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ON A RELATION BETWEEN NORMS OF THE MAXIMAL FUNCTION AND THE SQUARE FUNCTION OF A MARTINGALE

BY

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Abstract. Let Ω be a nonatomic probability space, let X be a Banach function space over Ω, and let $\mathcal{M}$ be the collection of all martingales on Ω. For $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}$, let Mf and Sf denote the maximal function and the square function of f, respectively. We give some necessary and sufficient conditions for X to have the property that if $f, g \in \mathcal{M}$ and $\|Mg\|_X \le \|Mf\|_X$, then $\|Sg\|_X \le C\|Sf\|_X$, where C is a constant independent of f and g.

1. Introduction. Let $(\Omega, \Sigma, \mathbb{P})$ be a nonatomic probability space. We denote by $\mathbb{F}$ the collection of all filtrations of $(\Omega, \Sigma, \mathbb{P})$, i.e., the collection of all sequences $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ of sub-σ-algebras of Σ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{Z}_+$. For each $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, we denote by $\mathcal{M}(\mathcal{F})$ the collection of all $\mathcal{F}$-martingales, and we let $\mathcal{M} = \bigcup_{\mathcal{F}\in\mathbb{F}} \mathcal{M}(\mathcal{F})$. For $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}$, the *maximal function* and the *square function* of f are defined by

$$Mf = \sup_{n\in\mathbb{Z}_+} |f_n| \quad \text{and} \quad Sf = \Big\{\sum_{n=0}^{\infty} (\Delta_n f)^2\Big\}^{1/2},$$

respectively, where $\Delta_n f = f_n - f_{n-1}$ for $n \ge 1$ and $\Delta_0 f = f_0$.

Let X be a Banach function space over Ω (see Definition 2.1 below). Recently the author gave in [6] some necessary and sufficient conditions for X to have the property that if $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is uniformly integrable and if $v = (v_n)$ is an $\mathcal{F}$-predictable process which is uniformly bounded by one in absolute value, then

$$\|M(v * f)\|_X \le C\|Mf\|_X,$$

where $v * f = ((v * f)_n)$ denotes the martingale transform of f by v, and where C is a constant which is independent of f and v. One such necessary and sufficient condition is that if $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$, and $\|Sg\|_X \le \|Sf\|_X$,

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then

$$\|Mg\|_X \le C\|Mf\|_X.$$

It is therefore natural to ask when X has the property that if $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$, and $\|Mg\|_X \le \|Mf\|_X$, then

$$\|Sg\|_X \le C\|Sf\|_X.$$

Our result gives necessary and sufficient conditions for X to have this property.

2. Preliminaries. Throughout this note, we assume that the probability space $(\Omega, \Sigma, \mathbb{P})$ is nonatomic, i.e., there is no $\mathbb{P}$-atom in Σ. In addition, we consider the interval $I := (0, 1]$ as a probability space equipped with Lebesgue measure ds.

Given two Banach spaces X and Y, we write $Y \hookrightarrow X$ to mean that Y is continuously embedded in X, i.e., $Y \subset X$ and the inclusion map is continuous.

DEFINITION 2.1. A *Banach function space* X over a probability space is a real Banach space of (equivalence classes of) random variables (i.e., measurable functions) having the following properties:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$.
(B2) If $|y| \le |x|$ a.s. and $x \in X$, then $y \in X$ and $\|y\|_X \le \|x\|_X$.
(B3) If $0 \le x_n \uparrow x$ a.s., $x_n \in X$ for all n, and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

We adopt the convention that $\|x\|_X = \infty$ if x is a random variable which does not belong to X.

Given two random variables x and y, we write $x \simeq_d y$ to mean that x and y have the same distribution.

DEFINITION 2.2. A Banach function space X is said to be *rearrangement-invariant* or *r.i.* if it has the property that whenever $x \simeq_d y$ and $x \in X$, then $y \in X$ and $\|x\|_X = \|y\|_X$.

By an *r.i. space* we mean a rearrangement-invariant Banach function space.

For example, Lebesgue spaces, Orlicz spaces, and Lorentz spaces are r.i. spaces, while weighted Lebesgue spaces with suitable weights are Banach function spaces which are not r.i. in general (cf. [3, Section 4]).

Let x be a random variable on Ω. The *nonincreasing rearrangement* of x is the function x^* on $I = (0, 1]$ defined by

$$x^*(t) = \inf\{\lambda > 0 \colon \mathbb{P}(\omega \in \Omega \colon |x(\omega)| > \lambda) \le t\}, \qquad t \in I,$$

with the convention that $\inf \emptyset = \infty$. Note that x^* is a nonincreasing right-continuous function whose distribution (with respect to Lebesgue measure) is the same as that of $|x|$. As a result, nonnegative random variables x and y have the same distribution if and only if $x^* = y^*$ on I.

The nonincreasing rearrangement ϕ^* of a measurable function ϕ on I is defined in the same way. Of course, if ϕ is nonincreasing, then $\phi^* = \phi$ a.s. on I.

Suppose that X is an r.i. space over Ω. The Luxemburg representation theorem ([1, Theorem 4.10, p. 62]) shows that there exists an r.i. space $\hat{X}$ over I which has the following properties:

- $x \in X$ if and only if $x^* \in \hat{X}$.
- $\|x\|_X = \|x^*\|_{\hat{X}}$ for all $x \in X$.

Such an r.i. space $\hat{X}$ is unique (cf. [1, p. 64]). For example, $\hat{L}_p(\Omega) = L_p(I)$.

Let $L_0(I)$ denote the linear space of real-valued measurable functions on I. If Z is a Banach function space over I, we denote by $B(Z)$ the collection of all linear operators T satisfying the following conditions:

- The domain of T contains Z and the range of T is contained in $L_0(I)$.
- The restriction of T to Z is a bounded operator from Z into itself.

