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A GENERAL FRAMEWORK FOR EXTENDING MEANS TO HIGHER ORDERS

BY

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Abstract. Although there is an extensive literature on various means of two positive operators and their applications, these means do not typically readily extend to means of three and more operators. It has been an open problem to define and prove the existence of these higher order means in a general setting. In this paper we lay the foundations for such a theory by showing how higher order means can be inductively defined and established in general metric spaces, in particular, in convex metric spaces. We consider uniqueness properties and preservation properties of these extensions, properties which provide validation to our approach. As our targeted application, we consider the positive operators on a Hilbert space under the Thompson metric and apply our methods to derive higher order extensions of a variety of standard operator means such as the geometric mean, the Gauss mean, and the logarithmic mean. That the operator logarithmic mean admits extensions of all higher orders provides a positive solution to a problem of Petz and Temesi [SIAM J. Matrix Anal. Appl. 27 (2005)].

1. Introduction. Formally, a *mean of order* n, or n*-mean* for short, on a set X is a function $\mu : X^n \to X$ satisfying $\mu(x, \dots, x) = x$ for all $x \in X$. It is frequently assumed in the definition that a mean is invariant under any permutation of variables; we call these *symmetric means*. Typically, a mean represents some type of averaging operator. The subject of means dates back into antiquity. The Greeks, motivated by their interest in proportions, defined up to eleven different means, the arithmetic, geometric, and harmonic being the best known.

In the twentieth century interest emerged in the theory of topological means, that is, symmetric means on topological spaces for which the mean operation is continuous. This work was pioneered by G. Aumann [2], who showed among other things that no sphere admits such a mean [3]. The problem of characterizing those spaces, particularly metric continua, that admit the structure of a topological mean has attracted considerable attention up to the present day. Although it has not been completely solved, numerous partial results exist, e.g., the result of Eckmann, Ganea, and Hilton [14] that a compact, connected polyhedron admitting the structure of a topological

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mean is contractible. B. Eckmann [13] had earlier introduced the important approach of using tools from algebraic topology to attack the problem. We do not pursue this particular topic further, since our interests lie elsewhere, but refer the reader to J. Charatonik's overview [11] of both older and recent work, where one finds a much fuller list of references related to these matters.

Operator means are of more recent vintage, but have a substantial literature that has grown out of foundational papers on the subject such as that of Kubo and Ando [18]. The theory has found a variety of applications, including the establishment of important inequalities, some of which find application in quantum-mechanical calculations. Such applications form part of the motivation for extending these operator means to higher orders than two, but finding a general method for doing this has been elusive, and the question of how to do this has remained an open problem. Recently, however, Ando, Li, and Mathias [1] proposed an attractive method for the case of the geometric mean on the positive (semi)definite Hermitian matrices.

Our purpose in this paper is to develop a method of extending means to higher orders that appears to offer a viable general approach. For the definition and method of extending, we favor a general version of the recent approach of Horwitz [16] for the case of means on the positive reals and of Ando, Li, and Mathias [1] for the case of the geometric mean on the positive (semi)definite Hermitian matrices. This approach has also been adopted and generalized beyond the case of the matrix geometric mean by Petz and Temesi in [24], [23], although in the general setting they only obtain existence of the higher order means for ordered tuples. Bhatia and Holbrook [6] have proposed an alternative generalization for the geometric mean via a geometric approach linked to work of É. Cartan (see [7] for a general discussion of the problem and their approach).

In this paper we show that the basic approach of Horwitz, of Ando, Li and Mathias, and of Petz and Temesi can be generalized to means on metric spaces and develop the theory of extensions in this context. The resulting theory is attractive for the generality of its results and for the resulting uniqueness and preservation properties of the extensions. The main theorems give rather general conditions that guarantee that extensions of all orders exist. Our main applications involve higher order means of positive operators on a Hilbert space. We are particularly interested in those cases in which one starts with a mean of two variables and inductively extends it to all dimensions greater than two.

In Section 2 we present our approach to extending means via limits of the "barycentric operator." (This method is called "symmetrization" in [24].) The extended means are invariant with respect to this operator and are typically characterized by this invariance. Section 3 contains a major result of the paper: means that are nonexpansive and coordinatewise contractive admit extensions to all higher orders.

In Section 4 a special case of the nonexpansive, coordinatewise contractive means is considered, namely convex means for which the mean is a metrically convex function assigning to any two points a metric midpoint. As explained in Section 5, Hadamard spaces (for which the metric satisfies the semiparallelogram law) form an important class of examples of metric spaces with associated convex mean. Our results yield higher order means that compute the "barycenter" for any finite set of points. These barycenters provide yet another intesting notion of a mean in Hadamard spaces, in addition to the previously defined "circumcenter" and the generalization of the geometric mean given by Bhatia and Holbrook [6].

In Section 6 we develop machinery for showing that certain types of iterated means are nonexpansive and coordinatewise contractive. Since many important means (e.g., the arithmetic-geometric mean) arise in this fashion, this is a useful and important result and adds great generality to our approach.

