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VECTOR-VALUED INVARIANT MEANS ON SPACES OF BOUNDED LINEAR MAPS

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Abstract. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a W^*-algebra. For a homomorphism Φ from $\mathcal{A}$ into $\mathcal{M}$, we introduce and study $\mathcal{M}$-valued invariant Φ-means on the space of bounded linear maps from $\mathcal{A}$ into $\mathcal{M}$. We establish several characterizations of existence of an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A},\mathcal{M})$. We also study the relation between existence of an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A},\mathcal{M})$ and amenability of $\mathcal{A}$. Finally, for a character ϕ of $\mathcal{A}$, we give some descriptions for ϕ-amenability of $\mathcal{A}$ in terms of $\mathcal{M}$-valued invariant Φ-means.

1. Introduction. Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{Y}$ be a Banach $\mathcal{A}$-bimodule. Then a linear map $D : \mathcal{A} \to \mathcal{Y}$ is a *derivation* if

$$D(ab) = D(a)\cdot b + a\cdot D(b) \quad (a, b \in \mathcal{A}).$$

For example, let $\zeta \in \mathcal{Y}$, and set

$$D_\zeta(a) = a\cdot\zeta - \zeta\cdot a \quad (a \in \mathcal{A}).$$

Then D_ζ is a derivation; these are *inner derivations*. The Banach algebra $\mathcal{A}$ is called *amenable* if every continuous derivation $D : \mathcal{A} \to \mathcal{Y}^*$ is an inner derivation, for all Banach $\mathcal{A}$-bimodules $\mathcal{Y}$; this important concept was introduced by Johnson [J], where it is proved that the group algebra $L^1(G)$ is amenable precisely when the locally compact group G is amenable; i.e., there is an invariant mean $m : L^\infty(G) \to \mathbb{C}$. Several interesting results have been obtained with the help of invariant means; see for example [BF], [FNS] and [G].

Invariant means on spaces of vector-valued functions on a locally compact group G were first considered by Husain and Wong [HW2]; they studied invariant means on $L^\infty(G, E^*)$ which take values in E^*, the continuous dual

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of a separated locally convex space E; see also [CL], [Di] and [HW1]. In fact, the definition of invariant mean on a space of vector-valued functions reduces to the usual one introduced by Greenleaf [Gr].

On the other hand, for a nonzero character ϕ on a Banach algebra $\mathcal{A}$, the interesting notion of ϕ-amenability of $\mathcal{A}$ was recently introduced and studied by Kaniuth, Lau and Pym [KLP1] and simultaneously by Monfared [M]; see also [DNS], [GNN], [HMT], [KLP2] and [NS]. More precisely, $\mathcal{A}$ is *ϕ-amenable* if there is an *invariant ϕ-mean* on $\mathcal{A}^*$, that is, a bounded linear functional $m : \mathcal{A}^* \to \mathbb{C}$ such that $\langle m, \phi\rangle = 1$ and $\langle m, f \cdot a\rangle = \langle m, f\rangle\langle \phi, a\rangle$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where $f \cdot a \in \mathcal{A}^*$ is defined by $\langle f \cdot a, b\rangle = \langle f, ab\rangle$ for all $b \in \mathcal{A}$. This is a considerable generalization of the concept of left amenability for a Lau algebra; the class of Lau algebras was introduced and studied by Lau [L1], who called them F-algebras; see also [DNN], [L2] and [LW]. Examples of Lau algebras include the group algebra and measure algebra of a locally compact group or hypergroup, as well as the Fourier algebra and the Fourier–Stieltjes algebra of a topological group; see also [L1] and [LL].

In this paper, for a W^*-algebra $\mathcal{M}$, we introduce and study $\mathcal{M}$-valued invariant Φ-means on spaces of bounded linear maps from $\mathcal{A}$ into $\mathcal{M}$ associated to a bounded nonzero homomorphism Φ from $\mathcal{A}$ into $\mathcal{M}$.

2. Preliminaries. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a W^*-algebra with identity element u. Let us denote by $\Delta(\mathcal{A}, \mathcal{M})$ the set of all bounded nonzero homomorphisms from $\mathcal{A}$ into $\mathcal{M}$ and by $\Delta_u(\mathcal{A}, \mathcal{M})$ the subset of all elements in $\Delta(\mathcal{A}, \mathcal{M})$ whose image contains u. It is clear that $\Delta(\mathcal{A})$, the spectrum of $\mathcal{A}$, is just $\Delta(\mathcal{A}, \mathbb{C})$ which is equal to $\Delta_1(\mathcal{A}, \mathbb{C})$.