For $T \in B(Z)$, we denote by $\|T\|_{B(Z)}$ the operator norm of the restriction of T to Z.

Our result will be stated in terms of the Boyd indices. We now recall their definition for an r.i. space. For each $s > 0$, define $D_s \colon L_0(I) \to L_0(I)$ by

$$(D_s\phi)(t) = \begin{cases} \phi(st) & \text{if } st \in I, \\ 0 & \text{if } st \notin I. \end{cases}$$

If Z is an r.i. space over I, then $D_s \in B(Z)$ for all $s > 0$. The *lower Boyd index* and the *upper Boyd index* of Z are defined by

$$\alpha_Z = \sup_{0<s<1} \frac{\log \|D_{1/s}\|_{B(Z)}}{\log s} \quad \text{and} \quad \beta_Z = \inf_{1<s<\infty} \frac{\log \|D_{1/s}\|_{B(Z)}}{\log s},$$

respectively. If X is an r.i. space over Ω, then the Boyd indices of X are defined by $\alpha_X = \alpha_{\hat{X}}$ and $\beta_X = \beta_{\hat{X}}$. For example, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1, \infty]$. More generally, we have $0 \le \alpha_X \le \beta_X \le 1$ for all r.i. spaces X. See [1, pp. 146–150 and p. 165] for details.

In the proof of our result, we will use an integral operator $\mathcal{P}$ and its formal adjoint $\mathcal{Q}$, which are defined as follows. If $\phi \in L_1(I)$, we let

$$(\mathcal{P}\phi)(t) = \frac{1}{t}\int_0^t \phi(s)\, ds, \qquad t \in I,$$

and if ϕ is integrable over $(t,1)$ for all $t \in I$, we let

$$(\mathcal{Q}\phi)(t) = \int_t^1 \frac{\phi(s)}{s}\, ds, \qquad t \in I.$$

It is easily checked that if $\phi \in L_1(I)$, then $\mathcal{Q}\phi \in L_1(I)$ and

$$\mathcal{P}(\mathcal{Q}\phi) = \mathcal{P}\phi + \mathcal{Q}\phi. \tag{2.1}$$

Suppose that X is an r.i. space over Ω. Shimogaki [7] proved (implicitly) that $\beta_X < 1$ if and only if $\mathcal{P} \in B(\hat{X})$, and that $\alpha_X > 0$ if and only if $\mathcal{Q} \in B(\hat{X})$ (see [1, pp. 150–153] for a proof).

3. The main result. Before stating the results, let us recall our convention that if X is a Banach function space and x is a random variable which does not belong to X, then $\|x\|_X = \infty$.

Our result is the following:

MAIN THEOREM. *Let X be a Banach function space over Ω. Then the following are equivalent:*

(i) *There exists a constant $C > 0$ such that if $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$, and $\|Mg\|_X \le \|Mf\|_X$, then*

$$\|Sg\|_X \le C\|Sf\|_X.$$

(ii) *There exists a constant $C > 0$ such that if $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$, and $Mg \le Mf$ a.s., then*

$$\|Sg\|_X \le C\|Sf\|_X.$$

(iii) *There exists a constant $C > 0$ such that if $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$, and $\|Sg\|_X \le \|Sf\|_X$, then*

$$\|Mg\|_X \le C\|Mf\|_X.$$

(iv) *There exists a constant $C > 0$ such that if $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$, and $Sg \le Sf$ a.s., then*

$$\|Mg\|_X \le C\|Mf\|_X.$$

(v) *There exists a constant $C > 0$ such that if $f \in \mathcal{M}$, then*

$$C^{-1}\|Sf\|_X \le \|Mf\|_X \le C\|Sf\|_X.$$

(vi) *X can be equivalently renormed so as to be an r.i. space such that $\alpha_X > 0$.*

It is known that the second inequality of (v) holds for all $f \in \mathcal{M}$ if and only if (vi) holds (see [5]). The Main Theorem extends this result.

The rest of this note is devoted to the proof of the Main Theorem. We begin by introducing the following notation:

- Given a random variable x on Ω and a real number λ, we denote by $\{x > \lambda\}$ the set of all $\omega \in \Omega$ for which $x(\omega) > \lambda$.
- Let ϕ be a function on I and let x be a random variable on Ω. If the range of x is contained in I, we denote by $\phi(x)$ the composition $\omega \mapsto \phi(x(\omega))$.
- If A is a subset of Ω or of I, we denote by $\mathbf{1}_A$ the indicator function of A.
- Let Z be a Banach function space over I. We denote by $\mathcal{D}(Z)$ the collection of all functions in Z which are nonnegative, nonincreasing, and right-continuous.

The proof of the Main Theorem requires a series of lemmas.

Lemma 3.1 ([4, Lemma 4]). *Let X be an r.i. space over Ω. Suppose there exists a constant $c > 0$ such that $\|\mathcal{Q}\psi\|_{\hat{X}} \leq c\|\psi\|_{\hat{X}}$ for all $\psi \in \mathcal{D}(\hat{X})$. Then $\mathcal{Q} \in B(\hat{X})$, or equivalently, $\alpha_X > 0$.*

Note that, since $(\Omega, \Sigma, \mathbb{P})$ is nonatomic, there exist nonnegative random variables ξ_1 and ξ_2 on Ω satisfying the following conditions (cf. [2, (5.6)]):

$$\{\xi_1 > 0\} \cap \{\xi_2 > 0\} = \emptyset, \tag{3.1}$$

$$\xi_1^*(t) = \xi_2^*(t) = \max\{1 - 2t, 0\} \quad \text{for all } t \in I. \tag{3.2}$$

A straightforward calculation yields the following:

Lemma 3.2. *Let ξ_1 and ξ_2 be as above, and let $\psi \in L_1(I)$. Define random variables x_1 and x_2 by letting*

$$x_1 = \psi(1 - \xi_1)\mathbf{1}_{\{\xi_1 > 0\}} \quad \text{and} \quad x_2 = \psi(1 - \xi_2)\mathbf{1}_{\{\xi_2 > 0\}},$$

and define families of sets $\{A_1(t) \colon t \in (0, 1/2]\}$ and $\{A_2(t) \colon t \in (0, 1/2]\}$ by letting

$$A_1(t) = \{\xi_1 > 1 - 2t\}, \quad t \in (0, 1/2],$$
$$A_2(t) = \{\xi_2 > 1 - 2t\}, \quad t \in (0, 1/2].$$

Then:

- *$\{|x_1| > 0\} \subset A_1(1/2)$ and $\{|x_2| > 0\} \subset A_2(1/2)$.*
- *$A_1(1/2) \cap A_2(1/2) = \emptyset$.*
- *$A_1(s) \subset A_1(t)$ and $A_2(s) \subset A_2(t)$ whenever $0 < s < t \leq 1/2$.*
- *$\mathbb{P}(A_1(t)) = \mathbb{P}(A_2(t)) = t$ for all $t \in (0, 1/2]$.*
- *$(x_1 - x_2)^*(t) = (x_1 + x_2)^*(t) = \psi^*(t)$ for all $t \in I$.*
- *$\mathbb{E}[x_1\mathbf{1}_{A_1(t)}] = \mathbb{E}[x_2\mathbf{1}_{A_2(t)}] = t(\mathcal{P}\psi)(2t)$ for all $t \in (0, 1/2]$.*

Lemma 3.3. *Let X be a Banach function space over Ω. If X is r.i. and if* (ii) *of the Main Theorem holds, then $\alpha_X > 0$.*

Proof. Suppose that (ii) holds. By Lemma 3.1, it suffices to prove that

$$\|\mathcal{Q}\psi\|_{\hat{X}} \leq C\|\psi\|_{\hat{X}} \quad \text{for all } \psi \in \mathcal{D}(\hat{X}), \tag{3.3}$$

where C is a constant which is independent of ψ. Fix $\psi \in \mathcal{D}(\hat{X})$ and let $\phi = \mathcal{Q}\psi - \psi$. Of course, we may assume that $\psi \not\equiv 0$, which implies that $(\mathcal{Q}\psi)(t) \to \infty$ as $t \to 0+$. Note that ψ, $\mathcal{Q}\psi$, and ϕ are integrable over I. Let $\varepsilon > 0$ and define a sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ of numbers in I as follows:

$$\begin{aligned} &t_0 = 1/2, \\ &t_n = \inf\{t \in I\colon (\mathcal{Q}\psi)(2t) < (\mathcal{Q}\psi)(2t_{n-1}) + \varepsilon/n\}, \qquad n \geq 1. \end{aligned}$$

Since $\mathcal{Q}\psi$ is continuous and $(\mathcal{Q}\psi)(t) \to \infty$ as $t \to 0+$, we have

$$(\mathcal{Q}\psi)(2t_n) = (\mathcal{Q}\psi)(2t_{n-1}) + \varepsilon/n \quad \text{for all } n \geq 1. \tag{3.4}$$

It is easy to check that $\{t_n\}$ is strictly decreasing and $t_n \to 0$ as $n \to \infty$.

Let ξ_1 and ξ_2 be random variables satisfying (3.1) and (3.2), and let $\{A_1(t)\colon t \in (0, 1/2]\}$ and $\{A_2(t)\colon t \in (0, 1/2]\}$ be the families of sets defined as in Lemma 3.2. We define random variables x_1, x_2, y_1, y_2, z_1, and z_2 as follows:

$$\begin{aligned} x_1 &= \phi(1 - \xi_1)\mathbf{1}_{\{\xi_1>0\}}, & x_2 &= \phi(1 - \xi_2)\mathbf{1}_{\{\xi_2>0\}}, \\ y_1 &= (\mathcal{Q}\psi)(1 - \xi_1)\mathbf{1}_{\{\xi_1>0\}}, & y_2 &= (\mathcal{Q}\psi)(1 - \xi_2)\mathbf{1}_{\{\xi_2>0\}}, \\ z_1 &= \psi(1 - \xi_1)\mathbf{1}_{\{\xi_1>0\}}, & z_2 &= \psi(1 - \xi_2)\mathbf{1}_{\{\xi_2>0\}}. \end{aligned}$$

Let $\Lambda_n = A_1(t_n) \cup A_2(t_n)$ for each $n \in \mathbb{Z}_+$. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ by

$$\mathcal{F}_n = \sigma(\{\Lambda \setminus \Lambda_n\colon \Lambda \in \Sigma\}), \qquad n \in \mathbb{Z}_+,$$

and define $g = (g_n) \in \mathcal{M}(\mathcal{F})$ by

$$g_n = \mathbb{E}[y_1 - y_2 \,|\, \mathcal{F}_n], \qquad n \in \mathbb{Z}_+.$$

Since $\mathbb{E}[y_1\mathbf{1}_{A_1(t_n)}] = \mathbb{E}[y_2\mathbf{1}_{A_2(t_n)}]$ by Lemma 3.2, it follows that

$$\begin{aligned} g_n &= \frac{\mathbf{1}_{\Lambda_n}}{\mathbb{P}(\Lambda_n)}\,\mathbb{E}[(y_1 - y_2)\mathbf{1}_{\Lambda_n}] + (y_1 - y_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \\ &= \frac{\mathbf{1}_{\Lambda_n}}{\mathbb{P}(\Lambda_n)}\,\{\mathbb{E}[y_1\mathbf{1}_{A_1(t_n)}] - \mathbb{E}[y_2\mathbf{1}_{A_2(t_n)}]\} + (y_1 - y_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \\ &= (y_1 - y_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \quad \text{a.s.} \end{aligned}$$

