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ASYMPTOTIC SPECTRAL DISTRIBUTIONS OF DISTANCE-k GRAPHS OF CARTESIAN PRODUCT GRAPHS

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Abstract. Let G be a finite connected graph on two or more vertices, and $G^{[N,k]}$ the distance-k graph of the N-fold Cartesian power of G. For a fixed $k \geq 1$, we obtain explicitly the large N limit of the spectral distribution (the eigenvalue distribution of the adjacency matrix) of $G^{[N,k]}$. The limit distribution is described in terms of the Hermite polynomials. The proof is based on asymptotic combinatorics along with quantum probability theory.

1. Introduction. Since Vershik [13] emphasized the importance of asymptotic problems in combinatorics, various approaches have been developed from different branches of mathematics. The main question in this context is to explore the limit behavior of a combinatorial object when it grows. *Asymptotic spectral analysis* of a growing graph is a subject in this line with wide applications to structural analysis of complex networks.

In this paper, we study a particular class of growing graphs naturally induced from the Cartesian powers of a finite connected graph. In fact, we will prove the following main result.

THEOREM 1.1. *Let $G = (V, E)$ be a finite connected graph with $|V| \geq 2$. For $N \geq 1$ and $k \geq 1$ let $G^{[N,k]}$ be the distance-k graph of $G^N = G \times \cdots \times G$ (N-fold Cartesian power) and $A^{[N,k]}$ its adjacency matrix. Then, for a fixed $k \geq 1$, the eigenvalue distribution of $N^{-k/2}A^{[N,k]}$ converges in moments as $N \to \infty$ to the probability distribution of*

$$\left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g), \tag{1.1}$$

where $\tilde{H}_k$ is the monic Hermite polynomial of degree k (see Section 2.4) and g is a random variable obeying the standard normal distribution $N(0,1)$.

It is noteworthy that the limit distribution (1.1) is obtained explicitly and is universal in the sense that it is independent of the details of a factor G.

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Namely, for large N, the spectral structure of the distance-k graph of G^N is dominated by the product structure. This shares a common nature with the central limit theorem in probability theory. In fact, we will prove the above result by applying quantum (non-commutative) probability theory [9], where central limit theorems of various kinds have been studied from algebraic and combinatorial viewpoints.

The study of asymptotic spectral distribution of $G^{[N,k]}$ for a large N limit appeared first in [11] where the case of $G = K_2$ (the complete graph on two vertices) and $k = 2$ was studied by means of quantum decomposition. Later in [12] the spectrum of the distance-k graph of $H(N, 2) = K_2^N$ was explicitly obtained in terms of the Krawtchouk polynomials for arbitrary $1 \leq k \leq N$. Then, by using certain limit formulas for the Krawtchouk polynomials, the asymptotic spectral distribution of the distance-k graph of $H(N, 2)$ was determined. The result is a special case of (1.1) with $|V| = 2$ and $|E| = 1$. In the recent paper [7] the above argument was extended to cover the distance-k graph of the Hamming graph $H(N, d) = K_d^N$. The result is again a special case of (1.1). The case of G being a star graph and $k = 2$ was discussed in [10]. During these studies it has been conjectured that the limit distribution does not depend on the detailed structure of G, as the central limit distribution of the sum of independent, identically distributed random variables is the normal (Gaussian) law independently of the distributions of the random variables. Our main result shows that this conjecture is true.

This paper is organized as follows. In Section 2, recalling some notions and notations in quantum probability, we prepare a useful result on the convergence of algebraic random variables (Proposition 2.2) and reformulate our main result (Theorem 2.6). In Section 3 we derive a combinatorial limit formula (Theorem 3.3), which is viewed as an extension of the commutative central limit theorem in quantum probability. In Section 4 we prove the main result. Our discussion is based on asymptotic estimation of combinatorial objects, along with the philosophy of Vershik [13].

Finally, we mention some relevant works. The distance-k graphs were introduced originally in the study of distance-regular graphs (see e.g. [2, 6]). The adjacency matrix of the distance-k graph of a finite graph G, say $D^{[k]} = A^{[1,k]}$, is nothing other than the k-distance matrix of G. Then the distance matrix D of G is defined by

$$D = \sum_{k=1}^{\infty} kD^{[k]},$$

where the right-hand side is in fact a finite sum. The spectrum of the distance matrix has been actively studied recently, in particular, in connection with spectral graph theory (see e.g. [5] and references cited therein). Asymptotic spectral analysis of the distance matrix will be an interesting research topic in this connection. Distance-k graphs are used to construct embeddings of graphs into metric spaces for measuring graph similarity, which has wide applications in statistical pattern recognition [4]. The asymptotic spectral analysis, being related to graph embeddings, is expected to contribute some applications in this line of research. It is also noteworthy that the probability distribution (1.1) is derived by Hora [8] from the asymptotic behavior of the Young graph (branching rule of representations of the symmetric groups).

2. Preliminaries

2.1. Algebraic probability space. An *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a $*$-algebra over the complex number field $\mathbb{C}$ with multiplication identity $1 = 1_{\mathcal{A}}$ and φ a state on it, i.e., $\varphi : \mathcal{A} \to \mathbb{C}$ is a $\mathbb{C}$-linear function on $\mathcal{A}$ satisfying $\varphi(1) = 1$ and $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. We do not assume any topological conditions. An element $a \in \mathcal{A}$ is called an (*algebraic*) *random variable*, and it is called *real* if $a^* = a$. It is known that

$$\varphi(a^*) = \overline{\varphi(a)}, \qquad a \in \mathcal{A},$$

and the Schwarz inequality holds:

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b), \qquad a, b \in \mathcal{A}.$$

In particular, for real random variables $a = a^*$ and $b = b^*$ we have

$$|\varphi(ab)|^2 \leq \varphi(a^2)\varphi(b^2).$$

A state φ is called *tracial* if

$$\varphi(ab) = \varphi(ba), \qquad a, b \in \mathcal{A}.$$

For a real random variable $a \in \mathcal{A}$ there exists a probability distribution μ on the real line $(-\infty, +\infty)$ such that

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \, \mu(dx), \qquad m = 0, 1, 2, \ldots.$$

The above μ is called the *spectral distribution* of a in the state φ. The existence of μ follows from the Hamburger theorem. However, uniqueness does not hold in general due to the famous indeterminate moment problem; for further details see [9, Chapter 1].

