COLLOQUIUM MATHEMATICUM

VOL. 113 2008 NO. 2

DIRECT LIMIT OF MATRIX ORDER UNIT SPACES

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Abstract. The notion of $\mathcal{F}$-approximate order unit norm for ordered $\mathcal{F}$-bimodules is introduced and characterized in terms of order-theoretic and geometric concepts. Using this notion, we characterize the inductive limit of matrix order unit spaces.

1. Introduction. A study of normed $\mathcal{F}$-bimodules as the direct limit of matrix normed spaces was suggested by B. E. Johnson, as an appropriate model to study the matricial theory of operator spaces. This idea was appreciated and justified by Effros and Ruan in [2]. The present authors extended this idea to the order-theoretic context. In [7, 8] they studied the direct limit of matrix ordered spaces and that of matricially Riesz normed spaces. Continuing this process, in this paper we discuss the direct limit of approximate matrix order unit spaces, studied by Karn and Vasudevan [3, 4]. We also consider the direct limit of matrix order unit spaces (studied by Choi and Effros [1]).

We recall the following notions discussed in [7, 8] (see also [2]).

Matricial notions. Let V be a complex vector space. Let $M_n(V)$ denote the set of all $n \times n$ matrices with entries from V. For $V = \mathbb{C}$, we denote $M_n(\mathbb{C})$ by M_n. For $\alpha = [\alpha_{ij}] \in M_n$ and $v = [v_{ij}] \in M_n(V)$ we define

$$\alpha v = \Big[\sum_{j=1}^{n} \alpha_{ij} v_{jk}\Big], \qquad v\alpha = \Big[\sum_{j=1}^{n} v_{ij} \alpha_{jk}\Big].$$

Then $M_n(V)$ is an M_n-bimodule for all $n \in \mathbb{N}$. In particular $M_n(V)$ is a complex vector space for all $n \in \mathbb{N}$. For $v \in M_n(V)$, $w \in M_m(V)$, we define

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+m}(V)$$

Next, we consider the family $\{M_n\}$. For each $n, m \in \mathbb{N}$ define $\sigma_{n,n+m} : M_n \to M_{n+m}$ by $\sigma_{n,n+m}(\alpha) = \alpha \oplus 0_m$. Then $\sigma_{n,n+m}$ is a vector space

2000 *Mathematics Subject Classification*: Primary 46L07.

Key words and phrases: approximate order unit, matricially Riesz normed spaces, positively bounded, $\mathcal{F}$-Riesz norm.

isomorphism with

$$\sigma_{n,n+m}(\alpha\beta) = \sigma_{n,n+m}(\alpha)\sigma_{n,n+m}(\beta).$$

Thus we can identify M_n with a subalgebra of M_{n+m} for every $m \in \mathbb{N}$. More generally, we may identify M_n with a subset of the set $\mathcal{F}$ of $\infty \times \infty$ complex matrices having all but a finite number of entries zero. In this sense, $\mathcal{F}$ may be considered as the direct or inductive limit of the family $\{M_n\}$. In other words,

$$\mathcal{F} = \bigcup_{n=1}^{\infty} M_n.$$

Let e_{ij} denote the $\infty \times \infty$ matrix with 1 at the (i,j)th entry and 0 elsewhere. Then the collection $\{e_{ij}\}$ is called the set of *matrix units* in $\mathcal{F}$. We write 1_n for $\sum_{i=1}^{n} e_{ii}$. For $i, j, k, l \in \mathbb{N}$, we have $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Note that for any $\alpha \in \mathcal{F}$, there exist complex numbers α_{ij} such that

$$\alpha = \sum_{i,j} \alpha_{ij} e_{ij} \quad \text{(a finite sum)}.$$

Thus $\mathcal{F}$ is an algebra.

For $\alpha = \sum_{i,j} \alpha_{ij} e_{ij} \in \mathcal{F}$, we define $\alpha^* = \sum_{i,j} \bar{\alpha}_{ji} e_{ij} \in \mathcal{F}$. Then $\alpha \mapsto \alpha^*$ is an involution. In other words, $\mathcal{F}$ is a $*$-algebra.

Definition 1.1. Let V be a complex vector space. Consider the family $\{M_n(V)\}$. For each $n, m \in \mathbb{N}$, define $T_{n,n+m} : M_n(V) \to M_{n+m}(V)$ by $T_{n,n+m}(v) = v \oplus 0_m$, $0_m \in M_m(V)$. Then $T_{n,n+m}$ is an injective homomorphism. Let $\mathcal{V}$ be the inductive limit of the directed family $\{M_n(V), T_{n,n+m}\}$. Then $\mathcal{V}$ is an $\mathcal{F}$-bimodule. We shall call $\mathcal{V}$ the *matricial inductive limit* or *direct limit* of V.

Definition 1.2. An $\mathcal{F}$-bimodule $\mathcal{V}$ is said to be *non-degenerate* if for every $v \in \mathcal{V}$ there exists an $n \in \mathbb{N}$ such that $1_n v 1_n = v$.