Section 7 presents categorical aspects of means and their extensions. In Section 8 a reverse construction is considered: given a mean, when is it an extension of a lower order mean? Certain uniqueness results flow out of these considerations. Connections between means, their extensions, and order are developed in Section 9. In particular, it is shown that the important property of order preservation carries over to the extended means.

The paper closes in Sections 10 and 11 with the study of means on the space of positive operators on a Hilbert space. It is shown, by applying our earlier results to the space of positive operators endowed with the Thompson metric, that certain iterated means are nonexpansive and coordinatewise contractive, hence extend to higher orders. This is true, for example, of the arithmetic-geometric and logarithmic means, as we show in Section 11, and hence they have higher order extensions. This conclusion for the logarithmic mean provides a positive solution of a problem of Petz and Temesi in [24].

2. Mean extensions. An *n-mean* on a set X is an n-ary operation (function) $\mu : X^n \to X$ that satisfies a generalized idempotency law: $\mu(x, \ldots, x) = x$ for all $x \in X$. A *mean* is just an n-mean for some $n \geq 2$. The mean is *symmetric* if it is invariant under permutations:

$$\mu(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \mu(x_1, \ldots, x_n) \quad \text{for any permutation } \pi \text{ on } \{1, \ldots, n\}.$$

(Note that for 2-means this means that the binary operation given by the mean is commutative.) A *topological n-mean* is a continuous n-mean μ on a Hausdorff topological space.

A principal goal of our work is to "extend" a (symmetric) n-mean on X to a (symmetric) $(n+1)$-mean. As mentioned in the introduction we pattern our approach after [16] and [1], an approach that has some relation to the process of compounding three means in three variables to obtain another such mean (see [8]).

DEFINITION 2.1. Given a set X and a k-mean $\mu : X^k \to X$, the *barycentric operator* $\beta = \beta_\mu : X^{k+1} \to X^{k+1}$ is defined by

$$\beta(\mathbf{x}) := (\mu(\pi_{\neq 1}\mathbf{x}), \dots, \mu(\pi_{\neq k+1}\mathbf{x})),$$

where $\mathbf{x} = (x_1, \dots, x_{k+1}) \in X^{k+1}$ and $\pi_{\neq j}\mathbf{x} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k+1}) \in X^k$. For a topological k-mean, we say that the barycentric map β is *power convergent* if for each $\mathbf{x} \in X^{k+1}$, we have $\lim_n \beta^n(\mathbf{x}) = (x^*, \dots, x^*)$ for some $x^* \in X$.

As a motivating geometric example for the terminology consider the 3-mean in $\mathbb{R}^3$ that assigns to any three points the centroid of the triangle for which they are the vertices, i.e., the point where the three medians meet. If we take now the four vertices of a 3-simplex or tetrahedron in $\mathbb{R}^3$, the barycentric operator applied to the 4-tuple consisting of the four vertices replaces each vertex with the centroid (barycenter) of the face opposite it, the face with vertices the remaining three vertices. Thus one may envision the tetrahedron with vertices the four centroids of the four faces as the result. Repeating this process, one obtains a shrinking family of tetrahedra whose intersection is the barycenter of the original tetrahedron, represented by the 4-tuple with all entries equal to that point.

REMARK 2.2. There is an alternative way that the barycentric map of a k-mean may be defined, namely instead of defining the ith coordinate of $\beta(\mathbf{x})$ by deleting the ith coordinate of $\mathbf{x} \in X^{k+1}$ and applying μ, we delete the coordinate $i^* := k + 2 - i$ and then apply μ. This means that we begin (from left to right) by first deleting coordinate $k + 1$, k, down to 1, instead of beginning by deleting coordinate 1 and continuing up to $k+1$. We denote this alternative barycentric map by β^*. One may define $\beta^*(\mathbf{x})$ alternatively by reversing the $(k + 1)$-tuple $\beta(\mathbf{x})$. Note that, as long as the mean μ is symmetric, both methods power converge to the same limit, provided one of them power converges. This equality of limits does not hold in general for nonsymmetric means however. The theories for β and β^* run parallel, so we restrict our attention to β with a few brief remarks concerning β^*.

We define our first notion of an extension in terms of the barycentric operator.

Definition 2.3. A mean $\nu : X^{k+1} \to X$ is a *β-invariant extension* of $\mu : X^k \to X$ if $\nu \circ \beta_\mu = \nu$, that is,

$$\nu(\mathbf{x}) = \nu(\mu(\pi_{\neq 1}\mathbf{x}), \dots, \mu(\pi_{\neq k+1}\mathbf{x})) \quad \text{for all } \mathbf{x} = (x_1, \dots, x_{k+1}) \in X^{k+1}. \tag{2.1}$$

The notion of a β-invariant extension was introduced by Horwitz [16], who called it type I invariance.