2.2. Convergence in moments. Let $(\mathcal{A}_n, \varphi_n)$, $n = 1, 2, \ldots$, and $(\mathcal{A}, \varphi)$ be algebraic probability spaces. We say that a sequence of real random variables $a_n \in \mathcal{A}_n$ *converges* to a real random variable $a \in \mathcal{A}$ *in moments* if

$$\lim_{n\to\infty} \varphi_n(a_n^m) = \varphi(a^m), \qquad m = 0, 1, 2, \ldots.$$

In this case we write

$$a_n \xrightarrow{\mathrm{M}} a$$

for simplicity.

Proposition 2.1. *If* $a_n \xrightarrow{\mathrm{M}} a$*, then* $p(a_n) \xrightarrow{\mathrm{M}} p(a)$ *for any polynomial* $p(x)$*.*

The above assertion is obvious. However, generalization to multivariable case is not trivial. For real random variables $a, b, \ldots, c \in \mathcal{A}$ the quantities of the form

$$\varphi(a^{\alpha_1} b^{\beta_1} \cdots c^{\gamma_1} \cdots a^{\alpha_i} b^{\beta_i} \cdots c^{\gamma_i} \cdots),$$

where $\alpha_i, \beta_i \ldots, \gamma_i$ are non-negative integers, are called *mth mixed moments* of $a, b, \ldots, c \in \mathcal{A}$ with $m = \sum_i (\alpha_i + \beta_i + \cdots + \gamma_i)$.

Proposition 2.2. *Let* $(\mathcal{A}_n, \varphi_n)$*,* $n = 1, 2, \ldots,$ *and* $(\mathcal{A}, \varphi)$ *be algebraic probability spaces. Let* $k \geq 1$ *be a fixed integer. Let* $a_n = a_n^*$*,* $z_{1n} = z_{1n}^*, \ldots, z_{kn} = z_{kn}^* \in \mathcal{A}_n$*,* $n = 1, 2, \ldots,$ *and* $a = a^* \in \mathcal{A}$ *be real random variables, and* $\zeta_1, \ldots, \zeta_k \in \mathbb{R}$*. Assume the following conditions hold:*

(i) $a_n \xrightarrow{\mathrm{M}} a$ *and* $z_{in} \xrightarrow{\mathrm{M}} \zeta_i 1$ *for* $i = 1, \ldots, k$*;*
(ii) φ_n *is a tracial state for* $n = 1, 2, \ldots$*;*
(iii) $\{a_n, z_{1n}, \ldots, z_{kn}\} \subset \mathcal{A}_n$ *have uniformly bounded mixed moments in the sense that*

$$C_m = \sup_n \max \left\{ |\varphi(a_n^{\alpha_1} z_{1n}^{\beta_1} \cdots z_{kn}^{\delta_1} \cdots a_n^{\alpha_i} z_{1n}^{\beta_i} \cdots z_{kn}^{\delta_i} \cdots)| \,; \begin{array}{l} \alpha_i, \beta_i, \ldots, \delta_i \geq 0 \text{ integers}, \\ \sum_i (\alpha_i + \beta_i + \cdots + \delta_i) = m \end{array} \right\} < \infty.$$

Then for any non-commutative polynomial $p(x, y_1, \ldots, y_k)$ *we have*

$$p(a_n, z_{1n}, \ldots, z_{kn}) \xrightarrow{\mathrm{M}} p(a, \zeta_1 1, \ldots, \zeta_k 1). \tag{2.1}$$

Remark 2.3. Strictly speaking, (2.1) is an abuse of notation because $p(a_n, z_{1n}, \ldots, z_{kn})$ is not necessarily real. We tacitly understand (2.1) to be

$$\lim_{n\to\infty} \varphi_n(p(a_n, z_{1n}, \ldots, z_{kn})^m) = \varphi(p(a, \zeta_1 1, \ldots, \zeta_k 1)^m), \qquad m = 1, 2, \ldots.$$

Remark 2.4. Obviously, condition (ii) in Proposition 2.2 may be replaced with

(ii′) φ_n restricted to the $*$-subalgebra generated by $\{a_n, z_{1n}, \ldots, z_{kn}\}$ is tracial.

Then we note that if $a_n, z_{1n}, \ldots, z_{kn}$ are mutually commutative, conditions (ii) and (iii) are redundant. In fact, as condition (ii′) is trivially satisfied, (ii) is redundant. For condition (iii) we first observe that

$$|\varphi(a_n^{\alpha_1} z_{1n}^{\beta_1} \cdots z_{kn}^{\delta_1} \cdots a_n^{\alpha_i} z_{1n}^{\beta_i} \cdots z_{kn}^{\delta_i} \cdots)| = |\varphi(a_n^{\alpha} z_{1n}^{\beta} \cdots z_{kn}^{\delta})|, \tag{2.2}$$

where $\alpha = \sum_i \alpha_i$, $\beta = \sum_i \beta_i, \dots, \delta = \sum_i \delta_i$. Applying the Schwarz inequality repeatedly we have

$$|\varphi(a_n^\alpha z_{1n}^\beta z_{2n}^\gamma \cdots z_{kn}^\delta)|^2 \le \varphi(a_n^{2\alpha})\varphi(z_{1n}^{2\beta} z_{2n}^{2\gamma} \cdots z_{kn}^{2\delta}),$$
$$|\varphi(z_{1n}^{2\beta} z_{2n}^{2\gamma} \cdots z_{kn}^{2\delta})|^2 \le \varphi(z_{1n}^{4\beta})\varphi(z_{2n}^{4\gamma} \cdots z_{kn}^{4\delta}), \quad \dots .$$

Finally, (2.2) is bounded by a product of moments of $a_n, z_{1n}, \dots, z_{kn}$, which remain finite as $n \to \infty$ since they are convergent sequences by (i). Thus, condition (iii) holds.