The matricial inductive limit of a complex vector space may be characterized in the following sense:

Theorem 1.3 ([2]). *The matricial inductive limit of a complex vector space is a non-degenerate $\mathcal{F}$-bimodule. Conversely, let $\mathcal{W}$ be a non-degenerate $\mathcal{F}$-bimodule. Put $W = e_{11}\mathcal{W}e_{11}$. Then W is a complex vector space and $\mathcal{W}$ is its matricial inductive limit. Moreover,*

(a) $M_n(W) \cong 1_n \mathcal{W} 1_n \cong W \otimes M_n$.
(b) $\mathcal{W} = \bigcup_{n=1}^{\infty} M_n(W) \cong W \otimes \mathcal{F}$.

Now we recall the relevant norm structure.

Definition 1.4. Let V be a complex vector space. Recall that $M_n(V)$ is an M_n-bimodule for all $n \in \mathbb{N}$. A *matrix norm* on V is a sequence $\{\|\cdot\|_n\}$

such that $\|\cdot\|_n$ is a norm on $M_n(V)$ for all $n \in \mathbb{N}$. We say that $(V, \{\|\cdot\|_n\})$ is a *matrix normed space* if $\|v \oplus 0_m\|_{n+m} = \|v\|_n$ and $\|\alpha v \beta\|_n \leq \|\alpha\| \, \|v\|_n \|\beta\|$ for all $v \in M_n(V)$, $\alpha, \beta \in M_n$ and $n, m \in \mathbb{N}$ (see [9]).

Definition 1.5. Let $\mathcal{V}$ be a non-degenerate $\mathcal{F}$-bimodule. Let $\|\cdot\|$ be a norm on $\mathcal{V}$. Then we say that $\|\cdot\|$ is an *$\mathcal{F}$-bimodule norm* on $\mathcal{V}$ if $\|\alpha v \beta\| \leq \|\alpha\| \, \|v\| \, \|\beta\|$ for any $\alpha, \beta \in \mathcal{F}$, $v \in \mathcal{V}$. In this case we say that $\mathcal{V}$ is a non-degenerate *normed $\mathcal{F}$-bimodule.*

Theorem 1.6. *Let $(V, \{\|\cdot\|_n\})$ be a matrix normed space. Let $\mathcal{V}$ be the matricial inductive limit of V. For each $v \in \mathcal{V}$, we define $\|v\|$ as follows: let $n \in \mathbb{N}$ be such that $1_n v 1_n = v$. Write $\|v\| = \|v\|_n$. This definition is independent of the choice of n and introduces an $\mathcal{F}$-bimodule norm on $\mathcal{V}$ such that $(\mathcal{V}, \|\cdot\|)$ is a non-degenerate normed $\mathcal{F}$-bimodule.*

Conversely, let $(\mathcal{W}, \|\cdot\|)$ be a non-degenerate normed $\mathcal{F}$-bimodule and let $W = 1_1 \mathcal{W} 1_1$ and $\|\cdot\|_n = \|\cdot\||_{M_n(W)}$ for all $n \in \mathbb{N}$. Then $(W, \{\|\cdot\|_n\})$ is a matrix normed space whose matricial inductive limit is $(\mathcal{W}, \|\cdot\|)$.

Next, we come to the order structure.

Definition 1.7. A *matrix ordered space* is a $*$-vector space V together with a cone $M_n(V)^+$ in $M_n(V)_{\mathrm{sa}}$ for all $n \in \mathbb{N}$ and with the following property: if $v \in M_n(V)^+$ and $\gamma \in M_{n,m}$ then $\gamma^* v \gamma \in M_m(V)^+$ for any $n, m \in \mathbb{N}$. Here $M_n(V)_{\mathrm{sa}}$ stands for the self-adjoint part of $M_n(V)$.

Definition 1.8. Let $\mathcal{W}$ be an $\mathcal{F}$-bimodule. Then a map $* : \mathcal{W} \to \mathcal{W}$ is called an *involution* on $\mathcal{W}$ if for all $v, w \in \mathcal{W}$ and $\alpha \in \mathcal{F}$,

(1) $(v^*)^* = v$,
(2) $(v + w)^* = v^* + w^*$,
(3) $(\alpha v)^* = v^* \alpha^*$, $(v\alpha)^* = \alpha^* v^*$.

In this case $\mathcal{W}$ is called a *$*$-$\mathcal{F}$-bimodule.* We set $\mathcal{W}_{\mathrm{sa}} = \{v \in \mathcal{W} \mid v = v^*\}$.

Definition 1.9. Let $\mathcal{V}$ be a $*$-$\mathcal{F}$-bimodule. Let $\mathcal{V}^+$ be a bimodule cone in $\mathcal{V}_{\mathrm{sa}}$, that is,

1. $v_1, v_2 \in \mathcal{V}^+ \Rightarrow v_1 + v_2 \in \mathcal{V}^+$.
2. $v \in \mathcal{V}^+$, $\alpha \in \mathcal{F} \Rightarrow \alpha^* v \alpha \in \mathcal{V}^+$.

