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# BOCHNER'S FORMULA FOR HARMONIC MAPS FROM FINSLER MANIFOLDS

BY

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**Abstract.** Let $\phi : (M,F) \to (N,h)$ be a harmonic map from a Finsler manifold to any Riemannian manifold. We establish Bochner's formula for the energy density of $\phi$ and maximum principle on Finsler manifolds, from which we deduce some properties of harmonic maps $\phi$.

**1. Introduction.** Let $(M,F)$ be a Finsler manifold, $SM$ the projective sphere bundle of $M$, with canonical projection map $\pi : SM \to M$ given by $(x,[y]) \mapsto x$, and let $S_xM := \pi^{-1}(x)$ be the projective sphere at $x$. We denote the pull-backs of $TM$ and $T^*M$ by $\pi^*TM$ and $\pi^*T^*M$, respectively. Let $\phi : (M,F) \to (N,h)$ be a smooth map from a Finsler manifold to a Riemannian manifold. The *energy density* of $\phi$ is the function $e(\phi) : SM \to \mathbb{R}$ defined by

$$e(\phi)(x,[y]) = \frac{1}{2}\sum_i h(\phi_* e_i, \phi_* e_i),$$

where $\{e_i\}$ is an orthonormal basis with respect to $g$ (the fundamental tensor of $F$) at $(x,[y])$. The *tension field* of $\phi$ is (see [3])

$$\tau(\phi) := -\langle d\phi, \dot{\eta}\rangle + \operatorname{Tr} Dd\phi \in \Gamma((\phi\circ\pi)^*TN), \tag{1.1}$$

where $\eta$ (resp. $Dd\phi$) denotes the Cartan form (resp. the second fundamental form) of $\phi$ and the dot "·" denotes the covariant derivative along the Hilbert form.

PROPOSITION 1 ([4]). *$\phi$ is a harmonic map if and only if $\tau(\phi) = 0$.*

In this paper, we shall establish Bochner's formula for $e(\phi)$ and the maximum principle on Finsler manifolds, from which we deduce some properties of harmonic maps $\phi$.

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**2. Bochner's formula.** Let

$$1 \le i, j, \ldots \le m, \quad 1 \le \mu, \nu, \ldots \le m-1, \quad 1 \le \alpha, \beta, \ldots \le n,$$

where $m = \dim M$ and $n = \dim N$.

Take a $g$-orthonormal frame field $\{e_i\}$ for $\pi^*TM$ and let $\{\omega_i\}$ be a local coframe. Let $\{\omega_{ij}\}$ and $\{{}^b\omega_{ij}\}$ be the Chern connection 1-form and the Berwald connection 1-form, respectively. We have (see [1])

$$ {}^b\omega_{ij} = \omega_{ij} + \dot{A}_{ij\mu}\omega_\mu, \tag{2.1}$$

$$ {}^bR^M_{ijkl} = R^M_{ijkl} + \dot{A}_{ijl|k} - \dot{A}_{ijk|l} + \dot{A}_{ipk}\dot{A}_{pjl} - \dot{A}_{ipl}\dot{A}_{pjk}, \tag{2.2}$$

where $A = A_{\lambda\mu\nu}\omega_\lambda\omega_\mu\omega_\nu$ is the Cartan tensor of $M$ and "$|$" denotes the horizontal covariant differentials with respect to the Chern connection.

Lemma 1 ([4]). *For $X = \sum_i x_i\omega_i \in \Gamma(\pi^*T^*M)$,*

$$\operatorname{div} X = \sum_i x_{i|i} + \sum_{\mu,\lambda} x_\mu P_{\lambda\lambda\mu},$$

*where $\operatorname{div} X$ denotes the divergence of $X$ on $SM$ with respect to the Riemannian metric $G$ on $SM$, and $P_{\lambda\mu\nu} = -\dot{A}_{\lambda\mu\nu}$ is the Landsberg curvature of $M$.*

Let $\phi : (M,F) \to (N,h)$ be a smooth map. Set $h = \sum_\alpha \theta_\alpha^2 \in \Gamma(\odot^2 T^*N)$ and $\phi^*\theta_\alpha = \sum_i a_{\alpha i}\omega_i$. The covariant differentials of $a_{\alpha i}$ with respect to the Berwald connection and the Chern connection are defined by, respectively,

$$\begin{aligned} Da_{\alpha i} &= da_{\alpha i} - \sum_j a_{\alpha j}\,{}^b\omega_{ij} - \sum_\beta a_{\beta i}\phi^*\theta_{\alpha\beta} \\ &= \sum_j a_{\alpha i,j}\omega_j + \sum_\mu a_{\alpha i;\mu}\omega_{m\mu}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} Da_{\alpha i} &= da_{\alpha i} - \sum_j a_{\alpha j}\omega_{ij} - \sum_\beta a_{\beta i}\phi^*\theta_{\alpha\beta} \\ &= \sum_j a_{\alpha i|j}\omega_j + \sum_\mu a_{\alpha i;\mu}\omega_{m\mu}. \end{aligned} \tag{2.4}$$

where "," denotes the horizontal covariant differentials with respect to the Berwald connection.