for all $n \in \mathbb{Z}_+$. Therefore $\Delta_n g = (y_1 - y_2)\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n}$ a.s. for all $n \geq 1$, and $\Delta_0 g = g_0 = 0$ a.s. Moreover, we have

$$Mg = Sg = |y_1 - y_2| = y_1 + y_2 \quad \text{a.s.}, \tag{3.5}$$

because $\mathbb{P}(\Lambda_n) = 2t_n \to 0$. Since $\mathcal{Q}\psi$ is nonincreasing, Lemma 3.2 implies

$$(Sg)^*(t) = (y_1 + y_2)^*(t) = (\mathcal{Q}\psi)(t) \quad \text{for all } t \in I. \tag{3.6}$$

Now define $h = (h_n) \in \mathcal{M}(\mathcal{F})$ by

$$h_n = \mathbb{E}[x_1 + x_2 \,|\, \mathcal{F}_n], \qquad n \in \mathbb{Z}_+.$$

Since $\mathcal{P}\phi = \mathcal{Q}\psi$ by (2.1), it follows from Lemma 3.2 that

$$
\begin{aligned}
h_n &= \frac{\mathbf{1}_{\Lambda_n}}{\mathbb{P}(\Lambda_n)}\mathbb{E}[(x_1 + x_2)\mathbf{1}_{\Lambda_n}] + (x_1 + x_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \\
&= \frac{\mathbf{1}_{\Lambda_n}}{2t_n}\{\mathbb{E}[x_1\mathbf{1}_{A_1(t_n)}] + \mathbb{E}[x_2\mathbf{1}_{A_2(t_n)}]\} + (x_1 + x_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \\
&= (\mathcal{P}\phi)(2t_n)\mathbf{1}_{\Lambda_n} + (x_1 + x_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \\
&= (\mathcal{Q}\psi)(2t_n)\mathbf{1}_{\Lambda_n} + (x_1 + x_2)\mathbf{1}_{\Omega\setminus\Lambda_n} \quad \text{a.s.}
\end{aligned}
\tag{3.7}
$$

for all $n \in \mathbb{Z}_+$. We need to estimate Sh. If $n = 0$, then $\Delta_n h = h_0 = 0$ a.s.; and if $n \geq 1$, then by (3.4),

$$
\begin{aligned}
\Delta_n h &= \big\{(\mathcal{Q}\psi)(2t_n) - (\mathcal{Q}\psi)(2t_{n-1})\big\}\mathbf{1}_{\Lambda_n} \\
&\quad + \{x_1 + x_2 - (\mathcal{Q}\psi)(2t_{n-1})\}\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} \\
&= \frac{\varepsilon}{n}\mathbf{1}_{\Lambda_n} + \{x_1 + x_2 - (\mathcal{Q}\psi)(2t_{n-1})\}\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} \quad \text{a.s.}
\end{aligned}
\tag{3.8}
$$

Since $2t_n \leq 1 - \xi_i < 2t_{n-1}$ on $A_i(t_{n-1}) \setminus A_i(t_n)$, we see from (3.4) that

$$
\begin{aligned}
&|x_i - (\mathcal{Q}\psi)(2t_{n-1})|\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)} \\
&\qquad = |(\mathcal{Q}\psi)(1 - \xi_i) - \psi(1 - \xi_i) - (\mathcal{Q}\psi)(2t_{n-1})|\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)} \\
&\qquad \leq \{|(\mathcal{Q}\psi)(1 - \xi_i) - (\mathcal{Q}\psi)(2t_{n-1})| + \psi(1 - \xi_i)\}\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)} \\
&\qquad \leq \{(\mathcal{Q}\psi)(2t_n) - (\mathcal{Q}\psi)(2t_{n-1}) + z_i\}\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)} \\
&\qquad \leq (\varepsilon/n + z_i)\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)}
\end{aligned}
$$

for $n \geq 1$ and $i = 1, 2$. Note that $x_i = z_i = 0$ on $A_j(t_{n-1})$ when $i \neq j$, and that $\Lambda_{n-1}\setminus\Lambda_n$ is the disjoint union of $A_1(t_{n-1})\setminus A_1(t_n)$ and $A_2(t_{n-1})\setminus A_2(t_n)$. Then we have

$$
\begin{aligned}
|x_1 + x_2 - (\mathcal{Q}\psi)(2t_{n-1})|\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} &= \sum_{i=1}^{2} |x_i - (\mathcal{Q}\psi)(2t_{n-1})|\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)} \\
&\leq \sum_{i=1}^{2} \left(\frac{\varepsilon}{n} + z_i\right)\mathbf{1}_{A_i(t_{n-1})\setminus A_i(t_n)} \\
&\leq (\varepsilon + z_1 + z_2)\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n}.
\end{aligned}
$$

Therefore by (3.8),

$$
\begin{aligned}
(Sh)^2 &= \sum_{n=1}^{\infty}\left\{\frac{\varepsilon^2}{n^2}\mathbf{1}_{\Lambda_n} + |x_1 + x_2 - (\mathcal{Q}\psi)(2t_{n-1})|^2\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n}\right\} \\
&\leq \varepsilon^2\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}(\varepsilon + z_1 + z_2)^2\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} \\
&= \kappa^2\varepsilon^2 + (\varepsilon + z_1 + z_2)^2 \quad \text{a.s.,}
\end{aligned}
$$

where $\kappa^2 = \sum_{n=1}^{\infty}(1/n^2) = \pi^2/6$. Thus

$$Sh \le \{\kappa^2\varepsilon^2 + (\varepsilon + z_1 + z_2)^2\}^{1/2} \le (\kappa+1)\varepsilon + z_1 + z_2 \quad \text{a.s.}$$

