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THE DIOPHANTINE EQUATION $(bn)^x + (2n)^y = ((b+2)n)^z$

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Abstract. Recently, Miyazaki and Togbé proved that for any fixed odd integer $b \geq 5$ with $b \neq 89$, the Diophantine equation $b^x + 2^y = (b+2)^z$ has only the solution $(x,y,z) = (1,1,1)$. We give an extension of this result.

1. Introduction. Let $\mathbb{N}$ be the set of positive integers. Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$ with $2 \mid b$. In 1956, Jeśmanowicz [J] conjectured that for any positive integer n, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z \tag{1.1}$$

has only the solution $(x,y,z) = (2,2,2)$. This conjecture is a famous unsolved problem in the field of exponential Diophantine equations. For related problems, see ([DC], [Le], [Miy], [TY]).

It is another interesting problem to find all triples (X,Y,Z) such that the Diophantine equation $X^x + Y^y = Z^z$, $x,y,z \in \mathbb{N}$ has only the solution $(x,y,z) = (1,1,1)$. Recently, Miyazaki and Togbé [MT] proved that for any fixed odd integer $b \geq 5$ with $b \neq 89$, the Diophantine equation $b^x + 2^y = (b+2)^z$ has only the solution $(x,y,z) = (1,1,1)$. Clearly, the Diophantine equation

$$(bn)^x + (2n)^y = ((b+2)n)^z \tag{1.2}$$

has the solution $(x,y,z) = (1,1,1)$.

In this paper, we obtain the following results.

THEOREM 1.1. *Let b be an odd integer with $b \geq 5$. If (x,y,z) is a solution of (1.2) with $(x,y,z) \neq (1,1,1)$, then $y < z < x$ or $x \leq z < y$.*

COROLLARY 1.2. *Let $b \geq 5$ be an odd prime power. If $\gcd(b,n) > 1$, then (1.2) has only the solution $(x,y,z) = (1,1,1)$.*

COROLLARY 1.3. *Let $b \geq 5$ be a prime power such that the order of 2 modulo b is even. If (x,y,z) is a solution of (1.2) with $(x,y,z) \neq (1,1,1)$, then $x \leq z < y$.*

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REMARK. Let $n = 5$, $b = 9$. Clearly, $45^2 + 10^3 = 55^2$. This example shows that (1.2) has a solution such that $1 < x = z < y$.

Throughout this paper, let $n \in \mathbb{N}$ and p be a prime. If n is not zero, there is a nonnegative integer e such that p^e divides n but p^{e+1} does not. We then denote $e = \upsilon_p(n)$.

2. Lemmas

LEMMA 2.1 (see [MT, Theorem 1.2]). *Let b be an odd positive integer with $b \geq 5$. Then the equation $b^x + 2^y = (b+2)^z$ has only the solution $(x, y, z) = (1, 1, 1)$ if $b \neq 89$, and the solutions $(x, y, z) = (1, 1, 1)$, $(1, 13, 2)$ if $b = 89$.*

LEMMA 2.2. *Let b be a positive integer. If $z \geq \max\{x, y\}$, then for any positive integer n, (1.2) has no solution other than $(x, y, z) = (1, 1, 1)$.*

Proof. If $z = 1$, then $x = y = 1$ and $(bn)^x + (2n)^y = ((b+2)n)^z$. If $z \geq 2$, then

$$(bn)^x + (2n)^y \leq (bn)^z + (2n)^z < ((b+2)n)^z. \blacksquare$$

3. Proof of Theorem 1.1. Let (x, y, z) be a solution of (1.2) with $(x, y, z) \neq (1, 1, 1)$. By Lemmas 2.1 and 2.2, we may assume that $n \geq 2$ and $z < \max\{x, y\}$. If $y < z < x$ or $x \leq z < y$, then we are done. Now we distinguish the following three remaining cases.

CASE 1: $z \leq y < x$. Then

$$n^{y-z}(b^x n^{x-y} + 2^y) = (b+2)^z. \tag{3.1}$$

If $\gcd(n, b+2) = 1$, then $y = z$ and $b^x n^{x-y} = (b+2)^y - 2^y$. Thus $y \geq 2$ and

$$b^{x-1} n^{x-y} = \sum_{i=1}^{y-1} \binom{y}{i+1} b^i 2^{y-i-1} + 2^{y-1} y. \tag{3.2}$$

Let p be a prime factor of b. Since $\gcd(b, 2) = 1$, we see from (3.2) that $p \mid y$. Further let $\upsilon_p(b) = \alpha$, $\upsilon_p(y) = \beta$. For $i = 1, \ldots, y-1$, let $\upsilon_p(i+1) = \gamma_i$. Then

$$\gamma_i \leq \left[\frac{\log(i+1)}{\log p}\right] \leq i - 1, \quad i = 1, \ldots, y-1.$$

