: bhsavita@pu.ac.in

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