COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

BY

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Abstract. Let \( a \) denote an ideal of a commutative Noetherian ring \( R \), and \( M \) and \( N \) two finitely generated \( R \)-modules with \( \text{pd} \ M < \infty \). It is shown that if either \( a \) is principal, or \( R \) is complete local and \( a \) is a prime ideal with \( \dim R/a = 1 \), then the generalized local cohomology module \( H^i_a(M,N) \) is \( a \)-cofinite for all \( i \geq 0 \). This provides an affirmative answer to a question proposed in [13].

1. Introduction. A generalization of local cohomology functors has been given by J. Herzog in [6]. Let \( a \) denote an ideal of a commutative Noetherian ring \( R \). For each \( i \geq 0 \), the functor \( H^i_a(\cdot, \cdot) \) is defined by \( H^i_a(M,N) = \lim_{\to} \text{Ext}^i_R(M/a^nM, N) \) for all \( R \)-modules \( M \) and \( N \). Clearly, this is a generalization of the usual local cohomology functor. The study of this concept was continued in [10], [2] and [12]. Recently, there is some new interest in generalized local cohomology (see e.g. [1], [13] and [14]).

In 1969, A. Grothendieck conjectured that if \( a \) is an ideal of \( R \) and \( N \) is a finitely generated \( R \)-module, then \( \text{Hom}_R(R/a, H^i_a(N)) \) is finitely generated for all \( i \geq 0 \). R. Hartshorne provided a counter-example to this conjecture in [5]. He defined a module \( N \) to be \( a \)-cofinite if \( \text{Supp}_R N \subseteq V(a) \) and \( \text{Ext}^i_R(R/a, N) \) is finitely generated for all \( i \geq 0 \), and he asked the following question.

QUESTION 1.1. Let \( a \) be an ideal of \( R \), and \( N \) a finitely generated \( R \)-module. When are \( H^i_a(N) \) \( a \)-cofinite for all \( i \geq 0 \)?

Hartshorne [5, Corollaries 6.3 and 7.7] proved that if \( a \) is an ideal of the complete regular local ring \( R \) and \( N \) is a finitely generated \( R \)-module, then \( H^i_a(N) \) is \( a \)-cofinite in two cases:

(i) (see [5, Corollary 6.3]) \( a \) is a principal ideal,

(ii) (see [5, Corollary 7.7]) \( a \) is a prime ideal with \( \dim R/a = 1 \).

This subject was studied by several authors afterward (see e.g. [8], [4] and [15]). The best result concerning cofiniteness of local cohomology is:
Theorem 1.2 ([8], [4], [15]). Let \(a\) be an ideal of \(R\), and \(N\) a finitely generated \(R\)-module. If either \(a\) is principal or \(R\) is local and \(\dim R/a = 1\), then \(H^i_a(N)\) is \(a\)-cofinite for all \(i \geq 0\).

S. Yassemi [13, Question 2.7] asked whether 1.2 holds for generalized local cohomology. The main aim of this paper is to extend 1.2 to that setting. More precisely, we prove the following.

Theorem 1.3. Let \(a\) be an ideal of the ring \(R\). Let \(M\) and \(N\) be two finitely generated \(R\)-modules with \(\text{pd} M < 1\). If either

(i) \(a\) is principal, or
(ii) \(R\) is complete local and \(a\) is a prime ideal with \(\dim R/a = 1\),

then \(H^i_a(M, N)\) is \(a\)-cofinite for all \(i \geq 0\).

All rings considered in this paper are assumed to be commutative Noetherian with identity. Our terminology follows the textbook [3].

2. Cofiniteness results. Let \(a\) denote an ideal of a ring \(R\). The generalized local cohomology is defined by

\[
H^i_a(M, N) = \lim_{n} \Ext^i_R(M/a^n M, N)
\]

for all \(R\)-modules \(M\) and \(N\). Note that this is in fact a generalization of the usual local cohomology, because if \(M = R\), then \(H^i_a(R, N) = H^i_a(N)\).