Then $(\mathcal{V}, \mathcal{V}^+)$ will be called an *ordered $\mathcal{F}$-bimodule.*

Theorem 1.10. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. Let $\mathcal{V}$ be the matricial inductive limit of V. Then $(\mathcal{V}, \mathcal{V}^+)$ is a non-degenerate ordered $\mathcal{F}$-bimodule, where $\mathcal{V}^+ = \bigcup_{n=1}^{\infty} M_n(V)^+$. Conversely, let $(\mathcal{W}, \mathcal{W}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule. Put $W = 1_1 \mathcal{W} 1_1$ and $M_n(W)^+ = 1_n \mathcal{W}^+ 1_n$ for all $n \in \mathbb{N}$. Then $(W, \{M_n(W)^+\})$ is a matrix ordered space with $\mathcal{W}^+ = \bigcup_{n=1}^{\infty} M_n(W)^+$.*

In the rest of the paper we will be dealing with non-degenerate ordered $\mathcal{F}$-bimodules. We introduce some more notations. We write $J_n = \sum_{i=1}^{n} e_{i,n+i}$ for any $n \in \mathbb{N}$. Note that $\|I_n\| = \|J_n\| = 1$ and $J_n I_n = 0$, $I_n J_n = J_n$, $J_n J_n = 0$, $J_n J_n^* = I_n$. Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule. Let $u_1, u_2 \in \mathcal{V}^+$, and find an $n \in \mathbb{N}$ such that $1_n u_1 1_n = u_1$ and $1_n u_2 1_n = u_2$. We denote $u_1 + J_n^* u_2 J_n$ by $(u_1, u_2)_n^+$. For any $v \in \mathcal{V}$ and $n \in \mathbb{N}$ with $1_n v 1_n = v$ we denote $I_n v J_n + J_n^* v^* I_n$ by $\mathrm{sa}_n(v)$.

NOTE. In the notation $(u_1, u_2)_n^+ \pm \mathrm{sa}_n(v) \in \mathcal{V}^+$, we say that $n \in \mathbb{N}$ is *suitable* provided $1_n u_1 1_n = u_1$, $1_n u_2 1_n = u_2$ and $1_n v 1_n = v$. This terminology will be used throughout the paper without any further explanation.

DEFINITION 1.11. Let $(\mathcal{V}, \mathcal{V}^+)$ be a positively generated non-degenerate ordered $\mathcal{F}$-bimodule. Let $\|\cdot\|$ be an $\mathcal{F}$-bimodule norm on $\mathcal{V}$. We say $\|\cdot\|$ is an *$\mathcal{F}$-Riesz norm* on $\mathcal{V}$ if for any $v \in \mathcal{V}$,

$$\|v\| = \inf\{\max(\|u_1\|, \|u_2\|) \mid (u_1, u_2)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+ \text{ for some } u_1, u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

DEFINITION 1.12. Let $(\mathcal{V}, \mathcal{V}^+)$ be an ordered $\mathcal{F}$-bimodule. We say that $\mathcal{V}^+$ is *proper* if $\mathcal{V} \cap (-\mathcal{V}^+) = \{0\}$, and *generating* if given $v \in \mathcal{V}$ there exist $v_0, v_1, v_2, v_3 \in \mathcal{V}^+$ such that $v = \sum_{k=0}^{3} i^k v_k$, where $i^2 = -1$.

DEFINITION 1.13. Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule such that $\mathcal{V}^+$ is proper and generating. Assume that $\|\cdot\|$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ such that $\mathcal{V}^+$ is norm closed. Then the triple $(\mathcal{V}, \mathcal{V}^+, \|\cdot\|)$ is called an *$\mathcal{F}$-Riesz normed bimodule.*

The following characterization of non-degenerate $\mathcal{F}$-Riesz normed bimodules can be obtained from [8].

THEOREM 1.14. *Let $(V, \{M_n(V)^+\}, \{\|\cdot\|_n\})$ be a matricially Riesz normed space. Let $(\mathcal{V}, \mathcal{V}^+)$ be the matricial inductive limit of the matrix ordered space $(V, \{M_n(V)^+\})$ and let $(\mathcal{V}, \|\cdot\|)$ be the matricial inductive limit of matrix normed space $(V, \{\|\cdot\|_n\})$. Then $(\mathcal{V}, \mathcal{V}^+, \|\cdot\|)$ is a non-degenerate $\mathcal{F}$-Riesz normed bimodule. Conversely, let $(\mathcal{W}, \mathcal{W}^+, \|\cdot\|)$ be a non-degenerate $\mathcal{F}$-Riesz normed bimodule. Let $W = 1_1 \mathcal{W} 1_1$ and $M_n(W)^+ = 1_n \mathcal{W}^+ 1_n$ and $\|\cdot\|_n = \|\cdot\||_{M_n(W)}$ for all $n \in \mathbb{N}$. Then $(W, \{M_n(W)^+\}, \{\|\cdot\|_n\})$ is a matricially Riesz normed space whose inductive limit is $(\mathcal{W}, \mathcal{W}^+, \|\cdot\|)$.*

2. Direct limit of approximate matrix order unit spaces. In this section, we discuss the notion of an approximate order unit and the consequent $\mathcal{F}$-norm in the context of ordered $\mathcal{F}$-bimodules.