For each Banach right $\mathcal{A}$-module $\mathcal{X}$, let $B(\mathcal{X}, \mathcal{M})$ denote the Banach space of all bounded linear maps from $\mathcal{X}$ into $\mathcal{M}$. For each $\Phi \in \Delta(\mathcal{A}, \mathcal{M}) \cup \{0\}$, we denote by $B(\mathcal{X}, \mathcal{M}_\Phi)$ the Banach $\mathcal{A}$-bimodule $B(\mathcal{X}, \mathcal{M})$ with the module actions

$$(S \cdot a)(\xi) = S(\xi)\Phi(a) \quad \text{and} \quad (a \cdot S)(\xi) = S(\xi \cdot a),$$

for all $S \in B(\mathcal{X}, \mathcal{M})$, $a \in \mathcal{A}$ and $\xi \in \mathcal{X}$.

Moreover, let us remark that $B(\mathcal{A}, \mathcal{M})$ is a Banach $\mathcal{A}$-bimodule with the actions of $\mathcal{A}$ on $B(\mathcal{A}, \mathcal{M})$ given by $(T \cdot a)(b) = T(ab)$ and $(a \cdot T)(b) = T(ba)$ for all $a, b \in \mathcal{A}$ and $T \in B(\mathcal{A}, \mathcal{M})$; see Dales [D] for more details.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a W^*-algebra. Then the following statements are equivalent:*

- (i) *$\mathcal{A}$ has a bounded right approximate identity.*
- (ii) *Any continuous derivation $D : \mathcal{A} \to B(\mathcal{X}, \mathcal{M}_0)$ is inner for all Banach right $\mathcal{A}$-modules $\mathcal{X}$.*

Proof. First, we note that $B(\mathcal{X},\mathcal{M})$ is isometrically isomorphic to $(\mathcal{X}\widehat{\otimes}\mathcal{L})^*$, where $\mathcal{L}$ is the unique predual of $\mathcal{M}$ and $\mathcal{X}\widehat{\otimes}\mathcal{L}$ is the projective tensor product of $\mathcal{X}$ and $\mathcal{L}$; see for example [D].

(i)⇒(ii). Suppose that $\mathcal{A}$ has a bounded right approximate identity (e_α), and let $\mathcal{X}$ be a Banach right $\mathcal{A}$-module. If $D:\mathcal{A}\to B(\mathcal{X},\mathcal{M}_0)$ is a continuous derivation, then

$$D(ab)=a\cdot D(b)$$

for all $a,b\in\mathcal{A}$, and $(D(e_\alpha))$ can be considered as a net in $(\mathcal{X}\widehat{\otimes}\mathcal{L})^*$. So, there exists a subnet $(D(e_\beta))$ of $(D(e_\alpha))$ such that $D(e_\beta)\overset{w^*}{\to}S$ for some $S\in B(\mathcal{X},\mathcal{M})$. On the one hand, for each $a\in\mathcal{A}$, $\xi\in\mathcal{X}$ and $\lambda\in\mathcal{L}$, we have

$$\langle a\cdot D(e_\beta),\xi\otimes\lambda\rangle=\langle D(e_\beta),\xi\cdot a\otimes\lambda\rangle=\langle S,\xi\cdot a\otimes\lambda\rangle=\langle a\cdot S,\xi\otimes\lambda\rangle.$$

It follows that $a\cdot D(e_\beta)\overset{w^*}{\to}a\cdot S$. On the other hand,

$$\|D(ae_\beta)-D(a)\|\le\|D\|\,\|ae_\beta-a\|\to 0.$$

Therefore,

$$a\cdot D(e_\beta)=D(ae_\beta)\overset{w^*}{\to}D(a).$$

So, $D(a)=a\cdot S$. Consequently, $D=D_S$ is inner.

(ii)⇒(i). Define the map $D:\mathcal{A}\to B(B(\mathcal{A},\mathcal{M}),\mathcal{M}_0)$ by

$$D(a)(T)=T(a)$$

for all $a\in\mathcal{A}$ and $T\in B(\mathcal{A},\mathcal{M})$. It is easy to check that

$$D:\mathcal{A}\to B(B(\mathcal{A},\mathcal{M}),\mathcal{M}_0)$$

is a continuous derivation. By assumption, there is $\mathbf{m}$ in $B(B(\mathcal{A},\mathcal{M}),\mathcal{M}_0)$ such that $D(a)=a\cdot\mathbf{m}$ for all $a\in\mathcal{A}$. Let u be the identity element of $\mathcal{M}$ and let λ_0 be an element in $\mathcal{L}$ such that $\langle u,\lambda_0\rangle=1$. Let E be a bounded linear functional on $\mathcal{A}^*$ defined by

$$\langle E,f\rangle=\langle\mathbf{m}(f\otimes u),\lambda_0\rangle$$

for all $f\in\mathcal{A}^*$, where $f\otimes u$ is the element in $B(\mathcal{A},\mathcal{M})$ defined by

$$\langle(f\otimes u)(a),\lambda\rangle=\langle f,a\rangle\langle u,\lambda\rangle$$

for all $a\in\mathcal{A}$ and $\lambda\in\mathcal{L}$. For each $a,b\in\mathcal{A}$ and $\lambda\in\mathcal{L}$ we obtain

$$\begin{aligned}\langle(((f\otimes u)\cdot a)(b),\lambda\rangle&=\langle(f\otimes u)(ab),\lambda\rangle=\langle f,ab\rangle\langle u,\lambda\rangle\\&=\langle f\cdot a,b\rangle\langle u,\lambda\rangle=\langle(f\cdot a\otimes u)(b),\lambda\rangle.\end{aligned}$$