Proof of Proposition 2.2. Since $p(a_n, z_{1n}, \dots, z_{kn})^m$, $m \ge 1$, is again a non-commutative polynomial in $a_n, z_{1n}, \dots, z_{kn}$, it is sufficient to prove that

$$\lim_{n\to\infty} \varphi_n(p(a_n, z_{1n}, \dots, z_{kn})) = \varphi(p(a, \zeta_1 1, \dots, \zeta_k 1)) \tag{2.3}$$

for all non-commutative polynomials p. Moreover, by the linearity of a state we need only prove (2.3) for all non-commutative monomials of the form

$$p(x, y_1, \dots, y_k) = x^{\alpha_1} y_1^{\beta_1} \cdots y_k^{\delta_1} \cdots x^{\alpha_i} y_1^{\beta_i} \cdots y_k^{\delta_i} \cdots . \tag{2.4}$$

We will prove this by induction on the degree m of the monomial. Here the degree of p in (2.4) is defined by

$$m = \sum_i (\alpha_i + \beta_i + \cdots + \delta_i).$$

For $m = 1$ we need to show that

$$\lim_{n\to\infty} \varphi_n(a_n) = \varphi(a), \qquad \lim_{n\to\infty} \varphi_n(z_{in}) = \varphi(\zeta_i 1).$$

But these are obvious by assumption (i) of $a_n \xrightarrow{\mathrm{M}} a$ and $z_{in} \xrightarrow{\mathrm{M}} \zeta_i 1$. Let $m \ge 1$ and suppose that (2.3) is true for all non-commutative monomials (2.4) of degree up to m. Now let $p(x, y_1, \dots, y_k)$ be a non-commutative monomial of degree $m+1$. We need to prove (2.3) for this monomial.

Case 1: $p(x, y_1, \dots, y_k) = x^{m+1}$. In this case (2.3) holds obviously by the assumption of $a_n \xrightarrow{\mathrm{M}} a$.

Case 2: $p(x, y_1, \dots, y_k) = x^\alpha y_i q$ with $\alpha \ge 0$ and $q = q(x, y_1, \dots, y_k)$ a non-commutative monomial of degree $m - \alpha$. For simplicity we set

$$p(a_n, z_{1n}, \dots, z_{kn}) = a_n^\alpha z_{in} w_n\,, \qquad p(a, \zeta_1 1, \dots, \zeta_k 1) = a^\alpha \zeta_i w.$$

Then we have

$$\begin{aligned}
&|\varphi_n(a_n^\alpha z_{in} w_n) - \varphi(a^\alpha \zeta_i w)| \\
&\quad \le |\varphi_n(a_n^\alpha z_{in} w_n) - \varphi_n(a_n^\alpha \zeta_i w_n)| + |\varphi_n(a_n^\alpha \zeta_i w_n) - \varphi(a^\alpha \zeta_i w)| \\
&\quad = |\varphi_n(a_n^\alpha (z_{in} - \zeta_i 1) w_n)| + |\zeta_i|\, |\varphi_n(a_n^\alpha w_n) - \varphi(a^\alpha w)| \\
&\quad = |\varphi_n((z_{in} - \zeta_i 1) w_n a_n^\alpha)| + |\zeta_i|\, |\varphi_n(a_n^\alpha w_n) - \varphi(a^\alpha w)|,
\end{aligned} \tag{2.5}$$

where the last identity is due to assumption (ii). The second term of (2.5) tends to 0 as $n \to \infty$ by the assumption of induction. For the first term we apply the Schwarz inequality to obtain

$$|\varphi_n((z_{in} - \zeta_i 1)w_n a_n^\alpha)|^2 \le \varphi_n((z_{in} - \zeta_i 1)^2)\varphi((w_n a_n^\alpha)^*(w_n a_n^\alpha)). \tag{2.6}$$

Since $(w_n a_n^\alpha)^*(w_n a_n^\alpha)$ is a monomial of degree $2m$, we have $\varphi((w_n a_n^\alpha)^*(w_n a_n^\alpha)) \le C_{2m}$ by the uniform boundedness assumption (iii). Then (2.6) becomes

$$|\varphi_n((z_{in} - \zeta_i 1)w_n a_n^\alpha)|^2 \le C_{2m}\{\varphi_n(z_{in}^2) - 2\zeta_i\varphi_n(z_{in}) + \zeta_i^2\} \to 0 \quad \text{as } n \to \infty.$$

Hence (2.5) tends to 0 as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \varphi_n(a_n^\alpha z_{in} w_n) = \varphi(a^\alpha \zeta_i w).$$

Thus, (2.3) holds for our monomial $p(x, y_1, \ldots, y_k) = x^\alpha y_i q$.