DEFINITION 2.1. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. Then an increasing net $\{e_\lambda\}_{\lambda \in D}$ in V^+ is called an *approximate order unit for V* if for any $v \in V$, there are $\alpha > 0$ and $\lambda \in D$ such that

$$\begin{bmatrix} \alpha e_\lambda & v \\ v^* & \alpha e_\lambda \end{bmatrix} \in M_2(V)^+.$$

When $e_\lambda = e$ for all λ, we say that e is an *order unit* for V.

Definition 2.2. Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate, ordered $\mathcal{F}$-bimodule. An increasing net $\{e_\lambda\}_{\lambda \in D}$ in $\mathcal{V}^+$ is called an *approximate order unit for* $\mathcal{V}$ if given a $v \in \mathcal{V}$, we can find $\lambda \in D$, $\alpha > 0$ and $N \in \mathbb{N}$ such that $(\alpha e_\lambda, \alpha e_\lambda)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$.

Remark. If $\mathcal{V}$ has an approximate order unit, then $\mathcal{V}^+$ is generating.

Construction. Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule. Put $V = e_{11}\mathcal{V}e_{11}$. Then by Theorem 1.10, $(V, \{M_n(V)^+\})$ is a matrix ordered space and $(\mathcal{V}, \mathcal{V}^+)$ is its matricial inductive limit. Let $\{e_\lambda\}_{\lambda \in D}$ be an approximate order unit for $\mathcal{V}$. For every $\lambda \in D$, put $e_\lambda^1 = e_{11}e_\lambda e_{11}$. Then the net $\{e_\lambda^1\}_{\lambda \in D}$ is an approximate order unit for V. Define, for each $n \in \mathbb{N}$,

$$e_\lambda^n = e_\lambda^1 \oplus \cdots \oplus e_\lambda^1 = \bigoplus_{i=1}^n e_\lambda^1.$$

Then e_λ^n has the following representation.

Lemma 2.3.

$$e_\lambda^n = \sum_{i=1}^n e_{i1} e_\lambda^1 e_{1i}.$$

Proof. For $p, q \in \mathbb{N}$, let $J_{p,q} = \sum_{i=1}^q e_{i,p+i}$, so that $J_{0,p} = I_p$, $J_{p,p} = J_p$, where $J_p = \sum_{i=1}^p e_{i,p+i}$. Let $u_1, \ldots, u_k \in \mathcal{V}$ with $1_{n_i} u_i 1_{n_i} = u_i$, $i = 1, \ldots, k$. Define

$$u_1 \oplus \cdots \oplus u_k = \sum_{i=0}^{k-1} (J_{\sum_{s=0}^i n_s, n_{i+1}})^* u_{i+1} (J_{\sum_{s=0}^i n_s, n_{i+1}}),$$

where $n_0 = 0$. This gives

$$e_\lambda^n = \bigoplus_{i=1}^n e_\lambda^1 = \sum_{i=0}^{n-1} J_{i,1}^* e_\lambda^1 J_{i,1} = \sum_{i=1}^n e_{i,1} e_\lambda^1 e_{1,i}.$$

The following two results will be needed.

Lemma 2.4. *If $\{e_\lambda\}_{\lambda \in D}$ is an approximate order unit for V, then $\{e_\lambda^n\}_{\lambda \in D}$ is an approximate order unit for $M_n(V)$ for all $n \in \mathbb{N}$.*

Proof. Follows from [4, Lemma 2.6].

Lemma 2.5. *Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule and $\{e_\lambda\}_{\lambda \in D}$ be an approximate order unit for $\mathcal{V}$. If $v \in \mathcal{V}$, then there exist $n \in \mathbb{N}$, $\lambda \in D$ and $\alpha > 0$ such that $(\alpha e_\lambda^n, \alpha e_\lambda^n)_n^+ \pm \mathrm{sa}_n(v) \in \mathcal{V}^+$.*

Proof. It suffices to show that for every $\mu \in D$, there exist $n \in \mathbb{N}$, $\lambda \in D$ and $\beta > 0$ such that $e_\mu \leq \beta e_\lambda^n$. Let $\mu \in D$. Find $n \in \mathbb{N}$ such that $1_n e_\mu 1_n = e_\mu$. Then $e_\mu \in M_n(V)$. Let $e_\mu^{ij} = e_{1i} e_\mu e_{j1}$ for all $i, j = 1, \ldots, n$. Then $e_\mu^{ij} \in V$ for all $i, j = 1, \ldots, n$. Thus for each pair (i, j), there are $\lambda_{ij} \in D$ and $\alpha_{ij} > 0$ such that $(\alpha_{ij} e^1_{\lambda_{ij}}, \alpha_{ij} e^1_{\lambda_{ij}})_1^+ \pm \mathrm{sa}_1(e_\mu^{ij}) \in M_2(V)^+ \subset \mathcal{V}^+$. Since D is directed, there is a $\lambda \in D$ such that $\lambda_{ij} \leq \lambda$ for all $i, j = 1, \ldots, n$. Let $\alpha = \max\{\alpha_{ij} \mid 1 \leq i, j \leq n\} > 0$. Thus $(\alpha e_\lambda^1, \alpha e_\lambda^1)_1^+ \pm \mathrm{sa}_1(e_\mu^{ij}) \in M_2(V)^+ \subset \mathcal{V}^+$ for all $i, j = 1, \ldots, n$. That is,