Definition 2.1. Let \(M\) be an \(R\)-module. The generalized ideal transform functor with respect to an ideal \(a\) of \(R\) is defined by

\[
D_a(M, \cdot) = \lim_{n} \Hom_R(a^n M, \cdot).
\]

Let \(R^i D_a(M, \cdot)\) denote the \(i\)th right derived functor of \(D_a(M, \cdot)\). One can check that there is a natural isomorphism \(R^i D_a(M, \cdot) \cong \lim_n \Ext^i_R(a^n M, \cdot)\). Thus, by considering the Ext long exact sequences induced by the short exact sequences

\[
0 \to a^n M \to M \to M/a^n M \to 0 \quad (n \in \mathbb{N}),
\]

we can deduce the following lemma.

Lemma 2.2. Let \(M\) be an \(R\)-module. For any \(R\)-module \(N\), there is an exact sequence

\[
0 \to H^i_a(M, N) \to \Hom_R(M, N) \to D_a(M, N) \to H^1_a(M, N) \to \cdots \to H^i_a(M, N) \to \Ext^i_R(M, N) \to R^i D_a(M, N) \to H^{i+1}_a(M, N) \to \cdots
\]

Moreover, if \(M\) has finite projective dimension, then there is a natural isomorphism \(H^{i+1}_a(M, N) \cong R^i D_a(M, N)\) for all \(i \geq \text{pd} M + 1\).
Let $a$ be an ideal of $R$, and $M$ an $R$-module. We say that $M$ is $a$-cofinite if $\text{Supp}_R M \subseteq V(a)$ and $\text{Ext}^i_R(R/a, M)$ is finitely generated for all $i \geq 0$.

**Lemma 2.3.** Suppose $M, N$ are two $R$-modules and $a$ an ideal of $R$. If $M$ is finitely generated, then $\text{Supp}_R H^i_a(M, N) \subseteq V(a)$ for all $i \geq 0$.

**Proof.** Let $p$ be a prime ideal of $R$. It follows from [9, Theorem 9.50] that

$$\text{Ext}^i_R(M/a^n M, N)_p \cong \text{Ext}^i_{R_p}(M_p/(a^n R_p)M_p, N_p)$$

for all $i \geq 0$. On the other hand, it is well known that the tensor product preserves direct limits (see e.g. [9, Corollary 2.20]). Thus

$$H^i_a(M, N)_p \cong R_p \otimes_R \lim_{n} \text{Ext}^i_R(M/a^n M, N)$$

$$\cong \lim_{n} \text{Ext}^i_{R_p}(M_p/(a^n R_p)M_p, N_p) \cong H^i_a(R_p, M, N_p).$$

This shows that $\text{Supp}_R H^i_a(M, N) \subseteq V(a)$, as required.  

**Lemma 2.4.** (i) If $M$ is a finitely generated $R$-module such that $\text{Supp}_R M \subseteq V(a)$, then $M$ is $a$-cofinite.

(ii) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of $R$-modules. Whenever two of $L$, $M$ or $N$ are $a$-cofinite, then so is the third.

**Proof.** (i) Since $M$ is finitely generated, it follows that $\text{Ext}^i_R(R/a, M)$ is finitely generated for all $i \geq 0$. Hence $M$ is $a$-cofinite, by definition.

(ii) This is well known and can be deduced easily by considering the long exact sequence

$$\cdots \rightarrow \text{Ext}^i_R(R/a, L) \rightarrow \text{Ext}^i_R(R/a, M) \rightarrow \text{Ext}^i_R(R/a, N)$$

$$\rightarrow \text{Ext}^{i+1}_R(R/a, L) \rightarrow \cdots.$$ 

**Lemma 2.5.** Let $a = Ra$ be a principal ideal of $R$, and $M$ and $N$ two finitely generated $R$-modules. Let $\text{Hom}_R(M, N)_a$ denote the localization of $\text{Hom}_R(M, N)$ with respect to the multiplicative closed subset $\{a^i : i \geq 0\}$ of $R$. Then

(i) there is a natural isomorphism $D_a(M, N) \cong \text{Hom}_R(M, N)_a$,

(ii) $H^i_a(M, N)$ is $a$-cofinite.