Therefore, for each $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$,

$$\begin{aligned}\langle a \cdot E, f\rangle &= \langle E, f \cdot a\rangle = \langle \mathbf{m}((f \cdot a) \otimes u), \lambda_0\rangle \\ &= \langle \mathbf{m}((f \otimes u) \cdot a), \lambda_0\rangle = \langle (a \cdot \mathbf{m})(f \otimes u), \lambda_0\rangle \\ &= \langle D(a)((f \otimes u)), \lambda_0\rangle = \langle (f \otimes u)(a), \lambda_0\rangle \\ &= \langle f, a\rangle\langle u, \lambda_0\rangle = \langle f, a\rangle.\end{aligned}$$

That is, E is a right identity for $\mathcal{A}^{**}$. Now, a standard argument shows that $\mathcal{A}$ has a bounded right approximate identity; see for example [R, Proposition 2.2.1]. ■

3. Vector-valued invariant $\varPhi$-means. We commence this section with the main object of the paper.

DEFINITION 3.1. Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{M}$ be a W^*-algebra with identity element u and let $\varPhi \in \varDelta(\mathcal{A}, \mathcal{M})$. We say that a bounded linear map $\mathbf{m} : B(\mathcal{A}, \mathcal{M}) \to \mathcal{M}$ is an *$\mathcal{M}$-valued invariant $\varPhi$-mean* on $B(\mathcal{A}, \mathcal{M})$ if for each $T \in B(\mathcal{A}, \mathcal{M})$ and $a \in \mathcal{A}$,

$$\mathbf{m}(\varPhi) = u \quad \text{and} \quad \mathbf{m}(T \cdot a) = \mathbf{m}(T)\varPhi(a).$$

Our first key result gives a characterization of existence of vector-valued invariant $\varPhi$-means.

PROPOSITION 3.2. *Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{M}$ be a W^*-algebra and let $\varPhi \in \varDelta_u(\mathcal{A}, \mathcal{M})$. Suppose that any continuous derivation $D : \mathcal{A} \to B(\mathcal{X}, \mathcal{M}_\varPhi)$ is inner for all Banach right $\mathcal{A}$-modules $\mathcal{X}$. Then there exists an $\mathcal{M}$-valued invariant $\varPhi$-mean on $B(\mathcal{A}, \mathcal{M})$.*

Proof. Since $\mathbb{C}\varPhi$ is a closed $\mathcal{A}$-submodule of $B(\mathcal{A}, \mathcal{M})$, we can consider the quotient space $\mathcal{X} = B(\mathcal{A}, \mathcal{M})/\mathbb{C}\varPhi$. Let π be the canonical mapping of $B(\mathcal{A}, \mathcal{M})$ onto $\mathcal{X}$, and define the $\mathcal{A}$-module monomorphism

$$\pi' : B(\mathcal{X}, \mathcal{M}_\varPhi) \to B(B(\mathcal{A}, \mathcal{M}), \mathcal{M}_\varPhi)$$

by

$$\pi'(S)(T) = S(\pi(T))$$

for all $S \in B(\mathcal{X}, \mathcal{M}_\varPhi)$ and $T \in B(\mathcal{A}, \mathcal{M})$. Since $\varPhi \in \varDelta_u(\mathcal{A}, \mathcal{M})$, there exists an element $a_0 \in \mathcal{A}$ such that $\varPhi(a_0) = u$. If $\mathbf{m}_0 \in B(B(\mathcal{A}, \mathcal{M}), \mathcal{M})$ is given by $\mathbf{m}_0(T) = T(a_0)$ for all $T \in B(\mathcal{A}, \mathcal{M})$, then $\mathbf{m}_0(\varPhi) = u$. Consider the inner derivation

$$D_{\mathbf{m}_0} : \mathcal{A} \to B(B(\mathcal{A}, \mathcal{M}), \mathcal{M}_\varPhi),$$

and note that $D_{\mathbf{m}_0}(a)(\varPhi) = 0$. Now, define the map $D : \mathcal{A} \to B(\mathcal{X}, \mathcal{M}_\varPhi)$ by $D(a)(\pi(T)) = D_{\mathbf{m}_0}(a)(T)$ for all $a \in \mathcal{A}$ and $T \in B(\mathcal{A}, \mathcal{M})$; this means that $\pi' \circ D = D_{\mathbf{m}_0}$. The fact that π' is a monomorphism shows that $D(a)$ is a unique element of $B(\mathcal{X}, \mathcal{M}_\varPhi)$ for all $a \in \mathcal{X}$, and that D is a continuous derivation. By assumption, D is inner and so there exists $S_0 \in B(\mathcal{X}, \mathcal{M}_\Phi)$ with $D = D_{S_0}$. Therefore

$$a \cdot \pi'(S_0) - \pi'(S_0) \cdot a = \pi'(a \cdot S_0 - S_0 \cdot a) = \pi'(D(a))$$
$$= D_{\mathbf{m}_0}(a) = a \cdot \mathbf{m}_0 - \mathbf{m}_0 \cdot a.$$

