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A CLASS OF IRREDUCIBLE POLYNOMIALS

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Abstract. Let
$$f(x) = x^n + k_{n-1}x^{n-1} + k_{n-2}x^{n-2} + \cdots + k_1x + k_0 \in \mathbb{Z}[x],$$
where
$$3 \le k_{n-1} \le k_{n-2} \le \cdots \le k_1 \le k_0 \le 2k_{n-1} - 3.$$
We show that $f(x)$ and $f(x^2)$ are irreducible over $\mathbb{Q}$. Moreover, the upper bound of $2k_{n-1} - 3$ on the coefficients of $f(x)$ is the best possible in this situation.

1. Introduction. In 1893, the Swedish actuary Gustaf Eneström published a paper on the location of zeros of a certain class of real-coefficient polynomials in a journal on pension insurance. Approximately twenty years later, Kakeya published a similar result, although his proof contained a mistake, which was subsequently corrected by Hurwitz. This result is known in the literature today as the Eneström–Kakeya Theorem. For more details concerning the history of this theorem, see [4]. We state this theorem for polynomials in $\mathbb{Z}[x]$.

THEOREM 1.1 (Eneström–Kakeya). *If*
$$f(x) = k_nx^n + k_{n-1}x^{n-1} + k_{n-2}x^{n-2} + \cdots + k_1x + k_0 \in \mathbb{Z}[x],$$
where
$$0 < k_n \le k_{n-1} \le k_{n-2} \le \cdots \le k_1 \le k_0,$$
then $f(x)$ has no zeros in the set $\{z \in \mathbb{C} : |z| < 1\}$.

We refer to the polynomials described in Theorem 1.1 as *Eneström–Kakeya polynomials*. While Theorem 1.1 gives information on the location of their zeros, it does not address the irreducibility of these polynomials over $\mathbb{Q}$. In fact, this question of irreducibility seems to be ignored in the literature. This might be due in part to the fact that none of the classical techniques or theorems for proving irreducibility applies to these polynomials. It is the purpose of this article to prove the irreducibility over $\mathbb{Q}$ of a certain subclass of the Eneström–Kakeya polynomials. We achieve this main result by first

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showing that if $f(x)$ has no zeros in the unit disk, and has a zero that is "large" enough, then $f(x)$ is irreducible over $\mathbb{Q}$. Then we show that the polynomials in question here do have these desired properties.

Throughout this article, all polynomials are assumed to be in $\mathbb{Z}[x]$. In addition, when we say irreducible or reducible, we mean irreducible or reducible over $\mathbb{Q}$.

2. Preliminaries. The following propositions are slightly modified versions of Lemma 5 in [2] and Lemma 3 in [3].

PROPOSITION 2.1. *Let $f(x)$ be monic with no zeros in the set $\{z \in \mathbb{C} : |z| \le 1\}$. If $f(x)$ has a zero α with $\lceil |\alpha| \rceil > |f(0)|/2$, then $f(x)$ is irreducible.*

We omit the proof of Proposition 2.1 as it is similar to the proof of the following proposition.

PROPOSITION 2.2. *Let $f(x)$ be monic with no zeros in the set $\{z \in \mathbb{C} : |z| \le 1\}$. If $f(x)$ has a zero $\alpha \in \mathbb{C} \setminus \mathbb{R}$ with $\lceil |\alpha^2| \rceil > |f(0)|/2$, then $f(x)$ is irreducible.*

Proof. Suppose that $f(x)$ is reducible. Then we can write $f(x)=h(x)k(x)$ for some monic positive-degree polynomials $h(x)$ and $k(x)$. Since $f(\alpha) = 0$ we may assume without loss of generality that $k(\alpha) = 0$. Since $\alpha \in \mathbb{C} \setminus \mathbb{R}$, we have $k(\overline{\alpha}) = 0$. Now let r be the degree of $h(x)$ and let $\alpha_1, \ldots, \alpha_r$ be the zeros of $h(x)$. Since $f(x)$ has no zeros in the set $\{z \in \mathbb{C} : |z| \le 1\}$, we know that $|\alpha_i| > 1$ for $1 \le i \le r$. Thus, $|h(0)| \ge 2$, since $h(x) \in \mathbb{Z}[x]$. Similarly, we see that

$$|k(0)| \ge \lceil |\alpha\overline{\alpha}| \rceil > \frac{|f(0)|}{2}.$$

Hence,

$$|f(0)| = |h(0)| \cdot |k(0)| > 2 \cdot \frac{|f(0)|}{2} = |f(0)|,$$

and this contradiction proves the proposition. ∎

Although the next proposition, due to the authors of this paper, appears in [3], we present a proof here for the sake of completeness.