**Proof.** (i) If $a$ is nilpotent, then it is clear that both $D_a(M, N)$ and $\text{Hom}_R(M, N)_a$ will vanish. Hence, we may and do assume that $a$ is not nilpotent. For all $i, j \in \mathbb{N}$ with $j \geq i$, let $\pi_{ij} : \text{Hom}_R(a^j M, N) \rightarrow \text{Hom}_R(a^i M, N)$ be defined by $\pi_{ij}(f) = f|_{a^i M}$ for $f \in \text{Hom}_R(a^j M, N)$. Also, denote the natural map $\text{Hom}_R(a^i M, N) \rightarrow D_a(M, N)$ by $\pi_i$. Recall that we defined $D_a(M, N)$ as the direct limit of the direct system $(\text{Hom}_R(a^i M, N), \pi_{ij})_{i,j \in \mathbb{N}}$.

Now define $\psi_i : \text{Hom}_R(a^i M, N) \rightarrow (\text{Hom}_R(M, N))_a$ by $\psi_i(f) = f \lambda_i/a^i$, where $\lambda_i : M \rightarrow a^i M$ is defined by $\lambda_i(m) = a^i m$ for $m \in M$. Clearly $\{\psi_i\}_{i \in \mathbb{N}}$
is a morphism of direct systems. Assume \( \psi : D_a(M, N) \to (\text{Hom}_R(M, N))_a \) is the homomorphism induced by \( \{\psi_i\}_{i \in \mathbb{N}} \). Thus for each \( g \in D_a(M, N) \), we have \( \psi(g) = \psi_i(f) \), where \( i \in \mathbb{N} \) and \( f \in \text{Hom}_R(a^iM, N) \) are such that \( \pi_i(f) = g \).

We show that \( \psi \) is an isomorphism. First, we show that \( \psi \) is injective. Suppose \( \psi(g) = 0 \) for some \( g \in D_a(M, N) \). There are \( i \in \mathbb{N} \) and \( f \in \text{Hom}_R(a^iM, N) \) such that \( g = \pi_i(f) \). Hence

\[
\psi(g) = \psi_i(f) = f\lambda_i/a^i = 0.
\]

Hence there is \( t \in \mathbb{N} \) such that \( a^i(f\lambda_i) = 0 \). Set \( j = i + t \). Then \( \pi_{ij}(f) = 0 \) and so

\[
g = \pi_i(f) = \pi_j(\pi_{ij}(f)) = 0.
\]

Next, we show that \( \psi \) is surjective. Let \( x_1, \ldots, x_t \) be a set of generators of \( M \). Let \( l \in (\text{Hom}_R(M, N))_a \). Then there are \( h \in \text{Hom}_R(M, N) \) and \( c \in \mathbb{N} \) such that \( l = h/a^c \). Since \( N \) is a Noetherian \( R \)-module, there exists an integer \( e \geq c \) such that \( (0 : N a^e) = (0 : N a^{e+j}) \) for all \( j \geq 0 \). Define \( f \in \text{Hom}_R(a^{2e}M, N) \) by \( f(a^{2e}x) = a^{2e-c}h(x) \) for \( x \in M \). If \( a^{2e}x = a^{2e}x' \) for some \( x \) and \( x' \) in \( M \), then \( h(x - x') \in (0 : N a^{2e}) \). Hence \( a^{2e-c}h(x) = a^{2e-c}h(x') \). Therefore \( f \) is well defined. Set \( g = \pi_{2e}(f) \). Then

\[
\psi(g) = \psi_{2e}(f) = f\lambda_{2e}/a^{2e} = h/a^c = l.
\]

Thus \( \psi \) is surjective.