Put $\mathbf{m} := \mathbf{m}_0 - \pi'(S_0)$. Then

$$\mathbf{m} \in B(B(\mathcal{A}, \mathcal{M}), \mathcal{M}),$$

$\mathbf{m}(\Phi) = u$ and $a \cdot \mathbf{m} = \mathbf{m} \cdot a$, and thus

$$\mathbf{m}(T \cdot a) = (a \cdot \mathbf{m})(T) = (\mathbf{m} \cdot a)(T) = \mathbf{m}(T)\Phi(a)$$

for all $a \in \mathcal{A}$ and $T \in B(\mathcal{A}, \mathcal{M})$. ∎

In the next result, we prove the converse of Proposition 3.2, just for certain $\Phi \in \Delta(\mathcal{A}, \mathcal{M})$.

THEOREM 3.3. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a W^*-algebra. For each epimorphism $\Phi \in \Delta(\mathcal{A}, \mathcal{M})$, the following statements are equivalent:*

(i) *Any continuous derivation $D : \mathcal{A} \to B(\mathcal{X}, \mathcal{M}_\Phi)$ is inner for all Banach right $\mathcal{A}$-modules $\mathcal{X}$.*
(ii) *There is an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A}, \mathcal{M})$.*

Proof. The implication (i)⇒(ii) follows from Proposition 3.2. To prove (ii)⇒(i), suppose that $\mathbf{m}$ is an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A}, \mathcal{M})$. Let $\mathcal{X}$ be a Banach right $\mathcal{A}$-module and let $D : \mathcal{A} \to B(\mathcal{X}, \mathcal{M}_\Phi)$ be an arbitrary continuous derivation. Then D defines the bounded linear map $D' : B(B(\mathcal{X}, \mathcal{M}_\Phi), \mathcal{M}) \to B(\mathcal{A}, \mathcal{M})$ by

$$D'(\Lambda)(a) = \Lambda(D(a))$$

for all $\Lambda \in B(B(\mathcal{X}, \mathcal{M}_\Phi), \mathcal{M})$ and $a \in \mathcal{A}$. Let also $\Upsilon : \mathcal{X} \to B(B(\mathcal{X}, \mathcal{M}_\Phi), \mathcal{M})$ denote the bounded linear map satisfying

$$\Upsilon(\xi)(S) = S(\xi)$$

for all $\xi \in \mathcal{X}$ and $S \in B(\mathcal{X}, \mathcal{M}_\Phi)$. Now, define $(D' \circ \Upsilon)' : B(B(\mathcal{A}, \mathcal{M}), \mathcal{M}) \to B(\mathcal{X}, \mathcal{M}_\Phi)$ by

$$(D' \circ \Upsilon)'(\mathbf{n})(\xi) = \mathbf{n}\big((D' \circ \Upsilon)(\xi)\big)$$

for all $\mathbf{n} \in B(B(\mathcal{A}, \mathcal{M}), \mathcal{M})$ and $\xi \in \mathcal{X}$. Set $S_0 := -(D' \circ \Upsilon)'(\mathbf{m}) \in B(\mathcal{X}, \mathcal{M}_\Phi)$. To end this part, we will show $D = D_{S_0}$. Since D is a derivation, it follows that

$$\begin{aligned} D' \circ \Upsilon(\xi \cdot a)(b) &= \Upsilon(\xi \cdot a)(D(b)) \\ &= D(b)(\xi \cdot a) \\ &= (a \cdot D(b))(\xi) \\ &= D(ab)(\xi) - (D(a) \cdot b)(\xi) \\ &= \Upsilon(\xi)(D(ab)) - D(a)(\xi)\Phi(b) \\ &= D' \circ \Upsilon(\xi)(ab) - D(a)(\xi)\Phi(b) \\ &= (D' \circ \Upsilon(\xi) \cdot a)(b) - D(a)(\xi)\Phi(b) \end{aligned}$$

for all $a \in \mathcal{A}$ and $\xi \in \mathcal{X}$. Thus

$$D' \circ \Upsilon(\xi \cdot a) = D' \circ \Upsilon(\xi) \cdot a - D(a)(\xi)\Phi,$$

where $D(a)(\xi)\Phi \in B(\mathcal{A}, \mathcal{M})$ is defined by

$$(D(a)(\xi)\Phi)(b) = D(a)(\xi)\Phi(b)$$

for all $b \in \mathcal{A}$. Since $\Phi \in \Delta(\mathcal{A}, \mathcal{M})$ is an epimorphism, for each $a \in \mathcal{A}$ and $\xi \in \mathcal{X}$ there exists an element $x_{a,\xi} \in \mathcal{A}$ such that $\Phi(x_{a,\xi}) = D(a)(\xi)$. Therefore