Notice that by Lemma 3.2, the nonincreasing rearrangement of the right-hand side is equal to

$$(\kappa+1)\varepsilon + (z_1+z_2)^*(t) = (\kappa+1)\varepsilon + \psi(t).$$

It follows that

$$(Sh)^*(t) \le (\kappa+1)\varepsilon + \psi(t) \quad \text{for all } t \in I. \tag{3.9}$$

Now let $f = (f_n)$ be the martingale defined by

$$f_n = h_n + \varepsilon, \quad n \in \mathbb{Z}_+. \tag{3.10}$$

We estimate Mf. Since $2t_n \le 1 - \xi_i < 2t_{n-1}$ on $A_i(t_{n-1}) \setminus A_i(t_n)$, we have

$$\begin{aligned}(\mathcal{Q}\psi)(2t_n)\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} &= \sum_{i=1}^{2}(\mathcal{Q}\psi)(2t_n)\mathbf{1}_{A_i(2t_{n-1})\setminus A_i(2t_n)} \\ &\ge \sum_{i=1}^{2}(\mathcal{Q}\psi)(1-\xi_i)\mathbf{1}_{A_i(2t_{n-1})\setminus A_i(2t_n)} \\ &= \sum_{i=1}^{2} y_i \mathbf{1}_{A_i(2t_{n-1})\setminus A_i(2t_n)} = (y_1+y_2)\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n}\end{aligned} \tag{3.11}$$

for all $n \ge 1$. From (3.4), (3.7), (3.10), and (3.11) we see that

$$\begin{aligned}Mf &= \sup_{n\in\mathbb{Z}_+}[\{(\mathcal{Q}\psi)(2t_n)+\varepsilon\}\mathbf{1}_{\Lambda_n} + |x_1+x_2+\varepsilon|\mathbf{1}_{\Omega\setminus\Lambda_n}] \\ &\ge \sup_{n\in\mathbb{Z}_+}\{(\mathcal{Q}\psi)(2t_n)+\varepsilon\}\mathbf{1}_{\Lambda_n} = \sum_{n=1}^{\infty}\{(\mathcal{Q}\psi)(2t_{n-1})+\varepsilon\}\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} \\ &\ge \sum_{n=1}^{\infty}(\mathcal{Q}\psi)(2t_n)\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} \ge \sum_{n=1}^{\infty}(y_1+y_2)\mathbf{1}_{\Lambda_{n-1}\setminus\Lambda_n} = y_1+y_2 \quad \text{a.s.}\end{aligned}$$

Therefore $Mg \le Mf$ a.s. by (3.5). Since

$$Sf = \Big\{\sum_{n=1}^{\infty}(\Delta_n f)^2 + f_0^2\Big\}^{1/2} = \Big\{\sum_{n=1}^{\infty}(\Delta_n h)^2 + \varepsilon^2\Big\}^{1/2} \le Sh + \varepsilon$$

and since we are assuming that (ii) of the Main Theorem holds, it follows from (3.9) that

$$\begin{aligned}\|Sg\|_X &\le C\|Sf\|_X \le C\|Sh+\varepsilon\|_X \le C\|Sh\|_X + C\varepsilon\|\mathbf{1}_\Omega\|_X \\ &\le C\|(Sh)^*\|_{\hat{X}} + C\varepsilon\|\mathbf{1}_\Omega\|_X \le C\|(\kappa+1)\varepsilon+\psi\|_{\hat{X}} + C\varepsilon\|\mathbf{1}_\Omega\|_X \\ &\le C(\kappa+2)\varepsilon\|\mathbf{1}_\Omega\|_X + C\|\psi\|_{\hat{X}}.\end{aligned}$$

On the other hand, $\|\mathcal{Q}\psi\|_{\hat{X}} = \|(Sg)^*\|_{\hat{X}} = \|Sg\|_X$ by (3.6). Thus

$$\|\mathcal{Q}\psi\|_{\hat{X}} \le C(\kappa+2)\varepsilon\|\mathbf{1}_\Omega\|_X + C\|\psi\|_{\hat{X}}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (3.3) holds, as required. ∎

LEMMA 3.4 ([4, Lemma 1]; [5, Lemma 5.1]). *Let X be a Banach function space over Ω and let $\mathcal{S}_+$ be the set of all nonnegative simple random variables on Ω. Then the following are equivalent:*

- *There exists a constant $c > 0$ such that if $x, y \in \mathcal{S}_+$, $x \simeq_d y$, and $\{x > 0\} \cap \{y > 0\} = \emptyset$, then $\|y\|_X \le c\|x\|_X$.*
- *X can be equivalently renormed so as to be r.i.*

LEMMA 3.5. *Let X be a Banach function space over Ω. If* (ii) *of the Main Theorem holds, then X can be equivalently renormed so as to be r.i.*

Proof. Suppose that $x, y \in \mathcal{S}_+$, $x \simeq_d y$, and $\{x > 0\} \cap \{y > 0\} = \emptyset$. By the previous lemma, it suffices to prove that

$$\|y\|_X \le c\|x\|_X, \tag{3.12}$$

where c is a constant which is independent of x and y. We can write

$$x = \sum_{k=1}^{\ell} \lambda_k \mathbf{1}_{A_k} \quad \text{and} \quad y = \sum_{k=1}^{\ell} \lambda_k \mathbf{1}_{B_k},$$

where $\lambda_k > 0$ for all $k = 1, \ldots, \ell$, and where $\{A_k\}_{k=1}^{\ell}$ and $\{B_k\}_{k=1}^{\ell}$ are pairwise disjoint sequences of Σ-measurable subsets of Ω such that

$$\mathbb{P}(A_k) = \mathbb{P}(B_k) > 0 \quad \text{for all } k = 1, \ldots, \ell$$

and

$$\Big(\bigcup_{k=1}^{\ell} A_k\Big) \cap \Big(\bigcup_{k=1}^{\ell} B_k\Big) = \emptyset.$$