$$\sum_{i,j=1}^{n} \gamma_{ij}((\alpha e_\lambda^1, \alpha e_\lambda^1)_1^+ \pm \mathrm{sa}_1(e_\mu^{ij}))\gamma_{ij}^* \in M_{2n}(V)^+ \subset \mathcal{V}^+,$$

where $\gamma_{ij} = e_{i1} + e_{n+j,2}$. This gives

$$\begin{aligned}
&\sum_{i,j=1}^{n} (e_{i1} + e_{n+j,2})[\alpha e_\lambda^1 + e_{21}\alpha e_\lambda^1 e_{12} \pm (e_{11} e_\mu^{ij} e_{12} + e_{21} e_\mu^{ij*} e_{11})](e_{1i} + e_{2,n+j}) \\
&\quad = \sum_{i,j=1}^{n} [e_{i1}\alpha e_\lambda^1 e_{1i} \pm e_{i1} e_\mu^{ij} e_{1,n+j} \pm e_{n+j,1} e_\mu^{ij*} e_{1i} + e_{n+j,1}\alpha e_\lambda^1 e_{1,n+j}] \\
&\quad = n\alpha \sum_{i=1}^{n} e_{i1} e_\lambda^1 e_{1i} + n\alpha \sum_{j=1}^{n} e_{n+j,1} e_\lambda^1 e_{1,n+j} \\
&\qquad \pm \sum_{i,j=1}^{n} e_{i1} e_\mu^{ij} e_{1,n+j} \pm \sum_{i,j=1}^{n} e_{n+j,1} e_\mu^{ij*} e_{1i}.
\end{aligned}$$

Therefore

$$(n\alpha e_\lambda^n, n\alpha e_\lambda^n)_n^+ \pm \mathrm{sa}_n(e_\mu) \in \mathcal{V}^+.$$

Since $e_\mu \in \mathcal{V}^+$, $e_\mu \leq n\alpha e_\lambda^n$.

Definition 2.6. Let $v \in \mathcal{V}$. Define

$$\|v\|^{\mathrm{a}} = \inf\{\alpha > 0 \mid (\alpha e_\lambda^N, \alpha e_\lambda^N)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+ \text{ for a suitable } N \in \mathbb{N} \text{ and } \lambda \in D\}.$$

In what follows, we shall show that $\|\cdot\|^{\mathrm{a}}$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$.

Lemma 2.7. *Let $v \in V^+$. Let $\alpha \in \mathcal{F}$ be such that $1_N \alpha 1_N = \alpha$. Then $\alpha^* v^N \alpha \leq \|\alpha\|^2 v^N$.*

Proof. It is easy to note that $v^N \alpha = \alpha v^N$, $\alpha^* v^N = v^N \alpha^*$, $\alpha^*\alpha \leq \|\alpha\|^2 I_N$. Let $\beta = (\|\alpha\|^2 I_N - \alpha^*\alpha)^{1/2}$. Then $1_N \beta 1_N = \beta$ so that $\beta v^N = v^N \beta$. Since $v^N \in \mathcal{V}^+$, we have $\beta^* v^N \beta \in \mathcal{V}^+$. In other words, $\beta^2 v^N \in \mathcal{V}^+$. Thus $(\|\alpha\|^2 I_N - \alpha^*\alpha) v^N \in \mathcal{V}^+$. Therefore, $\alpha^*\alpha v^N = \alpha^* v^N \alpha \leq \|\alpha\|^2 v^N$.

LEMMA 2.8. *If* $(u_1,u_2)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$, *then, for any* $k>0$,

$$(ku_1, k^{-1}u_2)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+.$$

Proof. We have

$$\left(\sqrt{k}\, I_n, \frac{1}{\sqrt{k}}\, I_n\right)_N^+ ((u_1,u_2)_N^+ \pm \mathrm{sa}_N(v))\left(\sqrt{k}\, I_n, \frac{1}{\sqrt{k}}\, I_n\right)_N^+ = (ku_1, k^{-1}u_2)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+.$$

LEMMA 2.9. $\|\cdot\|^{\mathrm{a}}$ *is an* $\mathcal{F}$*-bimodule seminorm.*

Proof. Let $v \in \mathcal{V}$, $\alpha,\beta \in \mathcal{F}$, $\alpha \neq 0, \beta \neq 0$. Given $\varepsilon > 0$, there exist $k>0$, $\lambda \in D$ and $N \in \mathbb{N}$ such that $(ke_\lambda^N, ke_\lambda^N)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$ and $k < \|v\|^{\mathrm{a}} + \varepsilon$. Without loss of generality, we may assume $1_N\alpha 1_N = \alpha$ and $1_N\beta 1_N = \beta$. Thus as in Lemma 2.10 of [8],

$$(\alpha k e_\lambda^N \alpha^*, \beta^* k e_\lambda^N \beta) \pm \mathrm{sa}_N(\alpha v \beta) \in \mathcal{V}^+.$$