$$(\Phi \cdot x_{a,\xi})(b) = \Phi(x_{a,\xi}b) = \Phi(x_{a,\xi})\Phi(b) = D(a)(\xi)\Phi(b)$$

for all $b \in \mathcal{A}$. Thus $\Phi \cdot x_{a,\xi} = D(a)(\xi)\Phi$. Furthermore, for each $a \in \mathcal{A}$ and $\xi \in \mathcal{X}$,

$$\begin{aligned} (a \cdot S_0)(\xi) &= -(D' \circ \Upsilon)'(\mathbf{m})(\xi \cdot a) \\ &= -\mathbf{m}((D' \circ \Upsilon)(\xi \cdot a)) \\ &= \mathbf{m}(D(a)(\xi)\Phi) - \mathbf{m}((D' \circ \Upsilon)(\xi) \cdot a) \\ &= \mathbf{m}(\Phi \cdot x_{a,\xi}) - \mathbf{m}((D' \circ \Upsilon)(\xi))\Phi(a) \\ &= \mathbf{m}(\Phi)\Phi(x_{a,\xi}) - \mathbf{m}((D' \circ \Upsilon)(\xi))\Phi(a) \\ &= D(a)(\xi) + S_0(\xi)\Phi(a), \end{aligned}$$

and hence $D(a) = a \cdot S_0 - S_0\Phi(a)$. Combining this with the equation $S_0 \cdot a = S_0\Phi(a)$, we obtain

$$D(a) = a \cdot S_0 - S_0 \cdot a = D_{S_0}(a)$$

for all $a \in \mathcal{A}$. Consequently, D is inner. ∎

The following result describes interaction between existence of a vector-valued invariant Φ-mean and amenability of Banach algebras.

Proposition 3.4. *Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{M}$ be a commutative W^*-algebra and let $\Phi \in \Delta_u(\mathcal{A}, \mathcal{M})$. If $\mathcal{A}$ is amenable, then there is an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A}, \mathcal{M})$.*

Proof. Suppose that $\mathcal{A}$ is amenable. Then for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$, any continuous derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is inner. Let $\mathcal{L}$ denote the predual of $\mathcal{M}$ and note that $(\mathcal{X}\widehat{\otimes}\mathcal{L})^* = B(\mathcal{X},\mathcal{M})$ as Banach spaces. Since $\mathcal{M}$ is a commutative W^*-algebra, the projective tensor product $\mathcal{X}\,\widehat{\otimes}\,\mathcal{L}$ with the module operations restricted from the dual Banach $\mathcal{A}$-bimodule $B(\mathcal{X},\mathcal{M}_\Phi)^*$ is a Banach $\mathcal{A}$-bimodule. It follows that any continuous derivation from $\mathcal{A}$ into $B(\mathcal{X},\mathcal{M}_\Phi)$ is inner. Now, by Proposition 3.2, we conclude that there exists an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A},\mathcal{M})$. ∎

Let G be a locally compact group, and let $L^1(G)$ be the group algebra of G, so that $L^1(G)$ is a Banach algebra for the convolution product defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dy \qquad (x \in G)$$

for $f, g \in L^1(G)$. By the *measure algebra* $M(G)$ of G, we mean the Banach algebra of bounded regular Borel measures on G; see [HR]. Let $\mathcal{M}$ be a W^*-algebra. Each $\Phi \in \Delta_u(L^1(G),\mathcal{M})$ has the extension $\widetilde{\Phi} \in \Delta_u(M(G),\mathcal{M})$ defined by $\widetilde{\Phi}(\mu) := \Phi(\mu * f_0)$ for all $\mu \in M(G)$, where $f_0 \in L^1(G)$ with $\Phi(f_0) = u$, and $\mu * f$ is an element in $L^1(G)$ defined by

$$(\mu * f)(x) = \int_G f(y^{-1}x)\,d\mu(y) \qquad (x \in G)$$

for all $f \in L^1(G)$. If $\Psi \in \Delta_u(M(G),\mathcal{M})$ is another extension of Φ, then for each $\mu \in M(G)$, we have

$$\widetilde{\Phi}(\mu) = \Phi(\mu * f_0) = \Psi(\mu * f_0) = \Psi(\mu)\Psi(f_0) = \Psi(\mu).$$

As a consequence, we have the following result.