(ii) Let \( \psi : D_a(M, N) \to (\text{Hom}_R(M, N))_a \) be as above. By part (i), [3, Theorem 2.2.4(i)] and 2.2 we have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & H^0_a(M, N) & \to & \text{Hom}_R(M, N) & \overset{f}{\to} & D_a(M, N) & \to & H^1_a(M, N) & \overset{g}{\to} & \text{Ext}_R(M, N) \\
\downarrow\text{id} & & \downarrow\psi & & & & & & \downarrow\text{id} \\
0 & \to & \Gamma_a(\text{Hom}_R(M, N)) & \to & \text{Hom}_R(M, N) & \overset{h}{\to} & (\text{Hom}_R(M, N))_a & \to & H^1_a(\text{Hom}_R(M, N)) & \to & 0
\end{array}
\]

Let \( K \) be the kernel of \( g \). We have \( K \cong \text{coker} f \) and \( H^1_a(\text{Hom}_R(M, N)) \cong \text{coker} h \). The map \( \psi \) induces an isomorphism \( \psi^* : \text{coker} f \to \text{coker} h \) defined by \( \psi^*(x + \text{im} f) = \psi(x) + \text{im} h \) for \( x + \text{im} f \in \text{coker} f \). Hence \( K \cong H^1_a(\text{Hom}_R(M, N)) \). Therefore \( K \) is \( a \)-cofinite, by 1.2. Now consider the exact sequence

\[
0 \to K \to H^1_a(M, N) \to \text{im} g \to 0.
\]

Since \( \text{Ext}_R^1(M, N) \) is finitely generated, it follows by 2.3 and 2.4(i) that \( \text{im} g \) is \( a \)-cofinite. Thus \( H^1_a(M, N) \) is \( a \)-cofinite, by 2.4(ii).

**Lemma 2.6.** Let \( a \) denote an ideal of the ring \( R \), and \( N \) an \( a \)-cofinite \( R \)-module. Suppose that for any finitely generated \( R \)-module \( M \) with \( \text{pd} M < \infty \), \( \text{Hom}_R(M, N) \) (resp. \( M \otimes_R N \)) is \( a \)-cofinite. Then \( \text{Ext}_R^1(M, N) \) (resp.
Tor^i_R(M, N)) is a-cotinfinite for all finitely generated R-modules M with pd M < ∞ and all i ≥ 0.

Proof. We prove only the a-cotinfineness of Ext^i_R(M, N) for i ≥ 0; the proof of the other part is similar. The proof proceeds by induction on t = pd M. For t = 0, the claim holds by assumption. Now, suppose t > 0. There is a short exact sequence

0 → K → R^n → M → 0.

From this sequence, we deduce the exact sequence

0 → Hom_R(M, N) → Hom_R(R^n, N) → Hom_R(K, N) → Ext^1_R(M, N) → 0,

and the isomorphisms Ext^{i+1}_R(M, N) ≅ Ext^i_R(K, N) for all i ≥ 1. Thus from induction hypothesis, Ext^{i+1}_R(M, N) is a-cotinfinite for all i ≥ 1. Note that pd K < t. Also, by using the above exact sequence, one can check easily that Ext^1_R(M, N) is a-cotinfinite. Therefore, the claim follows by induction.

Lemma 2.7. Let a denote an ideal of the ring R. Let M and N be two finitely generated R-modules with pd M < ∞. If either

(i) a is principal, or
(ii) R is complete local and a is a prime ideal with dim R/a = 1,

then Ext^p_R(M, H^q_a(N)) is a-cotinfinite for all p, q ≥ 0

Proof. First, we consider the case that a is principal. By [9, Theorem 11.38], there is a Grothendieck spectral sequence

E^{p,q}_2 := Ext^p_R(M, H^q_a(N)) → H^p+q_a(M, N).