Thus

$$\upsilon_p\left(\binom{y}{i+1} b^i 2^{y-i-1}\right) = \upsilon_p\left(y \binom{y-1}{i} \frac{b^i}{i+1} 2^{y-i-1}\right) \geq \beta + 1,$$
$$i = 1, \ldots, y-1.$$

This means that

$$\upsilon_p\Big(\sum_{i=1}^{y-1}\binom{y}{i+1}b^i 2^{y-i-1}+2^{y-1}y\Big)=\beta. \tag{3.3}$$

By (3.2) and (3.3) we have $\alpha(x-1)\le\beta$. Let p run through all distinct prime factors of b; we know that $b^{x-1}\mid y$, thus $b^{x-1}\le y$, which is impossible.

Now suppose that $\gcd(n,b+2)=d>1$. For any odd prime factor p of d, by $\gcd(p,b^x n^{x-y}+2^y)=1$, we have $\upsilon_p(n^{y-z})=\upsilon_p((b+2)^z)$. Let $\upsilon_p(n)=\theta_1$ and $\upsilon_p(b+2)=\theta_2$. By (3.1), we find that

$$\Big(\frac{n}{p^{\theta_1}}\Big)^{y-z}(b^x n^{x-y}+2^y)=\Big(\frac{b+2}{p^{\theta_2}}\Big)^z.$$

However, we also have

$$\Big(\frac{b+2}{p^{\theta_2}}\Big)^z<b^z<b^x<\Big(\frac{n}{p^{\theta_1}}\Big)^{y-z}(b^x n^{x-y}+2^y),$$

a contradiction.

Case 2: $z<x<y$. Then

$$n^{x-z}(b^x+2^y n^{y-x})=(b+2)^z. \tag{3.4}$$

If $\gcd(n,b+2)=1$, then by (3.4) and $n\ge 2$, we have $x=z$, a contradiction.

Now suppose that $\gcd(n,b+2)=d>1$. For any odd prime factor p of d, by $\gcd(p,b^x+2^y n^{y-x})=1$, we have $\upsilon_p(n^{x-z})=\upsilon_p\big((b+2)^z\big)$. As in the proof of Case 1, we deduce that (3.4) cannot hold.

Case 3: $z<x=y$. Then

$$n^{x-z}(b^x+2^x)=(b+2)^z. \tag{3.5}$$

Let $b+2=\prod_{i=1}^t q_i^{\alpha_i}$ be the standard prime factorization of $b+2$, where $\alpha_i\ge 1$. Since $n^{x-z}\mid(b+2)^z$, we have

$$\Big(\prod_{i=1}^t q_i^{\alpha_i}-2\Big)^x+2^x=\prod_{i=1}^t q_i^{\beta_i}, \tag{3.6}$$

where $\beta_i\ge 0$. We know that if all $\beta_i=0$, then (3.6) cannot hold. Thus there exists an i such that $\beta_i\ge 1$, hence x is odd. By (3.6) we have

$$\sum_{m=1}^x(-2)^{x-m}\binom{x}{m}\prod_{i=1}^t q_i^{\alpha_i m}=\prod_{i=1}^t q_i^{\beta_i}. \tag{3.7}$$

Since $z < x = y$, we have $x \geq 2$. For any $1 \leq j \leq t$ and $m \geq 2$, we see that

$$\begin{aligned}\upsilon_{q_j}\left(\binom{x}{m}\prod_{i=1}^{t} q_i^{\alpha_i m}\right) &= \upsilon_{q_j}\left(x\binom{x-1}{m-1}\frac{{q_j}^{\alpha_j m}}{m}\right)\\ &\geq \upsilon_{q_j}(x)+\alpha_j m-\upsilon_{q_j}(m) > \upsilon_{q_j}(x)+\alpha_j,\end{aligned}$$

and so

$$\upsilon_{q_j}\left(\sum_{m=1}^{x}(-2)^{x-m}\binom{x}{m}\prod_{i=1}^{t} q_i^{\alpha_i m}\right) = \upsilon_{q_j}(x)+\alpha_j.$$

By (3.7), we have $\beta_j = \upsilon_{q_j}(x)+\alpha_j$ for all $1 \leq j \leq t$. Thus

$$\left(\prod_{i=1}^{t} q_i^{\alpha_i}-2\right)^x + 2^x = \prod_{i=1}^{t} q_i^{\upsilon_{q_i}(x)+\alpha_i}.$$