Proposition 3.5. *Let G be a locally compact group and let $\mathcal{M}$ be a W^*-algebra. Let $\Phi \in \Delta_u(L^1(G),\mathcal{M})$ and $\widetilde{\Phi} \in \Delta_u(M(G),\mathcal{M})$ be the unique extension of Φ. If there is an $\mathcal{M}$-valued invariant Φ-mean on $B(L^1(G),\mathcal{M})$, then there exists an $\mathcal{M}$-valued invariant $\widetilde{\Phi}$-mean on $B(M(G),\mathcal{M})$.*

Proof. Let $\mathbf{m}$ be an $\mathcal{M}$-valued invariant Φ-mean on $B(L^1(G),\mathcal{M})$. Choose $f_0 \in L^1(G)$ such that $\Phi(f_0) = u$. For each $T \in B(M(G),\mathcal{M})$, define $\widetilde{\mathbf{m}}(T) = \mathbf{m}(T|_{L^1(G)})$. We show that $\mathbf{m}$ is an $\mathcal{M}$-valued invariant $\widetilde{\Phi}$-mean on $B(M(G),\mathcal{M})$. To this end, note that

$$\widetilde{\mathbf{m}}(\widetilde{\Phi}) = \mathbf{m}(\widetilde{\Phi}|_{L^1(G)}) = \mathbf{m}(\Phi) = u.$$

Moreover, for each $T \in B(M(G),\mathcal{M})$ and $\mu \in M(G)$ we have

$$(T \cdot \mu)|_{L^1(G)} \cdot f_0 = T|_{L^1(G)} \cdot (\mu * f_0).$$

Since $\Phi(f_0) = u$, we obtain

$$\begin{aligned}\widetilde{\mathbf{m}}(T\cdot\mu) &= \mathbf{m}((T\cdot\mu)|_{L^1(G)}) = \mathbf{m}((T\cdot\mu)|_{L^1(G)})\Phi(f_0)\\ &= \mathbf{m}((T\cdot\mu)|_{L^1(G)}\cdot f_0) = \mathbf{m}(T|_{L^1(G)}\cdot(\mu * f_0))\\ &= \mathbf{m}(T|_{L^1(G)})\Phi(\mu * f_0) = \mathbf{m}(T|_{L^1(G)})\widetilde{\Phi}(\mu)\widetilde{\Phi}(f_0)\\ &= \widetilde{\mathbf{m}}(T)\widetilde{\Phi}(\mu),\end{aligned}$$

as required. ∎

The following example shows that the converse of Proposition 3.4 is not true.

Example 3.6. Let G be a nondiscrete amenable locally compact group and let $\mathcal{M}$ be a commutative W^*-algebra. Then $L^1(G)$ is amenable, and by Proposition 3.4, there is an $\mathcal{M}$-valued invariant Φ-mean on $B(L^1(G),\mathcal{M})$ for all $\Phi \in \Delta_u(L^1(G),\mathcal{M})$. Moreover, by Proposition 3.5, there exists an $\mathcal{M}$-valued invariant $\widetilde{\Phi}$-mean on $B(M(G),\mathcal{M})$, where $\widetilde{\Phi}$ is a unique extension of Φ. However $M(G)$ is not amenable; see [DGH].

4. Characterizations of ϕ-amenability. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a W^*-algebra with identity u. For $\phi \in \Delta(\mathcal{A})$, we define the map $\phi u \in \Delta_u(\mathcal{A},\mathcal{M})$ by $(\phi u)(a) = \langle \phi, a\rangle u$ for all $a \in \mathcal{A}$. In this section, we give two descriptions of ϕ-amenability of $\mathcal{A}$ in terms of vector-valued invariant ϕu-means.

Proposition 4.1. *Let $\mathcal{A}$ be a Banach algebra, let $\phi \in \Delta(\mathcal{A})$ and let $\mathcal{M}$ be a W^*-algebra. Then $\mathcal{A}$ is ϕ-amenable if and only if there exists an $\mathcal{M}$-valued invariant ϕu-mean on $B(\mathcal{A},\mathcal{M})$.*

Proof. First, suppose that $\mathcal{A}$ is ϕ-amenable. Then there exists an element $m \in \mathcal{A}^{**}$ with

$$\langle m, \phi\rangle = 1 \quad \text{and} \quad \langle m, f\cdot a\rangle = \langle m, f\rangle\langle \phi, a\rangle$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Let $\mathcal{L}$ be the unique predual of the W^*-algebra $\mathcal{M}$. Fix $\lambda_0 \in \mathcal{L}$ with $\langle u, \lambda_0\rangle = 1$, and let $\widehat{\lambda}_0 \in \mathcal{M}^*$ be defined by $\langle \widehat{\lambda}_0, \omega\rangle = \langle \omega, \lambda_0\rangle$ for all $\omega \in \mathcal{M}$. Define $\mathbf{m} : B(\mathcal{A},\mathcal{M}) \to \mathcal{M}$ by

$$\mathbf{m}(T) = \langle m, \widehat{\lambda}_0 \circ T\rangle u$$

for all $T \in B(\mathcal{A},\mathcal{M})$. For each $a \in \mathcal{A}$, we have

$$\langle \widehat{\lambda}_0 \circ (\phi u), a\rangle = \langle \widehat{\lambda}_0, \langle \phi, a\rangle u\rangle = \langle \phi, a\rangle\langle \widehat{\lambda}_0, u\rangle = \langle \phi, a\rangle.$$