We have E^{p,q}_2 = 0 for q ≠ 0, 1, because, by [3, Theorem 3.3.1], H^q_a(N) = 0 for all q > 1. Since E^{p,0}_2 is finitely generated and Supp_R E^{p,0}_2 ⊆ V(a), it follows by 2.4(i) that E^{p,0}_2 is a-cotinfinite. Therefore, it is enough to show that E^{p,1}_2 is a-cotinfinite for all p ≥ 0. By [9, Corollary 11.44], we have an exact sequence

0 → E^{1,0}_2 → H^1_a(M, N) → E^{0,1}_2 → E^{2,0}_2 → H^2_a(M, N).

By 2.5(ii), H^1_a(M, N) is a-cotinfinite. Thus, it turns out that im g is a-cotinfinite, by 2.4(ii). From the exact sequence

0 → im g → E^{0,1}_2 → ker f → 0,

we deduce that E^{0,1}_2 is a-cotinfinite. Note that ker f is a finitely generated R-module. Therefore 2.6 implies that E^{p,1}_2 is a-cotinfinite for all p ≥ 0, because H^q_a(N) is a-cotinfinite by 1.2.

Now suppose that R is a complete local ring and a a prime ideal of R with dim R/a = 1. In view of 2.6, it suffices to show that Hom_R(M, H^q_a(N))
is $\mathfrak{a}$-cofinite for all finitely generated $R$-modules $M$ with $\text{pd} M < \infty$. We prove this claim by induction on $\text{pd} M = t$. The case $t = 0$, is clear by 1.2. Now assume that $t > 0$ and consider the exact sequence

$$0 \to K \to R^n \to M \to 0.$$ 

It follows that $\text{pd} K \leq t - 1$. This short exact sequence yields the exact sequence

$$0 \to \text{Hom}_R(M, H^q_a(N)) \to \text{Hom}_R(R^n, H^q_a(N)) \xrightarrow{f} \text{Hom}_R(K, H^q_a(N)).$$

Since by [4, Theorem 2] the subcategory of $\mathfrak{a}$-cofinite $R$-modules is abelian, it follows that $\ker f \cong \text{Hom}_R(M, H^q_a(N))$ is $\mathfrak{a}$-cofinite. Therefore the claim follows by induction.

**Theorem 2.8.** Let $\mathfrak{a}$ denote a principal ideal of the ring $R$. Let $M$ and $N$ be two finitely generated $R$-modules with $\text{pd} M < \infty$. Then $H^p_\mathfrak{a}(M, N)$ is $\mathfrak{a}$-cofinite for all $p \geq 0$.

**Proof.** By [9, Theorem 11.38], there is a Grothendieck spectral sequence

$$E^{p,q}_2 := \text{Ext}^p_R(M, H^q_\mathfrak{a}(N)) \Rightarrow H^{p+q}_\mathfrak{a}(M, N).$$

This implies the following exact sequence in view of [11, Ex. 5.2.2] (note that $E^{p,q}_2 = 0$ for $q \neq 0, 1$):

$$\to E^{p,0}_2 \xrightarrow{f} H^p_\mathfrak{a}(M, N) \xrightarrow{d} E^{p-1,1}_2 \to E^{p+1,0}_2 \xrightarrow{g} H^{p+1}_\mathfrak{a}(M, N) \to \cdots.$$ 

Now, $\text{im} f$ is a quotient of $E^{p,0}_2$ and so is finitely generated. Hence $\text{im} f$ is $\mathfrak{a}$-cofinite, by 2.3 and 2.4(i). Also, $\ker g$ is $\mathfrak{a}$-cofinite by the same reason. By considering the short exact sequence

$$0 \to \text{im} d \to E^{p-1,1}_2 \to \ker g \to 0,$$

we deduce that $\text{im} d$ is $\mathfrak{a}$-cofinite. Note that $E^{p-1,1}_2$ is $\mathfrak{a}$-cofinite by 2.7(i). Now from the short exact sequence

$$0 \to \text{im} f \to H^p_\mathfrak{a}(M, N) \to \text{im} d \to 0,$$

we deduce that $H^p_\mathfrak{a}(M, N)$ is $\mathfrak{a}$-cofinite for all $p \geq 0$. 