Let $\{s_n\}_{n\in\mathbb{Z}_+}$ be the sequence of real numbers defined by

$$s_n = \sum_{k=0}^{n} \frac{1}{k+1}, \qquad n \in \mathbb{Z}_+,$$

and fix an integer N such that $s_N \ge C\pi/\sqrt{6} + 1$, where C is the constant in (ii) of the Main Theorem. For each $k = 1, \ldots, \ell$, let $\{C_{k,n}\}_{n=0}^{N}$ be a finite sequence of Σ-measurable subsets of A_k such that

$$C_{k,0} = A_k, \quad C_{k,0} \supset C_{k,1} \supset C_{k,2} \supset \cdots \supset C_{k,N},$$

and

$$\mathbb{P}(C_{k,n}) = \frac{1}{2^n}\mathbb{P}(A_k) = \frac{1}{2^n}\mathbb{P}(B_k) \quad \text{for all } n = 1, \ldots, N.$$

Such sequences $\{C_{k,n}\}_{n=0}^N$ certainly exist, because $(\Omega, \Sigma, \mathbb{P})$ is nonatomic. Furthermore, for each $k = 1, \ldots, \ell$, define a sequence $\{D_{k,n}\}_{n=1}^N$ by

$$D_{k,n} = C_{k,n-1} \setminus C_{k,n}, \qquad n = 1, \ldots, N,$$

and define a sequence $\{E_{k,n}\}_{n=0}^N$ by

$$E_{k,0} = \emptyset, \qquad E_{k,n} = \bigcup_{m=1}^{n} D_{k,m}, \qquad n = 1, \ldots, N.$$

Then clearly $\mathbb{P}(D_{k,n}) = \mathbb{P}(C_{k,n})$ and $E_{k,n} = A_k \setminus C_{k,n}$ for all $k = 1, \ldots, \ell$ and all $n = 1, \ldots, N$. We define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ by letting

$$\begin{aligned}
\mathcal{F}_0 &= \sigma\Big(\bigcup_{k=1}^{\ell} \{B_k \cup C_{k,0}\}\Big),\\
\mathcal{F}_1 &= \sigma\Big(\bigcup_{k=1}^{\ell} \{B_k \cup C_{k,1}, D_{k,1}\}\Big),\\
\mathcal{F}_2 &= \sigma\Big(\bigcup_{k=1}^{\ell} \{B_k \cup C_{k,2}, D_{k,1}, D_{k,2}\}\Big),\\
&\ \vdots\\
\mathcal{F}_N &= \sigma\Big(\bigcup_{k=1}^{\ell} \{B_k \cup C_{k,N}, D_{k,1}, \ldots, D_{k,N}\}\Big),
\end{aligned}$$

and

$$\mathcal{F}_n = \Sigma, \qquad n \geq N + 1;$$

and we define a process $f = (f_n)$ by letting

$$\begin{aligned}
f_0 &= \sum_{k=1}^{\ell} \lambda_k s_0 \mathbf{1}_{B_k \cup C_{k,0}} = x + y,\\
f_1 &= \sum_{k=1}^{\ell} \left[\lambda_k \left(s_1 \mathbf{1}_{B_k \cup C_{k,1}} + \left(s_0 - \frac{3}{2} \right) \mathbf{1}_{D_{k,1}} \right) + f_0 \mathbf{1}_{E_{k,0}} \right],\\
f_2 &= \sum_{k=1}^{\ell} \left[\lambda_k \left(s_2 \mathbf{1}_{B_k \cup C_{k,2}} + \left(s_1 - \frac{2^2 + 1}{3} \right) \mathbf{1}_{D_{k,2}} \right) + f_1 \mathbf{1}_{E_{k,1}} \right],\\
&\ \vdots\\
f_N &= \sum_{k=1}^{\ell} \left[\lambda_k \left(s_N \mathbf{1}_{B_k \cup C_{k,N}} + \left(s_{N-1} - \frac{2^N + 1}{N + 1} \right) \mathbf{1}_{D_{k,N}} \right) + f_{N-1} \mathbf{1}_{E_{k,N-1}} \right],
\end{aligned}$$

and

$$f_n = f_N, \qquad n \geq N+1.$$

We claim that $f = (f_n)$ is an $\mathcal{F}$-martingale. To see this, let $2 \leq n \leq N$ and let $a_{k,n} = \mathbb{E}[f_n \mathbf{1}_{B_k \cup C_{k,n-1}}]$. It is easy to see that