So, $\mathbf{m}(\phi u) = \langle m, \phi\rangle u = u$. Furthermore,

$$\begin{aligned}\langle \widehat{\lambda}_0 \circ (T\cdot a), b\rangle &= \langle \widehat{\lambda}_0, (T\cdot a)(b)\rangle = \langle T(ab), \lambda_0\rangle\\ &= \langle \widehat{\lambda}_0 \circ T, ab\rangle = \langle (\widehat{\lambda}_0 \circ T)\cdot a, b\rangle,\end{aligned}$$

and

$$\mathbf{m}(T\cdot a)=\langle m,\widehat{\lambda}_0\circ(T\cdot a)\rangle u=\langle m,\widehat{\lambda}_0\circ T\rangle\langle\phi,a\rangle u=\mathbf{m}(T)(\phi u)(a)$$

for all $a,b\in\mathcal{A}$ and $T\in B(\mathcal{A},\mathcal{M})$. Thus $\mathbf{m}$ is an $\mathcal{M}$-valued invariant ϕu-mean.

For the converse, suppose that $\mathbf{m}\in B(B(\mathcal{A},\mathcal{M}),\mathcal{M})$ is an $\mathcal{M}$-valued invariant ϕu-mean and define the map $\Lambda:\mathcal{A}^*\to B(\mathcal{A},\mathcal{M})$ by

$$\Lambda(f)(a)=\langle f,a\rangle u$$

for all $f\in\mathcal{A}^*$ and $a\in\mathcal{A}$. We claim that $\widehat{\lambda}_0\circ\mathbf{m}\circ\Lambda\in\mathcal{A}^{**}$ is an invariant ϕ-mean. To prove this, for each $a,b\in\mathcal{A}$ and $f\in\mathcal{A}^*$, we note that

$$\Lambda(f\cdot a)(b)=\langle f\cdot a,b\rangle u=\langle f,ab\rangle u=\Lambda(f)(ab)=(\Lambda(f)\cdot a)(b).$$

So, we have

$$\begin{aligned}\langle\widehat{\lambda}_0\circ\mathbf{m}\circ\Lambda,f\cdot a\rangle&=\langle\widehat{\lambda}_0\circ\mathbf{m},\Lambda(f)\cdot a\rangle=\langle\widehat{\lambda}_0,\mathbf{m}(\Lambda(f)\cdot a)\rangle\\&=\langle\widehat{\lambda}_0,\mathbf{m}(\Lambda(f))\langle\phi,a\rangle u\rangle=\langle\phi,a\rangle\langle\widehat{\lambda}_0,\mathbf{m}(\Lambda(f))\rangle\\&=\langle\phi,a\rangle\langle\widehat{\lambda}_0\circ\mathbf{m}\circ\Lambda,f\rangle.\end{aligned}$$

Moreover, $\langle\widehat{\lambda}_0\circ\mathbf{m}\circ\Lambda,\phi\rangle=1$; indeed,

$$\langle\widehat{\lambda}_0\circ\mathbf{m}\circ\Lambda,\phi\rangle=\langle\widehat{\lambda}_0\circ\mathbf{m},\phi u\rangle=\langle\widehat{\lambda}_0,u\rangle=1.$$

Thus $\mathcal{A}$ is ϕ-amenable. ∎

PROPOSITION 4.2. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a W^*-algebra. For $\phi\in\Delta(\mathcal{A})$, the following three conditions are equivalent:*

(i) *$\mathcal{A}$ is ϕ-amenable.*
(ii) *Any continuous derivation $D:\mathcal{A}\to B(B(\ker(\phi),\mathcal{M}),\mathcal{M}_{\phi u})$ is inner.*
(iii) *Any continuous derivation $D:\mathcal{A}\to B(\mathcal{X},\mathcal{M}_{\phi u})$ is inner for all Banach right $\mathcal{A}$-modules $\mathcal{X}$.*

Proof. (i)⇒(ii) is obvious. The implication (i)⇒(iii) follows from Proposition 4.1 and Theorem 3.3. Moreover (iii)⇒(ii) is trivial.

To prove (ii)⇒(i), define the map $D:\mathcal{A}\to B(B(\ker(\phi),\mathcal{M}),\mathcal{M}_{\phi u})$ by

$$D(a)(\Xi)=\Xi(ab-\langle\phi,a\rangle b)$$

for all $\Xi\in B(\ker(\phi),\mathcal{M})$. Then D is a continuous derivation. So, by (ii), there exists $\mathbf{n}\in B(B(\ker(\phi),\mathcal{M}),\mathcal{M}_{\phi u})$ such that $D=D_{\mathbf{n}}$. Choose $b\in\mathcal{A}$ with $\langle\phi,b\rangle=1$. Define $\mathbf{m}\in B(B(\mathcal{A},\mathcal{M}),\mathcal{M})$ by $\mathbf{m}(T)=T(b)-\mathbf{n}(T|_{\ker(\phi)})$ for all $T\in B(\mathcal{A},\mathcal{M})$. Then