PROPOSITION 2.3. *Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Q}[x]$ and suppose that $a_i \neq 0$ and $a_j \neq 0$ for some $0 \le i < j \le n$. Suppose further that*

$$\sum_{\substack{0 \le k \le n \\ k \neq t}} |a_k| \le q^t |a_t| \tag{2.1}$$

for some $0 \le t \le n$ with $t \neq i$ and $t \neq j$, and some $q \in \mathbb{R}$ with $0 < q \le 1$. If $f(x)$ has a zero $\alpha \in \{z \in \mathbb{C} : q \le |z| \le 1\}$, then equality holds in (2.1) and $\alpha^{2(j-i)} = 1$.

Proof. Suppose that $f(\alpha) = 0$, where $\alpha \in \{z \in \mathbb{C} : q \le |z| \le 1\}$. Then

$$q^t|a_t| \le |a_t\alpha^t| = \Big|\sum_{\substack{0\le k\le n\\ k\ne t}} a_k\alpha^k\Big| \le \sum_{\substack{0\le k\le n\\ k\ne t}} |a_k\alpha^k| \le \sum_{\substack{0\le k\le n\\ k\ne t}} |a_k| \le q^t|a_t|. \tag{2.2}$$

Thus, equality holds in (2.1) and $|\alpha| = 1$. Moreover, since equality then holds in every step of (2.2), it follows that

$$|a_j\alpha^j + a_i\alpha^i| = |a_j\alpha^j| + |a_i\alpha^i| = \frac{|a_j\alpha^j| + |a_i\alpha^i|}{|\alpha^i|} = |a_j\alpha^{j-i}| + |a_i|,$$

which implies that $\alpha^{j-i} \in \mathbb{R}$. Hence, $\alpha^{j-i} = \pm 1$. ∎

3. The main result. For the remainder of this article, let

$$f(x) = x^n + k_{n-1}x^{n-1} + \cdots + k_1x + k_0,$$

where

$$3 \le k_{n-1} \le \cdots \le k_1 \le k_0 \le 2k_{n-1} - 3.$$

Our main goal is to establish the irreducibility of $f(x)$ and $f(x^2)$. We first prove a lemma concerning the location of a certain zero of $f(x)$.

LEMMA 3.1. *If $k_{n-1} \ge 4$, then $f(x)$ has a negative real zero α with $\lceil|\alpha|\rceil \ge k_{n-1} - 1$.*

Proof. For convenience of notation, we let $a = k_{n-1}$. Let

$$g(x) = x^{n-2} + ax^{n-3} + k_{n-2}x^{n-4} + \cdots + k_2,$$

so that

$$f(x) = x^2g(x) + k_1x + k_0.$$

Observe that the same conditions on the coefficients of $f(x)$ also hold for $g(x)$. The proof is presented in two cases according to the parity of n.

First suppose that n is odd. In this situation, we use induction to show that $f(2-a) \ge a-1$ and the conclusion of the lemma follows. The base case here is $n = 3$, so let

$$f(x) = x^3 + ax^2 + k_1x + k_0.$$

Then

$$\begin{aligned} f(2-a) &= (2-a)^3 + a(2-a)^2 + k_1(2-a) + k_0 \\ &\ge (2-a)^3 + a(2-a)^2 + k_1(2-a) + k_1 \\ &= (2-a)^3 + a(2-a)^2 + k_1(3-a) \\ &\ge (2-a)^3 + a(2-a)^2 + (2a-3)(3-a) \qquad \text{(since } a \ge 4) \\ &= a-1, \end{aligned}$$

which establishes the base case. Now suppose that $n > 3$ is odd. Since $\deg(g)$ is odd, we find by induction that $g(2-a) \geq a-1$. Hence,

$$\begin{aligned} f(2-a) - a + 1 &= (2-a)^2 g(2-a) + k_1(2-a) + k_0 - a + 1 \\ &\geq (2-a)^2(a-1) + (2a-3)(2-a) + 1 = (a-1)(a-3)^2 \\ &\geq 0 \quad \text{since } a > 1, \end{aligned}$$

which completes the proof in the case when n is odd.