Finally, we see from Cases 1 and 2 that (2.3) is also true for all non-commutative monomials $p(x, y_1, \ldots, y_k)$ of degree $m + 1$. ∎

2.3. Adjacency matrix as algebraic random variable. Let $G = (V, E)$ be a finite graph and A the adjacency matrix. Let $\mathcal{A}(G)$ be the adjacency algebra, i.e., the commutative $*$-algebra generated by A. Define the normalized trace by

$$\varphi_{\mathrm{tr}}(a) = \frac{1}{|V|}\operatorname{Tr} a, \qquad a \in \mathcal{A}(G). \tag{2.7}$$

Then φ_{tr} becomes a state on $\mathcal{A}(G)$ and the adjacency matrix A is regarded as a real random variable of the algebraic probability space $(\mathcal{A}(G), \varphi_{\mathrm{tr}})$.

Proposition 2.5. *The spectral distribution of the adjacency matrix A in the state φ_{tr} coincides with the eigenvalue distribution of the graph G. In other words,*

$$\varphi_{\mathrm{tr}}(A^m) = \int_{-\infty}^{+\infty} x^m \,\mu(dx), \qquad m = 0, 1, 2, \ldots,$$

where μ is the eigenvalue distribution of G.

The proof is obvious; however, the above relation is a clue to studying the eigenvalue distribution of a graph by means of quantum probabilistic techniques.

2.4. Main result. Let $G = (V, E)$ be a graph. For an integer $k \ge 1$ the *distance-k graph* of G is the graph $G^{[k]} = (V, E^{[k]})$ with

$$E^{[k]} = \{\{x, y\};\ x, y \in V,\ \partial_G(x, y) = k\},$$

where $\partial_G(x, y)$ is the graph distance of G. The distance-1 graph of G coincides with G itself.

Now we rephrase the main result. Let $G = (V, E)$ be a finite connected graph with $|V| \ge 2$. For $k \ge 1$ and $N \ge 1$ let $G^{[N,k]}$ be the distance-k graph of $G^N = G \times \cdots \times G$ (N-fold Cartesian power). In general, $G^{[N,k]}$ is not necessarily connected. The adjacency matrix $A^{[N,k]}$ of $G^{[N,k]}$ is considered as a real random variable on the algebraic probability space $(\mathcal{A}(G^{[N,k]}), \varphi_{\mathrm{tr}})$, where φ_{tr} is the normalized trace (see Section 2.3). The main result, Theorem 1.1, is equivalent to the following statement.

THEOREM 2.6. *Notations and assumptions being as above, we have*

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{\mathrm{M}} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g),$$

where $\tilde{H}_k$ is the monic Hermite polynomial (see below) and g is a random variable obeying the standard normal distribution $N(0,1)$.

Following the standard terminology (e.g., [1, 3]) the *Hermite polynomials* $\{H_n(x)\}$ are defined by the three-term recurrence relation

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad 2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x).$$

The *monic Hermite polynomials* appearing in Theorem 2.6 are defined by simple normalization:

$$\tilde{H}_n(x) = 2^{-n/2} H_n(x/\sqrt{2}), \qquad n = 0, 1, 2, \ldots.$$

Then we have

$$\tilde{H}_0(x) = 1, \qquad \tilde{H}_1(x) = x, \qquad x\tilde{H}_n(x) = \tilde{H}_{n+1}(x) + n\tilde{H}_{n-1}(x). \tag{2.8}$$

It is known that $\tilde{H}_n(x)$ become orthogonal polynomials with respect to the standard normal distribution $N(0,1)$. We remark that they are not normalized:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{H}_n(x)^2 e^{-x^2/2}\, dx = n!, \qquad n = 0, 1, 2, \ldots.$$

3. Convergence of tensor powers of algebraic random variables. Let $(\mathcal{A}, \varphi)$ be an arbitrary algebraic probability space. For $N \ge 1$ we consider the N-fold tensor power $(\mathcal{A}^{\otimes N}, \varphi^{\otimes N})$. From now on we write φ for $\varphi^{\otimes N}$. For a real random variable $b = b^* \in \mathcal{A}$ and $i \in \{1, \ldots, N\}$ we define $b_N(i) \in \mathcal{A}^{\otimes N}$ by

$$b_N(i) = \overbrace{1 \otimes \cdots \otimes 1 \otimes b \otimes 1 \otimes \cdots \otimes 1}^{N\text{ factors}},$$

where b appears at the ith position. Let $\mathcal{B}_N$ denote the $*$-algebra generated by $b_N(1), \ldots, b_N(N)$. Obviously, $\mathcal{B}_N$ becomes a commutative $*$-subalgebra of $\mathcal{A}^{\otimes N}$ and we have $\mathcal{B}_N = \mathcal{B}_1^{\otimes N}$. For mutually distinct $i_1, \ldots, i_n \in \{1, \ldots, N\}$

we define $b_N(i_1,\dots,i_n) \in \mathcal{B}_N$ by

$$\begin{aligned} b_N(i_1,\dots,i_n) &= b_N(i_1)\cdots b_N(i_n) \\ &= 1\otimes\cdots\otimes 1\otimes b\otimes 1\otimes\cdots\otimes 1\otimes b\otimes 1\otimes\cdots\otimes 1, \end{aligned}$$

where b appears at the i_1th, $\dots, i_n$th positions. Finally, for $1 \le n \le N$ we set

$$b^{(N,n)} = \sum_{1\le i_1<\cdots<i_n\le N} b_N(i_1,\dots,i_n) = \frac{1}{n!}\sum_{\substack{i_1,\dots,i_n \\ \neq}} b_N(i_1,\dots,i_n) \tag{3.1}$$

and for convenience

$$b^{(N,0)} = 1\otimes\cdots\otimes 1.$$

We are interested in the asymptotic spectral distribution of $b^{(N,n)}$ as $N\to\infty$. For $n=1$ the result is well known (see e.g. [9, Chapter 8]).