Proposition 2.4. *Assume that $\mu : X^k \to X$ is a topological k-mean and that the corresponding barycentric operator β is power convergent. Define $\widetilde{\mu} : X^{k+1} \to X$ by $\widetilde{\mu}(\mathbf{x}) = x^*$ where $\lim_n \beta^n(\mathbf{x}) = (x^*, \dots, x^*)$.*

(i) *$\widetilde{\mu} : X^{k+1} \to X$ is a $(k+1)$-mean on X that is a β-invariant extension of μ.*
(ii) *Any continuous mean on X^{k+1} that is a β-invariant extension of μ must equal $\widetilde{\mu}$.*
(iii) *If μ is symmetric, so is $\widetilde{\mu}$.*

Proof. (i) For $x \in X$, $\mathbf{x} = (x, \dots, x) \in X^{k+1}$, we have $\beta(\mathbf{x}) = \mathbf{x}$ by the idempotency of μ. Thus $\mathbf{x} = \lim_n \beta^n(\mathbf{x})$ and hence $\widetilde{\mu}(\mathbf{x}) = x$, i.e., $\widetilde{\mu}$ is a mean. Further, we have

$$\widetilde{\mu}(\beta(\mathbf{x})) = \pi_1(\lim_n \beta^n(\beta(\mathbf{x}))) = \pi_1(\lim_n \beta^{n+1}(\mathbf{x})) = \widetilde{\mu}(\mathbf{x}),$$

where π_1 is projection onto the first coordinate. Thus $\widetilde{\mu} \circ \beta = \widetilde{\mu}$, i.e., $\widetilde{\mu}$ defines a β-invariant extension of μ.

(ii) Suppose that ν is a continuous $(k+1)$-mean of X that is a β-invariant extension of μ. Since $\nu = \nu \circ \beta = \nu \circ \beta^n$ (by repeated application of the first equality), for $\mathbf{x} \in X^{k+1}$,

$$\nu(\mathbf{x}) = \nu(\beta(\mathbf{x})) = \nu(\beta^n(\mathbf{x})) = \nu(x^*, \dots, x^*) = x^* = \widetilde{\mu}(\mathbf{x}),$$

where $(x^*, \dots, x^*) = \lim_{n\to\infty} \beta^n(\mathbf{x})$.

(iii) If μ is symmetric, then β commutes with any permutation applied to the entries of $\mathbf{x} \in X^{k+1}$, hence also β^n, and thus one obtains the same limit with constant entry $\widetilde{\mu}(x)$ in either case. Hence $\widetilde{\mu} : X^{k+1} \to X$ is symmetric. ∎

Remark 2.5. If β^* is power convergent, then a β^*-invariant extension is defined in the manner of the previous proposition and the analogous proposition holds for β^*. However, β-invariant extensions need not be β^*-invariant and vice versa. But both notions collapse to the same one for symmetric means.

We seek a notion of mean extension that both allows one to deduce readily that a large number of properties transfer from a mean to its extension, and also is applicable to a wide variety of means. The preceding proposition provides the ingredients for this definition.

DEFINITION 2.6. A $(k+1)$-mean ν is a *β-extension* of a topological k-mean μ (or *β-extends* μ) if for each $\mathbf{x} \in X^{k+1}$, $\lim_n \beta^n(\mathbf{x}) = (\nu(\mathbf{x}), \ldots, \nu(\mathbf{x}))$. In this case we say that *β power converges* to ν, written $\beta_\mu^n \to \nu$.

We restate parts of Proposition 2.4 in terms of this definition.

COROLLARY 2.7. *If β_μ power converges, where μ is a topological mean, then it converges to a $(k+1)$-mean $\widetilde{\mu}$, which (by definition) is a β-extension of μ. Furthermore, $\widetilde{\mu}$, if continuous, is the unique β-invariant extension of μ.*

REMARK 2.8. A. Horwitz [16] and later D. Petz and R. Temesi [24] consider means on the positive reals and show that any continuous symmetric 2-mean that is strict ($\min(a,b) < \mu(a,b) < \max(a,b)$ for $a \neq b$) and order-preserving in each variable has a power convergent barycentric map, and hence has a unique β-extension to a 3-mean. Petz and Temesi point out that the argument for power convergence extends to higher order variables, and thus one can inductively define β-extensions for all $n > 2$ [24, Section 5]. For the arithmetic, geometric, and harmonic means the extensions yield the usual corresponding means of n-variables. To check this, one only has to note that they are continuous and are β-invariant extensions, then apply the previous corollary.

3. Power convergence. In this section we consider properties preserved by β-extensions and develop sufficient conditions for a topological mean to (recursively) admit a β-extension.

Given X equipped with a k-mean μ, a subset C is *convex* if $\mu(x_1, \ldots, x_k) \in C$ whenever $x_1, \ldots, x_k \in C$.