$$\begin{aligned} a_{k,n} &= \mathbb{E}\Big[\lambda_k\Big(s_n \mathbf{1}_{B_k\cup C_{k,n}} + \Big(s_{n-1} - \frac{2^n+1}{n+1}\Big)\mathbf{1}_{D_{k,n}}\Big)\mathbf{1}_{B_k\cup C_{k,n-1}}\Big] \\ &= \lambda_k s_{n-1}(1+2^{-n+1})\mathbb{P}(B_k). \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{E}[f_n \,|\, \mathcal{F}_{n-1}] = \sum_{k=1}^{\ell} \mathbb{E}[f_n \mathbf{1}_{B_k\cup C_{k,n-1}} + f_n \mathbf{1}_{E_{k,n-1}} \,|\, \mathcal{F}_{n-1}] \\ &= \sum_{k=1}^{\ell}\Big[\frac{\mathbf{1}_{B_k\cup C_{k,n-1}}}{\mathbb{P}(B_k \cup C_{k,n-1})}\, a_{k,n} + f_{n-1}\mathbf{1}_{E_{k,n-1}}\Big] \\ &= \sum_{k=1}^{\ell}\Big[\frac{\mathbf{1}_{B_k\cup C_{k,n-1}}}{(1+2^{-n+1})\mathbb{P}(B_k)}\, a_{k,n} + f_{n-1}\mathbf{1}_{E_{k,n-1}}\Big] \\ &= \sum_{k=1}^{\ell}[\lambda_k s_{n-1}\mathbf{1}_{B_k\cup C_{k,n-1}} + f_{n-1}\mathbf{1}_{E_{k,n-1}}] \\ &= \sum_{k=1}^{\ell}\Big[\lambda_k s_{n-1}\mathbf{1}_{B_k\cup C_{k,n-1}} + \lambda_k\Big(s_{n-2} - \frac{2^{n-1}+1}{n}\Big)\mathbf{1}_{D_{k,n-1}} + f_{n-1}\mathbf{1}_{E_{k,n-2}}\Big] \\ &= \sum_{k=1}^{\ell}\Big[\lambda_k\Big(s_{n-1}\mathbf{1}_{B_k\cup C_{k,n-1}} + \Big(s_{n-2} - \frac{2^{n-1}+1}{n}\Big)\mathbf{1}_{D_{k,n-1}}\Big) + f_{n-2}\mathbf{1}_{E_{k,n-2}}\Big] \\ &= f_{n-1} \quad \text{a.s.} \end{aligned}$$

In the same way, we have $\mathbb{E}[f_1 \,|\, \mathcal{F}_0] = x + y = f_0$ a.s. Thus $f = (f_n)$ is an $\mathcal{F}$-martingale.

We have to estimate Sf. Let

$$U = \{y > 0\} \cup \bigcup_{k=1}^{\ell} C_{k,N} = \bigcup_{k=1}^{\ell}(B_k \cup C_{k,N}) \quad \text{and} \quad V = \bigcup_{k=1}^{\ell} E_{k,N}.$$

Then

$$Sf = (Sf)\mathbf{1}_U + (Sf)\mathbf{1}_V. \tag{3.13}$$

The estimate of $(Sf)\mathbf{1}_U$ is easy. Indeed if $0 \leq j \leq N$, then

$$(\Delta_j f)\mathbf{1}_U = \sum_{k=1}^{\ell}(\Delta_j f)\mathbf{1}_{B_k\cup C_{k,N}} = \frac{1}{j+1}\sum_{k=1}^{\ell}\lambda_k \mathbf{1}_{B_k\cup C_{k,N}} = \frac{x+y}{j+1}\,\mathbf{1}_U,$$

and if $j > N$, then $\Delta_j f = 0$. Therefore

$$(3.14)\qquad (Sf)^2\mathbf{1}_U = \sum_{j=0}^{N}(\Delta_j f)^2\mathbf{1}_U = \sum_{j=0}^{N}\frac{(x+y)^2}{(j+1)^2}\,\mathbf{1}_U \le \frac{\pi^2(x^2+y^2)}{6}\,\mathbf{1}_U.$$

We now estimate $(Sf)\mathbf{1}_V$. If $0 \le j < m \le N$, then $D_{k,m} \subset C_{k,j}$ and hence $(\Delta_j f)\mathbf{1}_{D_{k,m}} = (\lambda_k/(j+1))\mathbf{1}_{D_{k,m}}$ for each $k = 1, \dots, \ell$. On the other hand, if $1 \le m < j \le N$, then $D_{k,m} \subset E_{k,j-1}$ and hence $(\Delta_j f)\mathbf{1}_{D_{k,m}} = 0$ for each $k = 1, \dots, \ell$. Moreover, $(\Delta_m f)\mathbf{1}_{D_{k,m}} = -((2^m+1)\lambda_k/(m+1))\mathbf{1}_{D_{k,m}}$ Thus, if $1 \le m \le N$, then

$$\begin{aligned}
(Sf)^2\mathbf{1}_{\bigcup_{k=1}^{\ell} D_{k,m}} &= \sum_{j=0}^{m-1}(\Delta_j f)^2\mathbf{1}_{\bigcup_{k=1}^{\ell} D_{k,m}} + (\Delta_m f)^2\mathbf{1}_{\bigcup_{k=1}^{\ell} D_{k,m}} \\
&= \sum_{k=1}^{\ell}\Big[\sum_{j=0}^{m-1}(\Delta_j f)^2\mathbf{1}_{D_{k,m}} + (\Delta_m f)^2\mathbf{1}_{D_{k,m}}\Big] \\
&= \sum_{k=1}^{\ell}\Big[\sum_{j=0}^{m-1}\frac{1}{(j+1)^2} + \frac{(2^m+1)^2}{(m+1)^2}\Big]\lambda_k^2\mathbf{1}_{D_{k,m}} \\
&\le K^2\sum_{k=1}^{\ell}\lambda_k^2\mathbf{1}_{D_{k,m}} = K^2x^2\mathbf{1}_{\bigcup_{k=1}^{\ell} D_{k,m}},
\end{aligned}$$

where K is a constant $> \pi/\sqrt{6}$ which depends only on the value of the constant C. (Indeed, we can take $K = (2^N+1)\pi/\sqrt{6}$.) Notice that

$$V = \bigcup_{m=1}^{N}\bigcup_{k=1}^{\ell} D_{k,m}$$

and that if $m \ne m'$, then

$$\Big(\bigcup_{k=1}^{\ell} D_{k,m}\Big) \cap \Big(\bigcup_{k=1}^{\ell} D_{k,m'}\Big) = \emptyset.$$