By Lemma 2.7, $\alpha k e_\lambda^N \alpha^* \leq k\|\alpha\|^2 e_\lambda^N$ and $\beta^* k e_\lambda^N \beta \leq k\|\beta\|^2 e_\lambda^N$. Thus $(k\|\alpha\|^2 e_\lambda^N, k\|\beta\|^2 e_\lambda^N)_N^+ \pm \mathrm{sa}_N(\alpha v\beta) \in \mathcal{V}^+$. From Lemma 2.8, using $\|\beta\|/\|\alpha\|$, we obtain $(k\|\alpha\|\,\|\beta\| e_\lambda^N, k\|\alpha\|\,\|\beta\| e_\lambda^N)_N^+ \pm \mathrm{sa}_N(\alpha v\beta) \in \mathcal{V}^+$. By definition $\|\alpha v\beta\|^{\mathrm{a}} \leq k\|\alpha\|\,\|\beta\| < \|\alpha\|\,\|\beta\|(\|v\|^{\mathrm{a}}+\varepsilon)$. Since $\varepsilon>0$ is arbitrary $\|\alpha v\beta\|^{\mathrm{a}} \leq \|\alpha\|\,\|v\|^{\mathrm{a}}\|\beta\|$. Hence $\|\cdot\|^{\mathrm{a}}$ is an $\mathcal{F}$-bimodule seminorm.

LEMMA 2.10. $\|\cdot\|^{\mathrm{a}}$ *is an* $\mathcal{F}$*-Riesz seminorm.*

Proof. Let $v \in \mathcal{V}$. Let $u_1, u_2 \in \mathcal{V}^+$ be such that $(u_1,u_2)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$ for a suitable $N \in \mathbb{N}$. Put $k = \max(\|u_1\|^{\mathrm{a}}, \|u_2\|^{\mathrm{a}})$. Let $\varepsilon > 0$. Then by definition there exists $\lambda \in D$ such that $u_1 \leq (k+\varepsilon)e_\lambda^N$, $u_2 \leq (k+\varepsilon)e_\lambda^N$, and $((k+\varepsilon)e_\lambda^N, (k+\varepsilon)e_\lambda^N)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$. That is, $\|v\|^{\mathrm{a}} \leq k+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|v\|^{\mathrm{a}} \leq \max(\|u_1\|^{\mathrm{a}}, \|u_2\|^{\mathrm{a}})$. Hence

$$\|v\|^{\mathrm{a}} \leq \inf\{\max(\|u_1\|^{\mathrm{a}}, \|u_2\|^{\mathrm{a}}) \mid (u_1,u_2)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+ \text{ for some } u_1,u_2 \in \mathcal{V}^+ \text{ and a suitable } N \in \mathbb{N}\}.$$

For any $\alpha > \|v\|^{\mathrm{a}}$, by definition there exist $\lambda \in D$ and $N \in \mathbb{N}$ such that $(\alpha e_\lambda^N, \alpha e_\lambda^N)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$. Since $\|e_\lambda^N\|^{\mathrm{a}} \leq 1$, we conclude that $\|\cdot\|^{\mathrm{a}}$ is an $\mathcal{F}$-Riesz seminorm.

DEFINITION 2.11. The $\mathcal{F}$-Riesz seminorm given by Definition 2.6 is called an $\mathcal{F}$*-approximate order unit seminorm.*

We recall the following notions from [8].

DEFINITION 2.12. Let $\mathcal{A} \subset \mathcal{V}^+$. Then $\mathcal{A}$ is called *positively bounded* if $v \in \mathcal{V}_{\mathrm{sa}}$ and $v + k_n a_n \in \mathcal{V}^+$ for all $n \in \mathbb{N}$ implies $v \in \mathcal{V}^+$, where $\{a_n\}$ is a sequence in $\mathcal{A}$ and $\{k_n\}$ is a sequence in $(0,\infty)$ with $\inf k_n = 0$.

DEFINITION 2.13. Let $\mathcal{A} \subset \mathcal{V}^+$. Then $\mathcal{A}$ is called *almost positively bounded* if $(k_n u_1^n, k_n u_2^n)_{N_n}^+ \pm \mathrm{sa}_{N_n}(v) \in \mathcal{V}^+$ for all $n \in \mathbb{N}$ implies $v = 0$, where $\{u_1^n\}$, $\{u_2^n\}$ are sequences in $\mathcal{A}$, $\{k_n\}$ is a sequence in $(0,\infty)$ with $\inf k_n = 0$, and $\{N_n\}$ is a sequence in $\mathbb{N}$.