Theorem 3.1 (Commutative law of large numbers). *For a real random variable $b=b^*\in\mathcal{A}$ we have*

$$b^{(N,1)}/N \xrightarrow{\mathrm{M}} \varphi(b) \qquad \text{as } N\to\infty. \tag{3.2}$$

Theorem 3.2 (Commutative central limit theorem). *For a real random variable $b=b^*\in\mathcal{A}$ with $\varphi(b)=0$ and $\varphi(b^2)=1$ we have*

$$b^{(N,1)}/\sqrt{N} \xrightarrow{\mathrm{M}} g \qquad \text{as } N\to\infty, \tag{3.3}$$

where g is a Gaussian random variable obeying the standard normal law $N(0,1)$.

The commutative independence of $b_N(1),\dots,b_N(N)$ is essential in the above statements. We recall that (3.3) means

$$\begin{aligned} \lim_{N\to\infty}\varphi\left(\left(\frac{b^{(N,1)}}{\sqrt{N}}\right)^m\right) &= m\text{th moment of } N(0,1) \\ &= \begin{cases} 0, & m \text{ odd}, \\ \dfrac{(2k)!}{2^k k!}, & m=2k \text{ even}, \end{cases} \qquad m=0,1,2,\dots. \end{aligned}$$

We are now in a position to state a generalization of Theorem 3.2.

Theorem 3.3. *Notations and assumptions being as in Theorem 3.2, we have*

$$\frac{b^{(N,n)}}{N^{n/2}} \xrightarrow{\mathrm{M}} \frac{1}{n!}\tilde{H}_n(g) \qquad \text{as } N\to\infty$$

for all $n=1,2,\dots$, where $\tilde{H}_n$ is the monic Hermite polynomial of degree n defined in Section 2.4.

Before going into the proof, we consider the case of $n=2$ in detail to grasp the situation. We keep in mind that $b=b^*\in\mathcal{A}$ with $\varphi(b)=0$ and

$\varphi(b^2) = 1$. Starting with the simple identities

$$b^{(N,1)}b^{(N,1)} = \Big(\sum_{1\le i\le N} b_N(i)\Big)\Big(\sum_{1\le i\le N} b_N(i)\Big) = \sum_{\substack{i_1,i_2\\ \neq}} b_N(i_1,i_2) + \sum_{1\le i\le N} b_N^2(i)$$

$$= 2b^{(N,2)} + N + \sum_{1\le i\le N}(b^2-1)_N(i),$$

we obtain

$$2\,\frac{b^{(N,2)}}{N} = \frac{b^{(N,1)}}{\sqrt{N}}\,\frac{b^{(N,1)}}{\sqrt{N}} - 1 - \frac{1}{N}\sum_{1\le i\le N}(b^2-1)_N(i). \tag{3.4}$$

For simplicity we set

$$a_N = \frac{b^{(N,1)}}{\sqrt{N}}, \qquad z_{N1} = \frac{1}{N}\sum_{1\le i\le N}(b^2-1)_N(i). \tag{3.5}$$

Then (3.4) becomes

$$2\,\frac{b^{(N,2)}}{N} = a_N^2 - 1 - z_{N1}. \tag{3.6}$$

Moreover, a_N and z_{N1} are commutative, and

$$a_N \xrightarrow{\mathrm{M}} g, \qquad z_{N1} \xrightarrow{\mathrm{M}} 0,$$

which follows from Theorem 3.2 and Theorem 3.1, respectively. Noting that (3.6) is a polynomial in a_N and z_{N1}, we apply Proposition 2.2 to obtain

$$2\,\frac{b^{(N,2)}}{N} \xrightarrow{\mathrm{M}} g^2 - 1 = \tilde{H}_2(g),$$

which proves Theorem 3.3 for $n = 2$.

Proof of Theorem 3.3. We need some notation. For $1 \le n \le N$ and $y \in \mathcal{A}$ we define $F^{(N,n)}(y) \in \mathcal{A}^{\otimes N}$ by

$$F^{(N,n)}(y) = \sum \overbrace{1\otimes\cdots\otimes b\otimes\cdots\otimes y\otimes\cdots\otimes b\otimes\cdots\otimes 1}^{N\text{ factors}}, \tag{3.7}$$

where b appears $n-1$ times and y just once, and the sum is taken over all possible arrangements. Then after a simple calculation we obtain

$$b^{(N,1)}b^{(N,n)} = (n+1)b^{(N,n+1)} + (N-n+1)b^{(N,n-1)} + F^{(N,n)}(b^2-1), \tag{3.8}$$

where $1 \le n < N$.

For simplicity we set

$$B_{Nn} = n!\,\frac{b^{(N,n)}}{N^{n/2}}, \qquad z_{Nn} = \frac{F^{(N,n)}(b^2-1)}{N^{(n+1)/2}}, \qquad 1 \le n \le N. \tag{3.9}$$

Obviously, these are members of $\mathcal{B}_N$. For $n = 1$ we have $B_{N1} = a_N$ (see also (3.5)). With these notations, (3.8) becomes

$$B_{N,n+1} = a_N B_{Nn} - nB_{N,n-1} + \frac{n(n-1)}{N} B_{N,n-1} - n! z_{Nn}. \tag{3.10}$$

We are going to show that for each $n = 1, 2, \dots$ there exists a polynomial $p_n(x, y_1, \dots, y_{n-1})$ (independent of N) such that

$$B_{Nn} - p_n(a_N, z_{N1}, z_{N2}, \dots, z_{N,n-1}) = Y_{Nn} \tag{3.11}$$

is a real random variable in $\mathcal{B}_N$ and $Y_{Nn} \xrightarrow{\mathrm{M}} 0$ as $N \to \infty$. The assertion for $n = 1$ is trivial with

$$p_1(x) = x, \qquad Y_{N1} = 0. \tag{3.12}$$

For $n = 2$ we see from (3.6) that

$$p_2(x, y_1) = x^2 - 1 - y_1, \qquad Y_{N2} = 0. \tag{3.13}$$

Suppose that the assertion holds up to $n \geq 2$. Then (3.10) becomes

$$\begin{aligned} B_{N,n+1} &= a_N(p_n + Y_{Nn}) - n(p_{n-1} + Y_{N,n-1}) \\ &\quad + \frac{n(n-1)}{N} (p_{n-1} + Y_{N,n-1}) - n! z_{Nn} \\ &= a_N p_n - np_{n-1} - n! z_{Nn} \\ &\quad + a_N Y_{Nn} - \left\{ n - \frac{n(n-1)}{N} \right\} Y_{N,n-1} + \frac{n(n-1)}{N} p_{n-1}. \end{aligned} \tag{3.14}$$