Lemma 2 ([4]). *The second fundamental form of $\phi : (M,F) \to (N,h)$ satisfies $a_{\alpha i|j} = a_{\alpha j|i}$ and $a_{\alpha i;\mu} = 0$ for all $\alpha, i, j, \mu$.*

Lemma 3. *For $X = \sum_i x_i\omega_i \in \Gamma(\pi^*T^*M)$, we have*

$$x_{i,j} = x_{i|j} + x_\mu P_{ij\mu}.$$

*Proof.* The covariant differentials of $x_i$ with respect to the Berwald connection and the Chern connection are defined by, respectively,

$$dx_i - x_j\,{}^b\omega_{ij} = x_{i,j}\omega_j + x_{i;\mu}\omega_{m\mu}, \tag{2.5}$$

$$dx_i - x_j\omega_{ij} = x_{i|j}\omega_j + x_{i;\mu}\omega_{m\mu}. \tag{2.6}$$

Using (2.1) and (2.5), we obtain

$$x_{i,j}\omega_j + x_{i;\mu}\omega_{m\mu} = dx_i - x_j\omega_{ij} + x_\mu P_{i\lambda\mu}\omega_\lambda. \tag{2.7}$$

Combining (2.6) and (2.7) completes the proof.

From (2.3) and Lemma 2, we have

$$\begin{aligned}
&da_{\alpha i,j}\wedge\omega_j + a_{\alpha i,j}d\omega_j\\
&= -da_{\alpha i}\wedge{}^b\omega_{ij} - a_{\alpha i}d\,{}^b\omega_{ij} + da_{\beta i}\wedge\phi^*\theta_{\alpha\beta} + a_{\beta i}\phi^*d\theta_{\alpha\beta}\\
&= (-a_{\alpha j,k}\omega_k - a_{\alpha k}\,{}^b\omega_{jk} + a_{\beta j}\phi^*\theta_{\alpha\beta})\wedge{}^b\omega_{ij}\\
&\quad - a_{\alpha j}\big(\tfrac{1}{2}\,{}^bR^M_{ijkl}\omega_k\wedge\omega_l + {}^bP_{ijk\mu}\omega_k\wedge\omega_{m\mu} + {}^b\omega_{ik}\wedge{}^b\omega_{kj}\big)\\
&\quad + (a_{\beta i,k}\omega_k + a_{\beta k}\,{}^b\omega_{ik} - a_{\gamma i}\phi^*\theta_{\beta\gamma})\wedge\phi^*\theta_{\alpha\beta}\\
&\quad + a_{\beta i}\phi^*\big(\tfrac{1}{2}K^N_{\alpha\beta\gamma\delta}\theta_\gamma\wedge\theta_\delta + \theta_{\beta\gamma}\wedge\theta_{\gamma\alpha}\big)\\
&= (-a_{\alpha j,k}\omega_k - a_{\alpha k}\,{}^b\omega_{jk} + a_{\beta j}\phi^*\theta_{\alpha\beta})\wedge\omega_{ij}\\
&\quad - a_{\alpha j}\big(\tfrac{1}{2}\,{}^bR^M_{ijkl}\omega_k\wedge\omega_l + {}^bP_{ijk\mu}\omega_k\wedge\omega_{m\mu} + {}^b\omega_{ik}\wedge{}^b\omega_{kj}\big)\\
&\quad + (a_{\beta i,k}\omega_k + a_{\beta k}\,{}^b\omega_{ik} - a_{\gamma i}\phi^*\theta_{\beta\gamma})\wedge\phi^*\theta_{\alpha\beta}\\
&\quad + a_{\beta i}\phi^*\big(\tfrac{1}{2}a_{\gamma k}a_{\delta l}K^N_{\alpha\beta\gamma\delta}\omega_k\wedge\omega_l + \theta_{\beta\gamma}\wedge\theta_{\gamma\alpha}\big).
\end{aligned} \tag{2.8}$$