Observation. Let $n \in \mathbb{N}$. Let $u_1, u_2, v \in M_n(V)$ with $u_1, u_2 \in M_n(V)^+$ and $(u_1, u_2)_n^+ \pm \mathrm{sa}_n(v) \in \mathcal{V}^+$. By a technique used in Lemma 2.10 of [8], for any $1 \leq i, j \leq n$, we have

$$(e_{1i}u_1e_{i1}, e_{1j}u_2e_{j1})_1^+ \pm \mathrm{sa}_1(e_{1i}ve_{j1}) \in \mathcal{V}^+.$$

Lemma 2.14. *Assume that $\{e_\lambda^1\}_{\lambda\in D}$ is almost positively bounded. Then $\|\cdot\|^{\mathrm{a}}$ is a norm on $\mathcal{V}$.*

Proof. Let $\|v\|^{\mathrm{a}}=0$ for some $v\in\mathcal{V}$ such that $1_Nv1_N = v$. For every $n \in \mathbb{N}$, there exists $\lambda_n \in D$ such that $\left(\frac{1}{n}e_{\lambda_n}^N, \frac{1}{n}e_{\lambda_n}^N\right)_N^+ \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$. Then by the above observation, for any $1 \leq i, j \leq N$, $\left(\frac{1}{n}e_{\lambda_n}^1, \frac{1}{n}e_{\lambda_n}^1\right)_1^+ \pm \mathrm{sa}_1(e_{1i}ve_{j1}) \in \mathcal{V}^+$. That is, $e_{1i}ve_{j1} = 0$ for any $1 \leq i, j \leq N$. Hence $v = 0$.

Lemma 2.15. *Let $\mathcal{V}^+$ be proper. If $\{e_\lambda^1\}_{\lambda\in D}$ is positively bounded, then it is almost positively bounded.*

Proof. Let $v \in \mathcal{V}$ and let $\{\lambda_n\}$, $\{\mu_n\}$ be sequences in D, $\{k_n\}$ be a sequence of positive numbers with $\inf k_n = 0$ and $\{N_n\}$ be a sequence in $\mathbb{N}$ such that

$$(k_ne_{\lambda_n}^{N_n}, k_ne_{\mu_n}^{N_n})_{N_n}^+ \pm \mathrm{sa}_{N_n}(v) \in \mathcal{V}^+$$

for all $n \in \mathbb{N}$. Let $M \in \mathbb{N}$ be such that $1_Mv1_M = v$. Then

$$(k_ne_{\lambda_n}^M, k_ne_{\lambda_n}^M)_{N_n}^+ \pm \mathrm{sa}_M(v) \in \mathcal{V}^+.$$

Again for each $n \in \mathbb{N}$, we can choose $\delta_n \in D$ such that $\delta_n \geq \lambda_n, \mu_n$. Let

$$z_n = (k_ne_{\delta_n}^M, k_ne_{\delta_n}^M)_M^+ \pm \mathrm{sa}_M(v) \in \mathcal{V}^+.$$

We claim that $v = 0$. Fix j, k such that $1 \leq j, k \leq M$. Then by the Observation, we have

$$z_n^{jk} = (k_ne_{\delta_n}^1, k_ne_{\delta_n}^1)_1^+ \pm \mathrm{sa}_1(e_{1j}ve_{k1}) \in \mathcal{V}^+$$

for all $n \in \mathbb{N}$. Now

$$\begin{aligned}(I_1 + J_1)z_n^{jk}(I_1 + J_1)^* &= k_ne_{\delta_n}^1 + k_ne_{\delta_n}^1 \pm (e_{1j}ve_{k1} + e_{1k}v^*e_{j1})\\ &= 2(k_ne_{\delta_n}^1 \pm \Re(e_{1j}ve_{k1}))\end{aligned}$$

and

$$\begin{aligned}(I_1 + iJ_1)z_n^{jk}(I_1 + iJ_1)^* &= k_ne_{\delta_n}^1 + k_ne_{\delta_n}^1 \pm i(e_{1j}ve_{k1} - e_{1k}v^*e_{j1})\\ &= 2(k_ne_{\delta_n}^1 \pm i\Im(e_{1j}ve_{k1})).\end{aligned}$$

Since $\{e_\lambda^1\}_{\lambda\in D}$ is positively bounded, we have $\pm\Re(e_{1j}ve_{k1}) \in \mathcal{V}^+$ and $\pm\Im(e_{1j}ve_{k1}) \in \mathcal{V}^+$. Since $\mathcal{V}^+$ is proper, $\Re(e_{1j}ve_{k1}) = 0$ and $\Im(e_{1j}ve_{k1}) = 0$ for all $1 \leq j, k \leq M$. That is, $v = 0$. Hence $\{e_\lambda^1\}_{\lambda\in D}$ is almost positively bounded.