Now assume that n is even. In this situation, we use induction to show that $f(2-a) \leq -1$ and the conclusion of the proposition follows. The base case here is $n = 4$ so let

$$f(x) = x^4 + ax^3 + k_2x^2 + k_1x + k_0.$$

Then

$$\begin{aligned} f(2-a) + 1 &= (2-a)^4 + a(2-a)^3 + k_2(2-a)^2 + k_1(2-a) + k_0 + 1 \\ &\leq (2-a)^4 + a(2-a)^3 + k_1(2-a)^2 + k_1(2-a) + k_0 + 1 \\ &= (2-a)^4 + a(2-a)^3 + k_1(2-a)(3-a) + k_0 + 1 \\ &\leq (2-a)^4 + a(2-a)^3 + k_0(2-a)(3-a) + k_0 + 1 \\ &\leq (2-a)^4 + a(2-a)^3 + (2a-3)(2-a)(3-a) + 2a - 2 \\ &= -(a-1)(a-4) \\ &\leq 0 \quad \text{since } a \geq 4, \end{aligned}$$

and the base case is verified. Now suppose that $n > 4$ is even. Since $\deg(g)$ is even, we can assume by induction that $g(2-a) \leq -1$. Hence,

$$\begin{aligned} f(2-a) + 1 &= (2-a)^2 g(2-a) + k_1(2-a) + k_0 + 1 \\ &\leq -(2-a)^2 + a(2-a) + 2a - 2 = -2(a-1)(a-3) \\ &\leq 0 \quad \text{since } a \geq 4. \ \blacksquare \end{aligned}$$

The bounds on the coefficients of $f(x)$ in Lemma 3.1 are the best possible. For example, if $k_{n-1} = 3$, then a counterexample is $f(x) = x^4 + 3x^3 + 3x^2 + 3x + 3$. Also, if the upper bound on the coefficients is $2k_{n-1} - 2$, then a counterexample is $f(x) = x^4 + 4x^3 + 6x^2 + 6x + 6$.

LEMMA 3.2. *The polynomial $f(x)$ has no zeros in the set $\{z \in \mathbb{C} : |z| \leq 1\}$.*

Proof. By Theorem 1.1, $f(x)$ has no zeros in $\{z \in \mathbb{C} : |z| < 1\}$. So suppose that $f(\alpha) = 0$ with $|\alpha| = 1$. Consider the polynomial

$$F(x) = (x-1)f(x) = x^{n+1} + (k_{n-1} - 1)x^n + \sum_{i=0}^{n-2}(k_i - k_{i+1})x^{i+1} - k_0.$$

Since

$$1+(k_{n-1}-1)+\sum_{i=0}^{n-2}(k_i-k_{i+1})=k_0,$$

and the coefficients of x^{n+1} and x^n in $F(x)$ are nonzero, it follows from Proposition 2.3 that $\alpha^2=1$. It is readily seen that $f(1)\neq 0$. Since

$$f(-1)=1+\sum_{i=0}^{n-2}(k_i-k_{i+1})>0$$

when n is even, and

$$f(-1)=k_{n-1}-1+\sum_{i=0}^{n-3}(k_i-k_{i+1})>0$$

when n is odd, the lemma is proven. ∎

REMARK 3.3. Lemma 3.2 also holds when the coefficients of $f(x)$ satisfy

$$0<k_{n-1}\le k_{n-2}\le\cdots\le k_0,$$

without any upper bound on k_0.

We are now in a position to prove the main result.