LEMMA 3.1. *If a topological mean admits a β-extension, then any closed set that is convex with respect to the mean is convex with respect to the extension.*

Proof. Let $\mu : X^k \to X$ be the given mean, and let $x_1, \ldots, x_{k+1} \in A$, a closed μ-convex set. Set $\mathbf{x} := (x_1, \ldots, x_{k+1})$. Then by convexity each coordinate of $\beta(\mathbf{x})$ is in A and by induction each coordinate of $\beta^n(\mathbf{x})$ is in A. Since A is closed, it follows that the coordinate limits, which are all $\widetilde{\mu}(x_1, \ldots, x_{k+1})$, belong to A, where $\widetilde{\mu}$ is the β-extension. ■

Recall that the *convex hull* of a set A is the smallest convex subset containing A, and can be obtained by intersecting all convex sets containing A. In a similar fashion in the case of a topological mean μ the *closed convex hull* can be obtained by intersecting all closed convex sets containing A or, as follows from the continuity of μ, by closing up the convex hull.

Definition 3.2. A topological mean is *locally convex* if there exists at each point a basis of (not necessarily open) neighborhoods that are convex. A metric topological mean is *uniformly locally convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that the diameter of the convex hull of A is less than ε whenever the diameter of A is less than δ. A metric topological mean is *closed ball convex* if all closed balls $\overline{B}_\varepsilon(x) := \{y \in X : d(x,y) \leq \varepsilon\}$ are convex for all $x \in X$.

Remark 3.3. Since in a topological mean the closure of a set A is convex whenever A is, and the closure has the same diameter, the convex sets in the definition of locally convex and uniformly locally convex may be taken to be closed if the space is regular, which is the case in the metric setting.

Lemma 3.4. *Given a topological mean on a metric space, closed ball convexity implies uniform local convexity, which in turn implies local convexity.*

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/4$. Then any set A of diameter less than δ is contained in a closed ball of radius less than 2δ around any point of A, which in turn has diameter less than $4\delta = \varepsilon$. If X is closed ball convex, then this closed ball is convex and hence contains the closed convex hull of A. The proof that uniformly locally convex implies locally convex is straightforward. ∎

The next lemma is an immediate consequence of Lemma 3.1.

Lemma 3.5. *Let X be equipped with a topological mean with a β-extension. If X is locally convex resp. metric and uniformly locally convex resp. metric and closed ball convex with respect to the given mean, then it is with respect to the extension.*

Lemma 3.6. *If ν is a β-extension of the topological mean μ and if X is locally convex and regular, then ν is continuous.*

Proof. Let $\mu : X^k \to X$ and let $\nu : X^{k+1} \to X$ be the β-extension. Let $\mathbf{x} = (x_1, \ldots, x_{k+1}) \in X^{k+1}$, let $x^* = \nu(\mathbf{x})$, and let U be an open set containing x^*. Pick a closed convex neighborhood V of x^* such that $V \subseteq U$. Since by hypothesis the sequence $\beta^n(\mathbf{x})$ converges to the $(k+1)$-string with entries x^*, we have $\beta^n(\mathbf{x}) \in V^{k+1}$ for some n large enough. By continuity of μ and hence of β^n, there exists W open in X^{k+1} containing $\mathbf{x}$ such that $\beta^n(W) \subseteq V^{k+1}$. For any $\mathbf{y} \in W$, we have $\beta^n(\mathbf{y}) \in V^{k+1}$, and hence $\beta^m(\mathbf{y}) \in V^{k+1}$ for all $m > n$ since V is convex. Since V is closed it follows that $\nu(\mathbf{y}) \in V$. Thus ν is continuous. ∎

Definition 3.7. Let $\mu : X^k \to X$ be a k-mean on a metric space X. For $\mathbf{x} = (x_1, \ldots, x_{k+1}) \in X^{k+1}$, set $|\mathbf{x}| = \{x_1, \ldots, x_{k+1}\}$, the underlying set of the $(k+1)$-tuple, and define the *diameter* $\Delta(\mathbf{x})$ of $\mathbf{x}$ by

$$\Delta(\mathbf{x}) = \operatorname{diam}|\mathbf{x}| = \sup\{d(x_i, x_j) : 1 \leq i, j \leq k+1\}.$$

The mean μ is *weakly β-contractive* if for each $\mathbf{x} \in X^{k+1}$ we have $\lim_n \Delta(\beta^n(\mathbf{x})) = 0$. For $0 < \varrho < 1$, we say that μ is *coordinatewise ϱ-contractive* if for any $\mathbf{x}, \mathbf{y} \in X^k$ that differ only in one coordinate, say $x_j \neq y_j$,

$$d(\mu(\mathbf{x}), \mu(\mathbf{y})) \leq \varrho d(x_j, y_j)$$

LEMMA 3.8. *If $\mu : X^k \to X$ is a coordinatewise ϱ-contractive mean for $0 < \varrho < 1$, then it is weakly β-contractive.*

Proof. Assume that $\mu : X^k \to X$ is a coordinatewise ϱ-contractive k-mean. We equip X^{k+1} with the sup metric

$$d\big((x_1, \dots, x_{k+1}), (y_1, \dots, y_{k+1})\big) := \max\{d(x_j, y_j) : 1 \leq j \leq k+1\}.$$