Hence, setting

$$\begin{aligned} p_{n+1}(x, y_1, \dots, y_n) &= xp_n(x, y_1, \dots, y_{n-1}) \\ &\qquad - np_{n-1}(x, y_1, \dots, y_{n-2}) - n!\, y_n \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} Y_{N,n+1} &= a_N Y_{Nn} - \left\{ n - \frac{n(n-1)}{N} \right\} Y_{N,n-1} \\ &\qquad + \frac{n(n-1)}{N} p_{n-1}(a_N, z_{N1}, z_{N2}, \dots, z_{N,n-2}), \end{aligned} \tag{3.16}$$

we have

$$B_{N,n+1} = p_{n+1}(a_N, z_{N1}, z_{N2}, \dots, z_{Nn}) + Y_{N,n+1}. \tag{3.17}$$

It is clear that $p_{n+1}(x, y_1, y_2, \dots, y_n)$ is a polynomial and that $Y_{N,n+1}$ is a real random variable in $\mathcal{B}_N$. In (3.16) we have

$$p_{n-1}(a_N, z_{N1}, z_{N2}, \dots, z_{N,n-2}) \xrightarrow{\mathrm{M}} p_{n-1}(g, 0, 0, \dots, 0),$$

which follows by Proposition 2.2 and the fact that $z_{Nn} \xrightarrow{\mathrm{M}} 0$ as $N \to \infty$ (Lemma 3.4 below). Hence, applying the induction assumption, we see that $Y_{N,n+1} \xrightarrow{\mathrm{M}} 0$. This completes the induction.

Finally, applying Proposition 2.2 to (3.11), we obtain

$$B_{Nn} \xrightarrow{\mathrm{M}} p_n(g, 0, \dots, 0), \qquad n = 1, 2, \dots. \tag{3.18}$$

On the other hand, we know from (3.12) and (3.13) that

$$p_1(x) = x, \qquad p_2(x, 0) = x^2 - 1.$$

Moreover, from (3.15) we have

$$p_{n+1}(x, 0, \dots, 0) = xp_n(x, 0, \dots, 0) - np_{n-1}(x, 0, \dots, 0).$$

Comparing this with the recurrence relation (2.8) satisfied by the monic Hermite polynomials, we see that

$$p_n(x, 0, \dots, 0) = \tilde{H}_n(x).$$

Consequently, it follows from (3.18) that

$$B_{Nn} \xrightarrow{\mathrm{M}} \tilde{H}_n(g), \qquad n = 1, 2, \dots. \ \blacksquare$$

Lemma 3.4. *For $n = 1, 2, \dots$ we have*

$$z_{Nn} = \frac{F^{(N,n)}(b^2 - 1)}{N^{(n+1)/2}} \xrightarrow{\mathrm{M}} 0.$$

Proof. We need to show that $\varphi(z_{Nn}^m) \to 0$ as $N \to \infty$ for fixed $m, n = 1, 2, \dots$. For $m = 1$ the assertion is obvious so we assume that $m \geq 2$. For simplicity we set $z = b^2 - 1$. By definition we have

$$F^{(N,n)}(z) = \sum 1 \otimes \cdots \otimes b \otimes \cdots \otimes z \otimes \cdots \otimes b \otimes \cdots \otimes 1,$$

where b appears $n - 1$ times and z just once, and the sum is taken over all possible arrangements. Then $\varphi[(F^{(N,n)}(z))^m]$ is the sum of all terms of the form

$$\varphi(1 \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes 1), \tag{3.19}$$

where $(*)$ is of the form $b^s z^t$ with $1 \leq s + t \leq m$. If one of the $(*)$'s is occupied by b or z (i.e., $s + t = 1$), the value of (3.19) is zero since $\varphi(b) = \varphi(z) = 0$. Hence $\varphi[(F^{(N,n)}(z))^m]$ is the sum of the terms (3.19) such that $(*)$ is of the form $b^s z^t$ with $2 \leq s + t \leq m$. We set

$$K = K_m = \max\{1, |\varphi(b^s z^t)|;\, 2 \leq s + t \leq m\}.$$

Suppose that nm is even. Let S be the sum of terms (3.19) with $(*)$ being of order 2, that is, b^2, bz or z^2, and write

$$\varphi[(F^{(N,n)}(z))^m] = S + R.$$

First each term constituting S is estimated as

$$|\varphi(1 \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes 1)| \leq K^{nm/2}.$$