Noting that

$$\prod_{i=1}^{t} q_i^{\upsilon_{q_i}(x)} \leq x,$$

we deduce

$$\left(\prod_{i=1}^{t} q_i^{\alpha_i}-2\right)^x + 2^x \geq \left(\prod_{i=1}^{t} q_i^{\alpha_i}-2\right)\prod_{i=1}^{t} q_i^{\upsilon_{q_i}(x)} + 2\prod_{i=1}^{t} q_i^{\upsilon_{q_i}(x)} = \prod_{i=1}^{t} q_i^{\upsilon_{q_i}(x)+\alpha_i}.$$

Equality holds only when $\prod_{i=1}^{t} q_i^{\upsilon_{q_i}(x)} = x = 1$, a contradiction.

This completes the proof of Theorem 1.1. ∎

4. Proof of Corollary 1.2. Noting that $\gcd(b, n) > 1$, by Theorem 1.1 we may assume that $n \geq 2$ and it is sufficient to eliminate the following two cases.

Case 1: $y < z < x$. Then

$$b^x n^{x-y} + 2^y = (b+2)^z n^{z-y}. \tag{4.1}$$

Noting that $n^{z-y} \mid 2^y$, we have $n^{z-y} = 2^t$ for some integer t with $1 \leq t \leq y$.

If $t < y$, then $\upsilon_2(b^x n^{x-y} + 2^y) > t = \upsilon_2((b+2)^z n^{z-y})$, a contradiction.

If $t = y$, then by (4.1), we know that there exists a positive integer r such that $b^x 2^r + 1 = (b+2)^z$. Since $b+1 \mid (b+2)^z - 1$ and $\gcd(b, b+1) = 1$, we have $b+1 | 2^r$. Thus, $b+1$ is a power of 2. Since $b \geq 5$ is an odd prime power, by Mihăilescu's famous theorem on the Catalan equation [Mih], we know that this is impossible.

Case 2: $x \leq z < y$. Then

$$b^x = n^{z-x}((b+2)^z - 2^y n^{y-z}). \tag{4.2}$$

If $x = z$, then by $\gcd(b, n) > 1$, we have $\gcd(b, b+2) > 1$, a contradiction.

If $x < z$, then by (4.2), we have $n \mid b^x$. Since b is an odd prime power, we deduce $\gcd(b, (b+2)^z - 2^y n^{y-z}) = 1$. Thus by (4.2), we find that $b^x = n^{z-x}$ and

$$2^y n^{y-z} = (b+2)^z - 1 = \sum_{i=1}^{z} \binom{z}{i} (b+1)^i.$$

By the proof of Case 1, we find that $b+1$ is not a power of 2. Hence there exists an odd prime factor q of $b+1$, thus

$$q \mid \sum_{i=1}^{z} \binom{z}{i} (b+1)^i.$$

However, noting that $\gcd(n, b+1) = 1$, we get $\gcd(q, 2^y n^{y-z}) = 1$, a contradiction.

This completes the proof of Corollary 1.2. ∎

5. Proof of Corollary 1.3. By Theorem 1.1, it is sufficient to prove that (1.2) has no solution (x, y, z) satisfying $y < z < x$. By Lemma 2.1, we may suppose that $n \geq 2$ and (1.2) has a solution (x, y, z) with $y < z < x$. Then

$$b^x n^{x-y} + 2^y = (b+2)^z n^{z-y}. \tag{5.1}$$

Noting that $n^{z-y} \mid 2^y$, we have $n^{z-y} = 2^t$ for some integer t with $1 \leq t \leq y$.

If $t < y$, then $v_2(b^x n^{x-y} + 2^y) > t = v_2((b+2)^z n^{z-y})$, a contradiction.

If $t = y$, then by (5.1), we know that there exists a positive integer r such that $b^x 2^r + 1 = (b+2)^z$. Since the order of 2 modulo b is even, we have $z \equiv 0 \pmod{2}$. Write $z = 2z_1$. Then

$$b^x 2^r = ((b+2)^{z_1} + 1)((b+2)^{z_1} - 1).$$

Noting that $\gcd((b+2)^{z_1} + 1, (b+2)^{z_1} - 1) = 2$ and b is a prime power, we have

$$b^x \mid (b+2)^{z_1} + 1 \quad \text{or} \quad b^x \mid (b+2)^{z_1} - 1;$$

but

$$b^x > b^{2z_1} > ((b+2)+1)^{z_1} \geq (b+2)^{z_1} + 1,$$

a contradiction.

This completes the proof of Corollary 1.3. ∎

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