Define the covariant derivative of $a_{\alpha i,j}$ by

$$a_{\alpha i,j,k}\omega_k + a_{\alpha i,j;\mu}\omega_{m\mu} = da_{\alpha i,j} - a_{\alpha k,j}\,{}^b\omega_{ik} - a_{\alpha i,k}\,{}^b\omega_{jk} + a_{\beta i,j}\phi^*\theta_{\alpha\beta}. \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$\begin{aligned}
&a_{\alpha i,j,k}\omega_j\wedge\omega_k - a_{\alpha i,j;\mu}\omega_j\wedge\omega_{m\mu}\\
&= \tfrac{1}{2}a_{\beta i}a_{\gamma j}a_{\delta k}K^N_{\alpha\beta\gamma\delta}\omega_j\wedge\omega_k - \tfrac{1}{2}\,{}^bR^M_{ijkl}\omega_k\wedge\omega_l - a_{\alpha j}\,{}^bP_{jik\mu}\omega_k\wedge\omega_{m\mu}.
\end{aligned} \tag{2.10}$$

Thus we have the following result:

LEMMA 4. *The second fundamental form of $\phi : (M,F) \to (N,h)$ satisfies*

$$\begin{cases} a_{\alpha i,j,k} - a_{\alpha i,k,j} = -a_{\beta i}a_{\gamma j}a_{\delta k}K^N_{\alpha\beta\gamma\delta} + a_{\alpha l}\,{}^bR^M_{ilkj},\\ a_{\alpha i,j;\mu} = a_{\alpha k}\,{}^bP_{kij\mu}. \end{cases}$$

Define $\omega := [D_{e_i}e(\phi)]\omega_i = a_{\alpha j}a_{\alpha j,i}\omega_i$. Then $\omega$ is a global section of $\pi^*T^*M$. By Lemmas 1–4, we have

$$\begin{aligned}(2.11)\quad \operatorname{div}\omega &= (a_{\alpha i}a_{\alpha i,j})_{|j} + a_{\alpha i}a_{\alpha i,\mu}P_{\lambda\lambda\mu}\\ &= (a_{\alpha i}a_{\alpha i,j})_{,j} = \langle D_{e_j}d\phi, D_{e_j}d\phi\rangle + a_{\alpha i}a_{\alpha i,j,j}\\ &= \langle D_{e_j}d\phi, D_{e_j}d\phi\rangle + a_{\alpha i}a_{\alpha j,i,j}\\ &= \langle D_{e_j}d\phi, D_{e_j}d\phi\rangle + a_{\alpha i}a_{\alpha j,j,i} - a_{\alpha i}a_{\beta j}a_{\gamma i}a_{\delta j}K^N_{\alpha\beta\gamma\delta} + a_{\alpha i}a_{\alpha l}{}^bR^M_{jlji}.\end{aligned}$$

From $\tau(\phi) = 0$, we have $a_{\alpha i|i} + a_{\alpha\mu}P_{\lambda\lambda\mu} = 0$ for all $\alpha$, thus by (2.9) and Lemma 3 we obtain

$$(2.12)\qquad a_{\alpha j,j,i} = 0.$$

Now (2.2), (2.11) and (2.12) yield

PROPOSITION 2 (Bochner's formula). *Let $\phi : (M,F) \to (N,h)$ be a harmonic map from a Finsler manifold to any Riemannian manifold. Then*

$$\begin{aligned}\operatorname{div}[D_{e_i}e(\phi)]\omega_i &= \operatorname{div}\omega\\ &= \langle D_{e_i}d\phi, D_{e_i}d\phi\rangle - \langle R^N(d\phi e_i, d\phi e_j)d\phi e_i, d\phi e_j\rangle + \langle d\phi(\mathrm{Ric}^M(e_l)), d\phi e_i\rangle,\end{aligned}$$

*where* $\mathrm{Ric}^M(X) = \sum_{i,j,k} R^M_{ijkj}x^ix^k/\langle X,X\rangle$ *for* $X = x_ie_i \in TM$.

PROPOSITION 3 (Maximum principle). *Let $M$ be a compact Finsler manifold and suppose $f \in C^\infty(SM)$ satisfies* $\operatorname{div}((D_{e_i}f)\omega_i) \ge 0$. *Then $f|_M$ is constant.*

*Proof.* Set $\bar f = f - f_{\min}$, where $f_{\min} = \min_{SM} f$. Then

$$\begin{aligned}(2.13)\qquad \int_{SM}\operatorname{div}((D_{e_i}\bar f^2)\omega_i) &= \int_{SM}\operatorname{div}(2\bar f f_i\omega_i)\\ &= \int_{SM}\{(2\bar f f_i)_{|i} + 2\bar f f_\mu P_{\lambda\lambda\mu}\}\\ &= \int_{SM}\{2f_i^2 + 2\bar f f_{i|i} + 2\bar f f_\mu P_{\lambda\lambda\mu}\}\\ &= \int_{SM}\{2f_i^2 + 2\bar f\operatorname{div}((D_{e_i}f)\omega_i)\}.\end{aligned}$$

By (2.13), since $f \in C^\infty(SM)$ satisfies $\operatorname{div}((D_{e_i}f)\omega_i) \ge 0$, we have $f|_M =$ const.