Summarizing the above results, we obtain the following

Theorem 2.16. *Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate ordered $\mathcal{F}$-bimodule such that $\mathcal{V}^+$ is proper. Let $\{e_\lambda\}_{\lambda\in D}$ be an approximate order unit for $\mathcal{V}$ such that $\{e^n_\lambda\}_{\lambda\in D}$ is positively bounded for all $n \in \mathbb{N}$. Then $\|\cdot\|^{\mathrm{a}}$ is an $\mathcal{F}$-Riesz norm on $\mathcal{V}$ such that $\mathcal{V}^+$ is $\|\cdot\|^{\mathrm{a}}$-closed.*

Definition 2.17. A non-degenerate ordered $\mathcal{F}$-bimodule $(\mathcal{V}, \mathcal{V}^+)$ with an $\mathcal{F}$-approximate order unit $\{e_\lambda\}_{\lambda\in D}$ is called an *$\mathcal{F}$-approximate order unit bimodule* if $\mathcal{V}^+$ is proper and $\{e^n_\lambda\}_{\lambda\in D}$ is positively bounded for each $n \in \mathbb{N}$. It will be denoted by $(\mathcal{V}, \{e_\lambda\}_{\lambda\in D})$.

Definition 2.18. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space satisfying the following conditions:

1. $\{e_\lambda\}_{\lambda\in D}$ is an approximate order unit for V.
2. V^+ is proper.
3. $\{e^n_\lambda\}_{\lambda\in D}$ is positively bounded in $M_n(V)^+$ for all $n \in \mathbb{N}$.

Then $(V, \{e_\lambda\}_{\lambda\in D})$ is called an *approximate order unit space* [6].

Now we can summarize the results of the section by giving the following characterization of the direct limit of approximate matrix order unit spaces.

Theorem 2.19. *Let $(V, \{e_\lambda\}_{\lambda\in D})$ be an approximate order unit space. Let $(\mathcal{V}, \mathcal{V}^+)$ be the matricial inductive limit of V. Then $(\mathcal{V}, \{e^n_\lambda\}_{\lambda\in D, n\in\mathbb{N}})$ is an $\mathcal{F}$-approximate order unit bimodule. Conversely, let $(\mathcal{V}, \{e_\lambda\}_{\lambda\in D})$ be an $\mathcal{F}$-approximate order unit bimodule. Let $V = e_{11}\mathcal{V}e_{11}$, $M_n(V)^+ = 1_n\mathcal{V}^+1_n$ and $e^1_\lambda = e_{11}e_\lambda e_{11}$ for all $\lambda \in D$. Then $(V, M_n(V)^+)$ is a matrix ordered space such that $(V, \{e^1_\lambda\}_{\lambda\in D})$ is an approximate order unit space.*

3. Direct limit of matrix order unit spaces

Definition 3.1. Let $(\mathcal{V}, \mathcal{V}^+)$ be a non-degenerate, ordered $\mathcal{F}$-bimodule. Then $e \in \mathcal{V}^+$ is called an *$\mathcal{F}$-order unit for $\mathcal{V}$* if given a $v \in \mathcal{V}$, we can find $\alpha > 0$ and $N \in \mathbb{N}$ such that $(\alpha e, \alpha e)^+_N \pm \mathrm{sa}_N(v) \in \mathcal{V}^+$.

We prove that in a non-degenerate ordered $\mathcal{F}$-bimodule $\mathcal{V}$, an $\mathcal{F}$-order unit cannot exist. Indeed, suppose $\mathcal{V}$ has an $\mathcal{F}$-order unit, say e. Then there exists an $n \in \mathbb{N}$ such that $1_n e 1_n = e$. Let $v \in \mathcal{V}, v \neq 0$. Then for some i and j, $e_{1i}ve_{j1} \neq 0$. Put $w = e_{n+1,i}ve_{j,n+1}$. Then $w \neq 0$ and for any $\alpha > 0$, $(\alpha e, \alpha e)^+_{n+1} \pm \mathrm{sa}_{n+1}(w) \notin \mathcal{V}^+$ provided $\mathcal{V}^+$ is proper. If possible, let $(\alpha e, \alpha e)^+_{n+1} \pm \mathrm{sa}_{n+1}(w) \in \mathcal{V}^+$. This means

$$(e_{1,n+1}, e_{1,n+1})^{+*}_{n+1}[(\alpha e, \alpha e)^+_{n+1} \pm \mathrm{sa}_{n+1}(w))](e_{1,n+1}, e_{1,n+1})^+_{n+1} \in \mathcal{V}^+$$

or

$$((0,0)^+_1 \pm \mathrm{sa}_1(e_{1,n+1}we_{1,n+1}) \in \mathcal{V}^+.$$

That is, $\mathrm{sa}_1(e_{1i}ve_{j1}) \in \mathcal{V}^+$, which is a contradiction. However, in a matrix ordered space an order unit can exist.

THEOREM 3.2. *Let* (V, e) *be a matrix order unit space (see* [1]*). Let* $(\mathcal{V}, \mathcal{V}^+)$ *be the matricial inductive limit of* V. *Then* $(\mathcal{V}, \{e^n\}_{n\in\mathbb{N}})$ *is an* $\mathcal{F}$-*approximate order unit bimodule.*

Acknowledgements. The authors are grateful to the referees for their valuable suggestions.

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Received 16 July 2005;
revised 22 January 2006
(4637)