THEOREM 3.4. *The polynomials $f(x)$ and $f(x^2)$ are irreducible.*

Proof. If $k_{n-1}=3$, then the theorem follows from Eisenstein's criterion with $p=3$. So assume that $k_{n-1}\ge 4$. By Lemma 3.1, we see that $f(x)$ has a negative real zero α with $\lceil|\alpha|\rceil\ge k_{n-1}-1$. From Lemma 3.2, we know that $f(x)$ has no zeros in the set $\{z\in\mathbb{C}:|z|\le 1\}$. It follows from Proposition 2.1 that $f(x)$ is irreducible since

$$\frac{f(0)}{2}=\frac{k_0}{2}\le\frac{2k_{n-1}-3}{2}<\frac{2k_{n-1}-2}{2}=k_{n-1}-1\le\lceil|\alpha|\rceil.$$

Now let $\alpha=\alpha_1,\ldots,\alpha_n$ be the zeros of $f(x)$. Write

$$f(x)=\prod_{j=1}^{n}(x-\alpha_j).$$

Then

$$f(x^2)=\prod_{j=1}^{n}(x\pm\sqrt{\alpha_j}).$$

Since $|\alpha_j|>1$, it follows that $f(x^2)$ has no zeros in the set $\{z\in\mathbb{C}:|z|\le 1\}$. Furthermore, since $\alpha\in\mathbb{R}$ is negative, we deduce that $\beta=\sqrt{\alpha}\in\mathbb{C}\setminus\mathbb{R}$ is a zero of $f(x^2)$ with

$$\lceil|\beta^2|\rceil=\lceil|\alpha|\rceil>f(0)/2.$$

Hence, $f(x^2)$ is irreducible by Proposition 2.2. ∎

With $k_{n-1} \geq 3$, the upper bound of $2k_{n-1} - 3$ on the coefficients of $f(x)$ in Theorem 3.4 is the best possible since $h(1-a) = 0$ if

$$h(x) = x^n + ax^{n-1} + ax^{n-2} + \cdots + ax^2 + (a+1)x + (2a-2).$$

With the restriction on the ordering of the coefficients of $f(x)$, the lower bound of 3 in Theorem 3.4 is also the best possible since if $k_{n-1} = 2$, then $2k_{n-2} - 3 = 1$, and the conditions on the coefficients cannot be satisfied. However, if this upper bound is relaxed when $k_{n-1} = 2$, then we get the following.

PROPOSITION 3.5. *Let*

$$p(x) = x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0,$$

where

$$2 = k_{n-1} \leq k_{n-2} \leq \cdots k_1 \leq k_0 \leq 3.$$

Then $p(x)$ is irreducible. Moreover, with $k_{n-1} = 2$, the upper bound of 3 *is the best possible.*

Proof. If $k_0 = 2$, then $p(x)$ is irreducible by Eisenstein's criterion with $p = 2$. So let $k_0 = 3$, and suppose that $p(x) = g(x)h(x)$, where $\deg(g) > 0$ and $\deg(h) > 0$. Since $p(0) = 3$, we may assume without loss of generality that $|g(0)| = 1$. However, by Remark 3.3, $p(x)$ has no zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$. Consequently, $g(x)$ has all of its zeros in $\{z \in \mathbb{C} : |z| > 1\}$, which contradicts the fact that $|g(0)| = 1$.

The upper bound of 3 is the best possible when $k_{n-1} = 2$ since

$$\begin{aligned} x^{2n} + 2x^{2n-1} + 2^{2n-2} + \cdots + 2x^{n+1} + 4x^n + 4x^{n-1} + \cdots + 4x + 4 \\ = (x^n + 2)(x^n + 2x^{n-1} + \cdots + 2x + 2). \quad \blacksquare \end{aligned}$$

4. Concluding remarks. Even though Theorem 1.1 deals only with positive coefficients, one might wonder if there is some sort of analog of Theorem 3.4 when the coefficients of $f(x)$ are negative. Indeed, the following theorem, which is due to Brauer [1], addresses this situation.

THEOREM 4.1 (Brauer). *Let*

$$h(x) = x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0,$$

where

$$k_{n-1} \leq k_{n-2} \leq \cdots \leq k_0 < 0.$$

Then $f(x)$ is irreducible.

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