We show by induction on n that for any $\mathbf{x} \in X^{k+1}$ and any two adjacent coordinates $(\beta^n(\mathbf{x}))_i$ and $(\beta^n(\mathbf{x}))_{i+1}$,

$$d((\beta^n(\mathbf{x}))_i, (\beta^n(\mathbf{x}))_{i+1}) \leq \varrho^n d(x_i, x_{i+1}).$$

For $n = 1$, we note by the coordinatewise ϱ-contractive property that

$$d((\beta(\mathbf{x}))_i, (\beta(\mathbf{x}))_{i+1}) = d(\mu(\pi_{\neq i}(\mathbf{x})), \mu(\pi_{\neq i+1}(\mathbf{x})) \leq \varrho d(x_i, x_{i+1}),$$

since $\pi_{\neq i}(\mathbf{x})$ and $\pi_{\neq i+1}(\mathbf{x})$ differ only in the ith coordinate, where they have the entries x_{i+1} and x_i resp. Assume the validity of the inductive hypothesis for n. Since $\beta^{n+1}(\mathbf{x}) = \beta(\beta^n(\mathbf{x}))$, we deduce from the case $n = 1$ that

$$d((\beta^{n+1}(\mathbf{x}))_i, (\beta^{n+1}(\mathbf{x}))_{i+1}) \leq \varrho \cdot d((\beta^n(\mathbf{x}))_i, (\beta^n(\mathbf{x}))_{i+1}).$$

By the inductive hypothesis, the latter is no more than $\varrho \cdot \varrho^n d(x_i, x_{i+1}) = \varrho^{n+1} d(x_i, x_{i+1})$. This completes the induction. We then conclude from the triangle inequality that since any two entries of $\beta^n(\mathbf{x})$ are at most k steps apart, we have $d(\beta^n(\mathbf{x})_i, \beta^n(\mathbf{x})_j) \leq k\varrho^n \Delta(\mathbf{x})$, and thus $\Delta(\beta^n(\mathbf{x})) \leq k\varrho^n \Delta(\mathbf{x})$. Therefore $\lim_n \Delta(\beta^n(\mathbf{x})) = 0$. ∎

Note that if β_μ is power convergent, then μ must be weakly β-contractive. The next proposition provides a converse.

PROPOSITION 3.9. *Let X be a complete metric space endowed with a weakly β-contractive k-mean μ. If X is uniformly locally convex, then β is power convergent, so that a β-extension of μ exists.*

Proof. For $\mathbf{x} \in X$, set $C_n(\mathbf{x})$ equal to the closed convex hull of $|\beta^n(\mathbf{x})|$. By hypothesis $\Delta(\beta^n(\mathbf{x})) = \operatorname{diam} |\beta^n(\mathbf{x})| \to 0$ and then by uniform local convexity $\operatorname{diam} C_n(\mathbf{x}) \to 0$. Note that since $C_n(\mathbf{x})$ is convex, it contains $|\beta^m(\mathbf{x})|$ for all $m > n$, and hence contains $C_m(\mathbf{x})$. Thus the collection $\{C_n(\mathbf{x})\}$ is a decreasing sequence of closed convex sets whose diameters converge to 0. Since X is a complete metric space the intersection consists of a single point $\{x^*\}$, and it is now easy to show that $\beta^n(\mathbf{x})$ converges to the $(k+1)$-tuple with all entries x^*. ∎

We next single out another property that will be useful for inductively β-extending a mean and give some useful equivalences.

LEMMA 3.10. *Let X be a metric space endowed with a k-mean μ. Endow X^k and X^{k+1} with the sup metric that takes the supremum of the distances between each of the corresponding coordinates. Then the following three conditions are equivalent*:

(1) *For all* $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k) \in X^k$,

$$d(\mu(\mathbf{x}), \mu(\mathbf{y})) \leq \max\{d(x_j, y_j) : 1 \leq j \leq k\}.$$

(2) *The mean* $\mu : X^k \to X$ *is Lipschitz with Lipschitz constant* 1 (*hence*, *in particular*, *is continuous*).

(3) *The map* $\beta : X^{k+1} \to X^{k+1}$ *is Lipschitz with Lipschitz constant* 1.

These conditions imply

(4) *X is closed ball convex.*

Proof. (1)⇔(2): The right-hand side of (1) is the definition of the sup metric, so the two statements are equivalent.

(2)⇒(3): In each coordinate the map β is a projection followed by μ, a composition of maps with Lipschitz constant 1, and thus has Lipschitz constant 1. Since this holds in each coordinate, it holds in the sup metric.

(3)⇒(2): Fixing some $z \in X$, we see for $\mathbf{x} \in X^k$ that $\mu(\mathbf{x}) = \pi_1(\beta(z, \mathbf{x}))$, and the right-hand side is a composition of maps of Lipschitz constant 1.