We need to count the number of such terms. The number of choices of places where $(*)$ appears is given by $\binom{N}{nm/2}$. Then the arrangements of b^2, bz, z^2 at a set of chosen places $(*)$ is bounded by $3^{nm/2}$. Hence

$$S \le K^{nm/2}\binom{N}{nm/2}3^{nm/2} \le C_1(m,n)N^{nm/2}$$

for some constant $C_1(m,n)$. If nm is odd, letting S be the sum of terms (3.19) with $(*)$ being of order 2 except one $(*)$ of order 3, we have

$$S \le K^{(nm-1)/2}\binom{N}{(nm-1)/2}7^{(nm-1)/2} \le C_2(m,n)N^{(nm-1)/2}$$

In any case we have

$$S = O(N^{[nm/2]}).$$

By a similar argument we see easily that the rest term R has a smaller order:

$$R = o(N^{[nm/2]}).$$

Consequently,

$$\varphi(z_{Nn}^m) = \varphi\left(\left(\frac{F^{(N,n)}(z)}{N^{(n+1)/2}}\right)^m\right) \le \frac{S+R}{N^{(n+1)m/2}} = O(N^{-m/2}),$$

which tends to zero as $N \to \infty$. ■

4. Proof of Theorem 2.6. Associated with the finite graph $G = (V,E)$, we consider the full matrix algebra $\mathcal{M}(V)$, that is, the $*$-algebra of matrices with index set $V \times V$. The adjacency algebra $\mathcal{A}(G)$ is a commutative $*$-subalgebra of $\mathcal{M}(V)$. The normalized trace φ_{tr} on $\mathcal{A}(G)$ defined in (2.7) is naturally extended to $\mathcal{M}(V)$ and is denoted by the same symbol. As $\mathcal{A}(G^{[N,k]})$ is a $*$-subalgebra of $\mathcal{M}(V)^{\otimes N}$, the normalized trace on $\mathcal{A}(G^{[N,k]})$ coincides with the restriction of the product state $\varphi_{\mathrm{tr}}^{\otimes N}$ on $\mathcal{M}(V)^{\otimes N}$, which is denoted by φ for simplicity hereafter.

Let A and $A^{[N,k]}$ be the adjacency matrices of G and $G^{[N,k]}$, respectively. Following the notation in (3.1) we set

$$A^{(N,n)} = \sum_{1 \le i_1 < \cdots < i_n \le N} A_N(i_1,\dots,i_n) = \frac{1}{n!}\sum_{\substack{i_1,\dots,i_n \\ \neq}} A_N(i_1,\dots,i_n)$$

and define a real random variable $C(N,k)$ by

$$A^{[N,k]} = A^{(N,k)} + C(N,k). \tag{4.1}$$

We will first show that

$$\frac{C(N,k)}{N^{k/2}} \xrightarrow{\mathrm{M}} 0 \qquad \text{as } N \to \infty. \tag{4.2}$$

REMARK 4.1. As is easily seen, the adjacency matrix of G^N is given by

$$A^{(N,1)} = \sum_{i=1}^{N} A_N(i) = \sum_{i=1}^{N} \overbrace{1 \otimes \cdots \otimes A \otimes \cdots \otimes 1}^{N \text{ factors}},$$

where A sits at the ith position. Therefore, we have

$$A^{[N,1]} = A^{(N,1)}.$$

However, for $k \geq 2$, $A^{[N,k]} = A^{(N,k)}$ does not hold in general while it is easily seen to hold when G is a complete graph and $1 \leq k \leq N$. The N-fold Cartesian power of the complete graph K_d (where d stands for the number of vertices) is called a *Hamming graph* and is denoted by $H(N,d)$. The eigenvalue distribution of the distance-k graph of $H(N,d)$ is obtained by means of the Krawtchouk polynomials (see [12] for $d = 2$ and [7] for an arbitrary d).

LEMMA 4.2. *Let $G = (V,E)$ be a finite connected graph and let the distance between two vertices $\xi, \eta \in V$ be denoted by $\partial_G(\xi,\eta)$. Then for the N-fold Cartesian power G^N we have*

$$\partial_{G^N}(x,y) = \sum_{i=1}^{N} \partial_G(\xi_i,\eta_i),$$

where $x = (\xi_1,\ldots,\xi_N), y = (\eta_1,\ldots,\eta_N) \in V^N$.

Proof. Straightforward. ∎

It is convenient to introduce the distance matrix of G. For $k = 1,2,\ldots$ let $D^{[k]}$ be the *k-distance matrix* of G, which is a matrix indexed by $V \times V$ and defined by

$$(D^{[k]})_{xy} = \begin{cases} 1, & \partial_G(x,y) = k, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $D^{[k]}$ is the adjacency matrix of the distance-k graph of G. By definition $A = D^{[1]}$ and $A^{[1,k]} = D^{[k]}$.

We need a concise expression for $C(N,k)$. For illustration we consider the case of $k = 2$. For two vertices $x = (\xi_1,\ldots,\xi_N)$, $y = (\eta_1,\ldots,\eta_N) \in V^N$,

$$\partial_{G^N}(x,y) = \sum_{i=1}^{N} \partial_G(\xi_i,\eta_i) = 2$$

if and only if one of the following two cases occurs:

(i) there exist $1 < i_1 < i_2 \leq N$ such that $\partial_G(\xi_{i_1},\eta_{i_1}) = \partial_G(\xi_{i_2},\eta_{i_2}) = 1$ and $\partial_G(\xi_j,\eta_j) = 0$ for all $j \neq i_1, i_2$;

(ii) there exists $1 \leq i \leq N$ such that $\partial_G(\xi_i,\eta_i) = 2$ and $\partial_G(\xi_j,\eta_j) = 0$ for all $j \neq i$.