## 3. The properties

THEOREM 1. *Let $\phi : (M,F) \to (N,h)$ be a harmonic map from a compact Finsler manifold to a Riemannian manifold. Suppose* $\mathrm{Ric}^M \ge 0$ *and* $\mathrm{Riem}^N \le 0$. *Then*

(a) *$\phi$ is totally geodesic.*
(b) *If* $\mathrm{Ric}^M$ *is strictly positive definite at some point, then $\phi$ is constant.*

(c) *If* $\mathrm{Riem}^N < 0$, *then* $\phi$ *is either constant or of rank one*, *in which case its image is a closed geodesic.*

Remark 1. When $(M,F)$ is a Riemannian manifold, this theorem becomes the theorem of J. Eells and J. H. Sampson [3].

*Proof.* Integrating the formula of Proposition 2, we get

$$\int_{SM} \langle D_{e_i} d\phi, D_{e_i} d\phi\rangle = \int_{SM} \{\langle R^N(d\phi e_j, d\phi e_j) d\phi e_i, d\phi e_j\rangle - \langle d\phi(\mathrm{Ric}^M(e_i)), d\phi e_i\rangle\}. \tag{3.1}$$

The left-hand side of (3.1) is nonnegative and the right-hand side of (3.1) is nonpositive, so that $D_{e_i} d\phi = 0$ for all $i$ and $\phi$ is totally geodesic.

If $\mathrm{Ric}^M > 0$ at a point $x \in M$, then by (3.1) we get $e(\phi) = 0$ at $x$. On the other hand, by the formula of Proposition 2, $\mathrm{div}(D_{e_i} e(\phi)\omega_i) = \mathrm{div}\,\omega \ge 0$, and Proposition 3 shows that $e(\phi)|_M$ is constant, hence $e(\phi)|_M \equiv 0$ and $\phi$ is constant by Lemma 2.

If $\mathrm{Riem}^N < 0$, then $\langle R^N(d\phi e_i, d\phi e_j) d\phi e_i, d\phi e_j\rangle = 0$ implies that the rank of $\phi$ is zero or one. In the first case, $\phi$ is constant; and in the second case, the fact that $\phi$ is totally geodesic implies that the image of $\phi$ is a closed geodesic.

Theorem 2. *Let* $\phi : (M,F) \to (N,h)$ *be a harmonic map from a compact Finsler manifold to a Riemannian manifold. Suppose* $\mathrm{Ric}^M \ge a > 0$ *and* $\mathrm{Riem}^N \le b$ $(b > 0)$. *If*

$$\max\{\text{the rank of } \phi\} \le p \qquad (p \ge 2)$$

*and*

$$e(\phi) \le \frac{pa}{2(p-1)b},$$

*then* $\phi$ *is either constant or totally geodesic*; *in particular*, *when* $e(\phi) \le a/2b$, $\phi$ *is constant.*

Remark 2. When $(M,F)$ is a Riemannian manifold, this theorem becomes the theorem of H. C. J. Sealey [5].

*Proof.* Fix a point $x \in M$ and diagonalize $(\phi^* h)$ at the point $x$ so that $\langle d\phi e_i, d\phi e_j\rangle = \lambda_i \delta_{ij}$. Suppose the rank of $\phi$ is $q$. By the Schwarz inequality and Proposition 2, we obtain

$$\begin{aligned}\mathrm{div}\,\omega &= \langle D_{e_i} d\phi, D_{e_i} d\phi\rangle + R^M_{ij}\langle d\phi e_i, d\phi e_j\rangle \\ &\quad - (|d\phi e_i|^2 |d\phi e_j|^2 - \langle d\phi e_i, d\phi e_j\rangle\langle d\phi e_i, d\phi e_j\rangle)\mathrm{Riem}^N(d\phi e_i, d\phi e_j)\end{aligned} \tag{3.2}$$

$$\geq \langle D_{e_i} d\phi, D_{e_i} d\phi\rangle + 2ae(\phi) - b\Big[4(e(\phi))^2 - \sum_{s=1}^{q} \lambda_s^2\Big]$$

$$\geq \langle D_{e_i} d\phi, D_{e_i} d\phi\rangle + 2e(\phi)\left[a - \frac{2(p-1)}{p}\, be(\phi)\right] \geq 0.$$

By Proposition 3 and (3.2), $e(\phi)$ is constant and

$$\langle D_{e_i} d\phi, D_{e_i} d\phi\rangle \equiv 0, \tag{3.3}$$

$$e(\phi)\left[a - \frac{2(p-1)}{p} be(\phi)\right] \equiv 0. \tag{3.4}$$

If $e(\phi) \leq a/2b$, then by (3.4),we get $e(\phi) \equiv 0$.

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