(2)⇒(4): For $\varepsilon > 0$ and $x \in X$, $y_1, \ldots, y_k \in X^k$, for $\mathbf{y} := (y_1, \ldots, y_k)$ we have

$$d(x, \mu(\mathbf{y})) \leq d(\mu(x, \ldots, x), \mu(\mathbf{y})) \leq d((x, \ldots, x), \mathbf{y}) = \max_i d(x, y_i) \leq \varepsilon$$

provided $d(x, y_i) \leq \varepsilon$ for all i. Thus X is closed ball convex. ∎

DEFINITION 3.11. A k-mean μ on a metric space X is called *nonexpansive* if it satisfies, for all $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k) \in X^k$,

$$d(\mu(\mathbf{x}), \mu(\mathbf{y})) \leq \max\{d(x_j, y_j) : 1 \leq j \leq k\}, \tag{3.2}$$

or equivalently condition (2) or (3) of the preceding lemma.

LEMMA 3.12. *If μ is a nonexpansive k-mean on a metric space X and if μ has a β-extension $\widetilde{\mu}$, then $\widetilde{\mu}$ is nonexpansive.*

Proof. Let $\pi_1 : X^{k+1} \to X$ denote projection onto the first coordinate. For $\mathbf{x} \in X^{k+1}$,

$$\widetilde{\mu}(\mathbf{x}) = \pi_1(\lim_n \beta^n(\mathbf{x})) = \lim_n (\pi_1 \circ \beta^n)(\mathbf{x}).$$

Since $\widetilde{\mu}$ is the pointwise limit of Lipschitz maps $\pi_1 \circ \beta^n$ of Lipschitz constant 1 (by Lemma 3.10), it is itself Lipschitz of constant 1. ∎

The next proposition is the principal tool that allows us to extend means inductively to higher order.

PROPOSITION 3.13. *Let X be a complete metric space equipped with a nonexpansive, coordinatewise ϱ-contractive* $(0<\varrho<1)$ *k-mean $\mu : X^k \to X$, $k \geq 2$. Then the barycentric operator β is power convergent, and hence there exists a (unique) continuous $(k+1)$-mean $\widetilde{\mu} : X^{k+1} \to X$ that β-extends μ. Furthermore, $\widetilde{\mu} : X^{k+1} \to X$ is nonexpansive and coordinatewise ϱ-contractive.*

Proof. By Lemma 3.8, μ is weakly contractive. Since μ is nonexpansive, by Lemma 3.10, X is closed ball convex, hence uniformly locally convex (Lemma 3.4), and thus β is power convergent and has a β-extension to a $(k+1)$-mean $\widetilde{\mu}$ by Proposition 3.9. By Lemma 3.12 the mean $\widetilde{\mu}$ is nonexpansive and hence continuous (Lemma 3.10(2)).

To finish we show that $\widetilde{\mu}$ is coordinatewise ϱ-contractive. Let $\mathbf{x}, \mathbf{y} \in X^{k+1}$ differ only in the jth coordinate: $x_j \neq y_j$. Then by coordinatewise ϱ-contractivity

$$d((\beta(\mathbf{x}))_i, (\beta(\mathbf{y}))_i) = d(\mu(\pi_{\neq i}(\mathbf{x})), \mu(\pi_{\neq i}(\mathbf{y}))) \leq \varrho d(x_j, y_j),$$

since $\pi_{\neq i}(\mathbf{x})$ and $\pi_{\neq i}(\mathbf{y})$ differ in at most one coordinate, and are then x_j and y_j in that coordinate. Since the inequality holds for each i, we have $d(\beta(\mathbf{x}), \beta(\mathbf{y})) \leq \varrho d(x_j, y_j)$. Since β is nonexpansive by Lemma 3.10, we conclude that $d(\beta^n(\mathbf{x}), \beta^n(\mathbf{y})) \leq \varrho d(x_j, y_j)$ for all n. Taking the limit as $n \to \infty$, we obtain $d(\widetilde{\mu}(\mathbf{x}), \widetilde{\mu}(\mathbf{y})) \leq \varrho d(x_j, y_j)$. ∎

The next theorem is the culmination of this section. It follows from a straightforward induction using the preceding proposition.

THEOREM 3.14. *Let X be a complete metric space equipped with a nonexpansive, coordinatewise ϱ-contractive* $(0 < \varrho < 1)$ *k-mean $\mu : X^k \to X$, $k \geq 2$. Then there exists a unique family of continuous means $\mu_n : X^n \to X$, one for every $n > k$, such that each is a β-extension of the previous one. Furthermore, each μ_n is nonexpansive and coordinatewise ϱ-contractive.*

EXAMPLE 3.15. Consider on $\mathbb{R}$ the mean $\mu(x,y) = sx + (1-s)y$, where $0 < s < 1$. Set $\varrho = \max\{s, 1-s\}$. Then it is an elementary calculation to verify that μ is coordinatewise ϱ-contractive and nonexpansive. Hence μ inductively β-extends to an n-mean for all $n > 2$. For example, if $m(x,y) = \frac{2}{3}x + \frac{1}{3}y$, then one varifies that $m(x,y,z) = \frac{2}{5}x + \frac{7}{20}y + \frac{1}{4}z$ is a β-invariant extension of m, and hence must be its three-variable β-extension. If one uses the alternative barycentric operator β^*, then one obtains the extension $m_3(x,y,z) = \frac{4}{7}x + \frac{2}{7}y + \frac{1}{7}z$.