Then we have

$$(3.15)\qquad (Sf)^2\mathbf{1}_V = \sum_{m=1}^{N}(Sf)^2\mathbf{1}_{\bigcup_{k=1}^{\ell} D_{k,m}} \le \sum_{m=1}^{N} K^2x^2\mathbf{1}_{\bigcup_{k=1}^{\ell} D_{k,m}} = K^2x^2\mathbf{1}_V.$$

Since $K^2 > \pi^2/6$, it follows from (3.13)–(3.15) that

$$\begin{aligned}
(Sf)^2 &\le \frac{\pi^2(x^2+y^2)}{6}\,\mathbf{1}_U + K^2x^2\mathbf{1}_V \\
&\le \frac{\pi^2y^2}{6} + K^2x^2(\mathbf{1}_U + \mathbf{1}_V) = \frac{\pi^2y^2}{6} + K^2x^2.
\end{aligned}$$

Thus we obtain

$$Sf \le \frac{\pi y}{\sqrt{6}} + Kx. \tag{3.16}$$

Now, for each $k = 1, \ldots, \ell$, let $\{B_{k,1}, B_{k,2}\}$ be a partition of B_k such that

$$\mathbb{P}(B_{k,1}) = \mathbb{P}(B_{k,2}) = \tfrac{1}{2}\mathbb{P}(B_k),$$

and let y_1 and y_2 be the random variables defined by

$$y_1 = s_N \sum_{k=1}^{\ell} \lambda_k \mathbf{1}_{B_{k,1}} \quad \text{and} \quad y_2 = s_N \sum_{k=1}^{\ell} \lambda_k \mathbf{1}_{B_{k,2}}.$$

Define $g = (g_n) \in \mathcal{M}(\mathcal{F})$ by $g_n = \mathbb{E}[y_1 - y_2 \,|\, \mathcal{F}_n]$, $n \in \mathbb{Z}_+$. Then

$$g_n = \begin{cases} 0 & \text{if } 0 \le n \le N, \\ y_1 - y_2 & \text{if } n \ge N+1, \end{cases} \qquad \text{a.s.}$$

and hence

$$Mg = Sg = |y_1 - y_2| = s_N y \quad \text{ a.s.}$$

Since $(Mf)\mathbf{1}_{\{y>0\}} = s_N y$, it follows that $Mg \le Mf$ a.s.

Suppose that (ii) of the Main Theorem holds. Then (3.16) yields

$$s_N \|y\|_X = \|Sg\|_X \le C\|Sf\|_X \le \frac{C\pi}{\sqrt{6}}\|y\|_X + CK\|x\|_X.$$

By the choice of N, we obtain

$$\|y\|_X \le (s_N - C\pi/\sqrt{6})\|y\|_X \le CK\|x\|_X.$$

Thus (3.12) holds with $c = CK$. This completes the proof. ∎

Remark. Let X be a Banach function space over Ω. In [5] the author proved that if the inequality

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \le C\|Sf\|_X \tag{3.17}$$

holds for all $f \in \mathcal{M}$, then X can be equivalently renormed so as to be an r.i. space such that $\alpha_X > 0$. We can now give a simpler proof that if (3.17) holds for all $f \in \mathcal{M}$, then X can be renormed so as to be r.i.

Let $f = (f_n)$ be the martingale used in the proof of Lemma 3.5, and suppose that (3.17) holds. Since $|f_n|\,\mathbf{1}_{\{y>0\}} = s_n y$ for $n = 0, 1, \ldots, N$, it follows from (3.16) and (3.17) that

$$\begin{aligned} s_N\|y\|_X = \sup_{0\le n\le N} \|f_n \mathbf{1}_{\{y>0\}}\|_X &\le \sup_{n\in\mathbb{Z}_+} \|f_n\|_X \\ &\le C\|Sf\|_X \le \frac{C\pi}{\sqrt{6}}\|y\|_X + CK\|x\|_X. \end{aligned}$$

As in the proof of Lemma 3.5, we see that (3.12) holds with $c = CK$. Thus X can be equivalently renormed so as to be r.i.

Proof of the Main Theorem. It has been proved in [6] that (iii), (iv), (v), and (vi) are equivalent; moreover it is clear that (i) implies (ii). So it suffices to show that (ii) implies (vi) and (v) implies (i).

(ii)⇒(vi). Suppose that (ii) holds. Then Lemma 3.5 implies that X can be renormed so as to be r.i. Hence we may apply Lemma 3.3 to deduce that $\alpha_X > 0$. Thus (vi) holds.

(v)⇒(i). Suppose that $\mathcal{F} \in \mathbb{F}$, $f, g \in \mathcal{M}(\mathcal{F})$ and $\|Mg\|_X \le \|Mf\|_X$. If (v) holds, then

$$\|Sg\|_X \le C\|Mg\|_X \le C\|Mf\|_X \le C^2\|Sf\|_X.$$

Thus (i) holds. This completes the proof. ∎

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REFERENCES

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[2] K. M. Chong and M. N. Rice, *Equimeasurable Rearrangements of Functions*, Queen's Papers in Pure Appl. Math. 28, Queen's Univ., Kingston, ON, 1971.

[3] M. Kikuchi, *Characterization of Banach function spaces that preserve the Burkholder square-function inequality*, Illinois J. Math. 47 (2003), 867–882.

[4] M. Kikuchi, *A necessary and sufficient condition for certain martingale inequalities in Banach function spaces*, Glasgow Math. J. 49 (2007), 431–447.

[5] M. Kikuchi, *On the Davis inequality in Banach function spaces*, Math. Nachr. 281 (2008), 697–709.

[6] M. Kikuchi, *On some inequalities for martingale transforms in Banach function spaces*, Acta Sci. Math. (Szeged), to appear.

[7] T. Shimogaki, *Hardy–Littlewood majorants in function spaces*, J. Math. Soc. Japan 17 (1965), 365–373.

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