We then have

$$A^{[N,2]} = \sum_{1\le i_1<i_2\le N} D^{[1]}(i_1,i_2) + \sum_{1\le i\le N} D^{[2]}(i) = A^{(N,2)} + C(N,2).$$

The above argument is applied to $A^{[N,k]}$ for a general k. For $k \ge 1$ we set

$$\Lambda(k) = \Big\{\lambda = (j_1, j_2, \dots);\ j_h \ge 0 \text{ are integers such that } \sum_{h=1}^{\infty} h j_h = k\Big\}.$$

An element of $\Lambda(k)$ is called a *partition* of k. For $\lambda = (j_1, j_2, \dots) \in \Lambda(k)$ we define

$$C(\lambda) = \sum 1 \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes 1,$$

where $(*)$ is occupied by $D^{[h]}$ precisely j_h times $(h = 1, 2, \dots)$ and the sum is taken over all possible arrangements. For $\lambda_0 = (k, 0, 0, \dots)$, we have

$$C(\lambda_0) = \sum_{1\le i_1<\cdots<i_k\le N} D^{[1]}(i_1, \dots, i_k) = A^{(N,k)}.$$

LEMMA 4.3. *For $k \ge 1$, we have*

$$A^{[N,k]} = A^{(N,k)} + \sum_{\lambda\in\Lambda(k)\setminus\{\lambda_0\}} C(\lambda). \tag{4.3}$$

Proof. For $x = (\xi_1, \dots, \xi_N)$ and $y = (\eta_1, \dots, \eta_N) \in V^N$, we have $(A^{[N,k]})_{xy} = 1$, that is, $\partial_{G^N}(x, y) = k$ if and only if

$$\sum_{i=1}^{N} \partial_G(\xi_i, \eta_i) = k.$$

Then by counting the number of pairs (ξ_i, η_i) having the same distance h, we come to (4.3) with no difficulty. ∎

LEMMA 4.4. *For $\lambda \in \Lambda(k) \setminus \{\lambda_0\}$ we have*

$$C(\lambda)/N^{k/2} \xrightarrow{\mathrm{M}} 0.$$

Proof. Let $\lambda = (j_1, j_2, \dots)$ and $J = \sum j_h$. Let $M(m)$ be the maximum of the absolute values of the mixed moments of $D^{[1]}, D^{[2]}, \dots$ of degree $\le m$. By explicit expansion

$$C(\lambda)^m = \sum 1 \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes 1,$$

where $(*)$ is a non-commutative monomial in $D^{[1]}, D^{[2]}, \dots$ of degree at most m. When computing the value $\varphi(C(\lambda)^m)$, the terms having a monomial $D^{[h]}$ of degree 1 do not contribute since $\varphi(D^{[h]}) = 0$. Hence we need to consider only the terms where every $(*)$ is a monomial of degree at least 2. We write

$$\varphi(C(\lambda)^m) = S + R,$$

with

$$S = \sum \varphi(1 \otimes \cdots \otimes (*) \otimes \cdots \otimes (*) \otimes \cdots \otimes 1),$$

where all $(*)$'s are monomials of degree 2 or all $(*)$'s are monomials of degree 2 except one which is of degree 3 according to the parity of mJ. For S we see that the number of choices of places where $(*)$ appears is given by $\binom{N}{[mJ/2]}$. Then the arrangements of $D^{[1]}, D^{[2]}, \ldots$ at the chosen places are in a finite number $c(m, J)$ depending on m and J although the explicit expression is not simple. Hence

$$|S| \le M(m)^{[mJ/2]} \binom{N}{[mJ/2]} c(m,k) \le C_1(m,k) N^{[mJ/2]}$$

for some constant $C_1(m,k)$. Similarly, the number R of choices of places where $(*)$ appears is $\le \binom{N}{[mJ/2]-1}$ so that

$$|R| = o(N^{[mJ/2]}).$$

We note that for $\lambda \in \varLambda(k) \backslash \{\lambda_0\}$,

$$J = \sum_h j_h < \sum_h h j_h = k.$$

Hence $J - k \le -1$ and

$$\left[\frac{mJ}{2}\right] - \frac{km}{2} \le -\frac{m}{2}.$$

Consequently,

$$\varphi\left(\left(\frac{C(\lambda)}{N^{k/2}}\right)^m\right) \le \frac{C_1(m,k)N^{[mJ/2]} + o(N^{[mJ/2]})}{N^{km/2}} = O(N^{-m/2}) \to 0. \ \blacksquare$$

Proof of Theorem 2.6. By Lemma 4.3 we have

$$A^{[N,k]} = A^{(N,k)} + C(N,k), \tag{4.4}$$

$$C(N,k) = \sum_{\lambda \in \varLambda(k)\backslash\{\lambda_0\}} C(\lambda). \tag{4.5}$$

Upon applying Theorem 3.3 to $A^{(N,k)}$ we take normalization into account. Note first that

$$\varphi(A) = 0, \qquad \varphi(A^2) = \frac{2|E|}{|V|}.$$

In fact, $\varphi(A^2)$ is the mean degree of G. Noting that $A/\sqrt{\varphi(A^2)}$ is a normalized real random variable, we apply Theorem 3.3 to obtain

$$\frac{A^{(N,k)}}{N^{k/2}\varphi(A^2)^{k/2}} \xrightarrow{\mathrm{M}} \frac{1}{k!}\,\tilde{H}_k(g).$$

Therefore,

$$\frac{A^{(N,k)}}{N^{k/2}} \xrightarrow{\mathrm{M}} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\, \tilde{H}_k(g). \tag{4.6}$$

On the other hand, for $C(N,k)$ in (4.4) we have

$$\frac{C(N,k)}{N^{k/2}} \xrightarrow{\mathrm{M}} 0 \tag{4.7}$$

by Lemma 4.4 and Proposition 2.2. Finally, the assertion follows from (4.6) and (4.7) with the help of Proposition 2.2 again. ■

Remark 4.5. During the above argument we needed to restrict ourselves to the tracial states, although the combinatorial limit formula in Theorem 3.3 holds for a general state. This restriction is reasonable to obtain the eigenvalue distribution of a graph since the normalized trace on the adjacency algebra is related to the eigenvalue distribution of the graph (see Section 2.3). However, it is plausible that our argument can be modified to cover a general case, for example, a vector state (sometimes called a vacuum state) on the adjacency algebra. The work is now in progress.

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