4. Convex means. In general, a metric space may have none, one, or many midpoints between two given points in the space. (Recall that m is a *midpoint* of a and b if $d(m,a) = d(m,b) = \frac{1}{2}d(a,b)$.) We wish to consider the setting where possibly many midpoints may exist, but there is a distinguished midpoint, and these distinguished midpoints appear in a "convex" manner.

DEFINITION 4.1. A symmetric mean $\mu : X \times X \to X$, written $\mu(x,y) = x\#y$, on a complete metric space X is called a *convex mean* if it satisfies the *basic convexity condition*

$$d(x\#z, y\#z) \le \frac{1}{2}\, d(x,y) \quad \text{for all } x,y,z \in X. \tag{4.3}$$

LEMMA 4.2. *For a convex mean, $x\#y$ is a metric midpoint for all x,y.*

Proof. By the basic convexity condition, $d(x\#y, y = y\#y) \le \frac{1}{2}d(x,y)$ and similarly $d(x\#y, x) \le \frac{1}{2}d(x,y)$. Thus

$$d(x,y) \le d(x, x\#y) + d(x\#y, y) \le \frac{1}{2}\, d(x,y) + \frac{1}{2}\, d(x,y) = d(x,y).$$

It follows that each inequality is an equality, so $d(x, x\#y) + d(x\#y, y) = d(x,y)$. Hence adding together the two inequalities in the first line of the proof gives an equality, so each inequality is an equality. ∎

Note that in the case when there is only one metric midpoint between two points x,y, it must be the case that $x\#y$ is that midpoint.

The next proposition gives a useful equivalence for convexity.

PROPOSITION 4.3. *Let (X,d) be a complete metric space equipped with a symmetric mean μ. Then μ is a convex mean if and only if*

$$d(x\#y, u\#v) \le \frac{1}{2}\, d(x,u) + \frac{1}{2}\, d(y,v) \quad \textit{for all } x,y,u,v \in X.$$

Proof. For a convex mean, $d(x\#y, u\#y) \le \frac{1}{2}d(x,u)$ and $d(u\#y, u\#v) \le \frac{1}{2}d(y,v)$. The condition of the theorem now follows by adding these inequaties and an application of the triangle inequality.

The reverse implication follows by applying the inequality of the theorem to $d(x\#z, y\#z)$. ∎

PROPOSITION 4.4. *A convex mean inductively β-extends to a symmetric, nonexpansive, coordinatewise $\frac{1}{2}$-contractive n-mean for every $n > 2$.*

Proof. Note that the definition of a convex mean is that of a symmetric coordinatewise $\frac{1}{2}$-contractive 2-mean. Proposition 4.3 further implies that it is nonexpansive, since

$$d(x\#y, u\#v) \le \frac{1}{2}\, d(x,u) + \frac{1}{2}\, d(y,v) \le \max\{d(x,u), d(y,v)\}.$$

Thus by Theorem 3.14 we obtain inductively a β-extension for every n that is nonexpansive and coordinatewise $\frac{1}{2}$-contractive. By Proposition 2.4 each extension is symmetric. ■

EXAMPLE 4.5. Let X be a Banach space (or a closed convex subset thereof) and define the symmetric 2-mean $\mu(x,y) = \frac{1}{2}(x+y)$. This is the midpoint with respect to the norm metric, and is easily seen to be a convex mean. Setting $\mu_k(x_1,\ldots,x_k) = \frac{1}{k}\sum_{i=1}^{k} x_i$, one verifies directly that μ_{k+1} is the β-extension of μ_k, so μ inductively β-extends to the standard arithmetic mean μ_n for all n.

5. Hadamard spaces. A metric space X is said to satisfy the *semi-parallelogram law* if for any two points $x_1, x_2, \in X$, there exists $z \in X$ that satisfies, for all $x \in X$,

$$d(x_1,x_2)^2 + 4d(x,z)^2 \le 2d(x,x_1)^2 + 2d(x,x_2)^2.$$

It follows readily that z is the unique midpoint between x_1 and x_2. An *Hadamard space* (occasionally called a *Bruhat–Tits space*) is a complete metric space that satisfies the semiparallelogram law.

Using a metric notion for an upper bound of curvature (geodesic triangles in the metric space satisfy certain inequalities when compared with test triangles), one calls a metric space a CAT(κ)-*space* if it is a geodesic space (each pair of points can be connected by a metric geodesic) satisfying the curvature bound condition for the real number κ (see [4, Chapter I] or [9, Section II.1]). The CAT(0)-spaces are the *non-positively curved* spaces. A metric space has an alternative characterization as an Hadamard space: it is a simply connected, complete, geodesic CAT(0)-space (see [4, Proposition 5.1, Chapter I] or [9, Exercise 1.9, Section II.1]).