**Theorem 2.9.** Let $\mathfrak{p}$ denote a prime ideal of the complete local ring $(R, \mathfrak{m})$ with $\dim R/\mathfrak{p} = 1$, and $M, N$ two finitely generated $R$-modules with $\text{pd} M < \infty$. Then $H^i_\mathfrak{p}(M, N)$ is $\mathfrak{p}$-cofinite for all $i \geq 0$.

**Proof.** There is a spectral sequence

$$E^{p,q}_2 := \text{Ext}^p_R(M, H^q_\mathfrak{p}(N)) \Rightarrow H^{p+q}_\mathfrak{p}(M, N) = E^n.$$ 

It follows from 2.7(ii) that $E^{p,q}_2$ is $\mathfrak{p}$-cofinite for all $p, q$. By considering the sequence

$$\cdots \to E^{p-2, q+1}_2 \xrightarrow{d^{p-2,q+1}} E^{p,q}_2 \xrightarrow{d^{p,q}} E^{p+2,q-1}_2 \to \cdots,$$
we deduce that $\text{im } d_2^{p,q-2, q+1}$ and $\text{ker } d_2^{p,q}$ are $p$-cofinite, by [4, Theorem 2]. Hence $E_3^{p,q} = \text{ker } d_2^{p,q} / \text{im } d_2^{p,q-2, q+1}$ is $p$-cofinite. By imitating this argument we find that $E_r^{p,q} = \text{ker } d_r^{p,q} / \text{im } d_r^{p-r+1,q+r-2}$ is $p$-cofinite for all $r > 0$ and so $E^{p,q}_\infty$ is $p$-cofinite for all $p, q \geq 0$. There is a filtration

$$E^n = E^n_0 \supseteq \cdots \supseteq E^n_p \supseteq \cdots \supseteq E^n_n \supseteq E^n_{n+1} = 0,$$

such that $E^n_p/E^n_{p+1} \cong E^{p,n-p}_\infty$. Thus $E^n_n$ is $p$-cofinite. Now, by applying 2.4(ii) repeatedly to the short exact sequences

$$0 \to E^n_{p+1} \to E^n_p \to E^{p,n-p}_\infty \to 0, \quad p = 0, 1, \ldots, n - 1,$$

we deduce that $E^n$ is $p$-cofinite, as required.

Many results concerning local cohomology in positive prime characteristic can be extended to generalized local cohomology. In particular the main results of [7] also hold for generalized local cohomology.

**Theorem 2.10.** Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$, and $\mathfrak{a}$ an ideal of $R$. Let $\mathfrak{p}$ be a prime ideal of $R$, and $M$ a finitely generated $R$-module. Then

(i) $\mu^i(\mathfrak{p}, H^j_{\mathfrak{a}}(M, R)) \leq \mu^i(\mathfrak{p}, \text{Ext}^j_R(M/\mathfrak{a}M, R))$ for all $j \geq 0$. In particular $\mu^i(\mathfrak{p}, H^j_{\mathfrak{a}}(M, R))$ is finite for all $j \geq 0$ and all $i \geq 0$.

(ii) $\text{Ass}_R(H^j_{\mathfrak{a}}(M, R)) \subseteq \text{Ass}_R(\text{Ext}^j_R(M/\mathfrak{a}M, R))$ and so $\text{Ass}_R(H^j_{\mathfrak{a}}(M, R))$ is finite for all $j \geq 0$.

**Proof.** The proof is a straightforward adaptation of the proof of [7, Theorem 2.1 and Corollary 2.3].

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