$$\mathbf{m}(\phi u)=(\phi u)(b)-\mathbf{n}((\phi u)|_{\ker(\phi)})=u;$$

furthermore, for each $a \in \mathcal{A}$ and $T \in B(\mathcal{A}, \mathcal{M})$, we have

$$\begin{aligned}\mathbf{m}(T \cdot a) &= (T \cdot a)(b) - \mathbf{n}(T_0 \cdot a) \\ &= (T \cdot a)(b) - T_0(ab - \langle \phi, a \rangle b) - (\mathbf{n} \cdot a)(T_0) \\ &= \langle \phi, a \rangle T(b) - (\mathbf{n} \cdot a)(T_0) \\ &= \langle \phi, a \rangle T(b) - \mathbf{n}(T_0)\langle \phi, a \rangle \\ &= \langle \phi, a \rangle \mathbf{m}(T),\end{aligned}$$

where $T_0 = T|_{\ker(\phi)}$. So, $\mathbf{m} \in B(B(\mathcal{A}, \mathcal{M}), \mathcal{M})$ is an $\mathcal{M}$-valued invariant ϕu-mean. Consequently, $\mathcal{A}$ is ϕ-amenable by Proposition 4.1. ■

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REFERENCES

[BF] J. W. Baker and M. Filali, *Minimal ideals in group algebras and their biduals*, Math. Proc. Cambridge Philos. Soc. 120 (1996), 475–488.

[CL] C. Chou and A. T. Lau, *Vector-valued invariant means on spaces of bounded operators associated to a locally compact group*, Illinois J. Math. 45 (2001), 581–602.

[D] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. 24, Oxford Sci.Publ., Clarendon Press, New York, 2000.

[DGH] H. G. Dales, F. Ghahramani and A. Ya. Helemskii, *The amenability of measure algebras*, J. London Math. Soc. (2) 66 (2002), 213–226.

[DNS] M. Dashti, R. Nasr-Isfahani and S. Soltani Renani, *Character amenability of Lipschitz algebra*, Canad. Math. Bull. (2012), to appear.

[DNN] S. Desaulniers, R. Nasr-Isfahani and M. Nemati, *Common fixed point properties and amenability of a class of Banach algebras*, J. Math. Anal. Appl. 402 (2013), 536–544.

[Di] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leur applications*, Acta Sci. Math. (Szeged) 12 (1950), 213–227.

[FNS] M. Filali, M. Neufang and M. Sangani Monfared, *On ideals in the bidual of the Fourier algebra and related algebras*, J. Funct. Anal. 258 (2010), 3117–3133.

[GNN] E. Ghaderi, R. Nasr-Isfahani and M. Nemati, *Some notions of amenability for certain products of Banach algebras*, Colloq. Math. 130 (2013), 147–157.

[G] E. E. Granirer, *Geometric and topological properties of certain w^* compact convex sets which arise from the study of invariant means*, Canad. J. Math. 37 (1985), 107–121.

[Gr] F. P. Greenleaf, *Invariant Means on Topological Groups and their Applications*, Van Nostrand Math. Stud. 16, Van Nostrand, New York, 1969.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer, Berlin, 1963.

[HMT] Z. Hu, M. S. Monfared and T. Traynor, *On character amenable Banach algebras*, Studia Math. 193 (2009), 53–78.

[HW1] T. Husain and J. C. S. Wong, *Invariant means on vector valued functions I*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 27 (1973), 717–727.

[HW2] T. Husain and J. C. S. Wong, *Invariant means on vector valued functions II*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 27 (1973), 729–742.

[J] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).

[KLP1] E. Kaniuth, A. T.-M. Lau and J. Pym, *On ϕ-amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc. 144 (2008), 85–96.

[KLP2] E. Kaniuth, A. T.-M. Lau and J. Pym, *On character amenability of Banach algebras*, J. Math. Anal. Appl. 344 (2008), 942–955.

[L1] A. T.-M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fund. Math. 118 (1983), 161–175.

[L2] A. T.-M. Lau, *Uniformly continuous functionals on Banach algebras*, Colloq. Math. 51 (1987), 195–205.

[LL] A. T.-M. Lau and J. Ludwig, *Fourier–Stieltjes algebra of a topological group*, Adv. Math. 229 (2012), 2000–2023.

[LW] A. T.-M. Lau and J. C. S. Wong, *Invariant subspaces for algebras of linear operators and amenable locally compact groups*, Proc. Amer. Math. Soc. 102 (1988), 581–586.

[M] M. S. Monfared, *Character amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc. 144 (2008), 697–706.

[NS] R. Nasr-Isfahani and S. Soltani Renani, *Character contractibility of Banach algebras and homological properties of Banach modules*, Studia Math. 202 (2011), 205–225.

[R] V. Runde, *Lectures on Amenability*, Lecture Notes in Math. 1774, Springer, Berlin, 2002.

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