What is important for our current purposes is the following remark:

REMARK 5.1. Let X be an Hadamard metric space and define a 2-mean by $\mu(x,y)$ being the unique midpoint between x and y. This defines a convex mean in the sense of the preceding section; see [4, Proposition 5.4, Chapter I] or [9, Proposition 2.2, Section II.2]. Hence by the preceding section this mean may be β-extended to an n-mean for every $n > 2$. This result provides a new and interesting operation for Hadamard spaces, namely assigning a "barycenter" to any finite set, extending the operation of taking the midpoint for any two-element set.

A wide variety of Hadamard spaces and constructions for new Hadamard spaces from old appear in [4] and [9]. Some examples include Hadamard manifolds (simply connected complete Riemannian manifolds with nonpositive sectional curvature), particularly simply connected symmetric spaces of noncompact type, finite-dimensional hyperbolic geometries over the reals, complexes, and quaternions, symmetric cones, Tits buildings, and various examples obtained by coning and gluing.

Of particular interest to us is the example of the manifold of positive-definite matrices endowed with the usual Riemannian metric called the trace metric. This metric yields an Hadamard manifold and the midpoint mean operation in this case is precisely the geometric mean of the two positive definite matrices; see [19] and the references there. Using the fact that the length metric satisfies the semiparallelogram law, hence is a convex metric with the midpoint operation being a convex mean, we obtain the following alternative derivation of the principal result of [1]:

COROLLARY 5.2. *Let X denote the set of positive definite real or complex matrices equipped with the Riemannian trace metric. Then the midpoint operation for the corresponding length metric, which is precisely the geometric mean, defines a convex* 2*-mean, which* (*by Proposition* 4.4) *β-extends to an n-mean for each $n > 2$.*

6. Iterated means. A standard construction technique for means is iteration, the arithmetic-geometric mean being the best known example. In this section we develop machinery for showing that certain iterated means are coordinatewise ϱ-contractive and nonexpansive, hence admit β-extensions of all orders. We apply this machinery to operator means in a later section.

DEFINITION 6.1. Let λ, ν be 2-means on a complete metric space X. Starting with $\lambda_1 = \lambda$ and $\nu_1 = \nu$, we give two different induction schemes for obtaining sequences of means $\{\lambda_n\}$ and $\{\nu_n\}$:

(i) $\lambda_{n+1}(x,y) = \lambda(\lambda_n(x,y), \nu_n(x,y))$, $\nu_{n+1}(x,y) = \nu(\lambda_n(x,y), \nu_n(x,y))$;
(ii) $\nu_{n+1}(x,y) = \nu(\lambda_n(x,y), \nu_n(x,y))$, $\lambda_{n+1}(x,y) = \lambda(\lambda_n(x,y), \nu_{n+1}(x,y))$.

If there exists a 2-mean μ such that $\lim_n \lambda_n(x,y) = \mu(x,y) = \lim_n \nu_n(x,y)$ for all $x, y \in X$, then μ is called the *iterated composition* in case (i) and the *skewed iterated composition* in case (ii) of λ and ν, and denoted $\mu = \lambda * \nu$ resp. $\mu = \lambda *_{\mathrm{s}} \nu$.

We begin with a useful lemma that ensures convergence.

LEMMA 6.2. *Let $\{x_n\}, \{y_n\}$ be sequences in a complete metric space X satisfying one of the following two conditions*:

(i) *for each $k \geq 1$, x_{k+1} is a midpoint of x_k and y_k and $d(x_{k+1}, y_{k+1}) \leq d(x_{k+1}, y_k)$, or*
(ii) *for each $k \geq 1$, x_{k+1} is a midpoint of x_k and y_{k+1} and $d(x_k, y_{k+1}) \leq d(x_k, y_k)$.*

Then both sequences are Cauchy and converge to the same point.

Proof. Assume (i). For any $n \geq 1$, we have by hypothesis

$$d(x_{n+1}, y_{n+1}) \leq d(x_{n+1}, y_n) = \frac{1}{2}\, d(x_n, y_n),$$

where the last equality follows from that the fact that x_{n+1} is a midpoint of x_n, y_n. Similarly,

$$d(x_{n+1}, x_n) = \frac{1}{2}\, d(x_n, y_n).$$

It follows by induction resp. induction and the triangle inequality that

$$d(x_n, y_n) \leq \frac{1}{2^{n-1}}\, d(x_1, y_1)$$

resp.

$$d(x_{n+k}, x_n) \leq \left(\sum_{i=0}^{k-1} \frac{1}{2^{n+i}}\right) d(x_1, y_1) < \frac{1}{2^{n-1}}\, d(x_1, y_1).$$

Thus the sequence $\{x_n\}$ is Cauchy, and hence converges, and the sequence $\{y_n\}$ must also approach the same limit.

Part (ii) follows by applying part (i) to the sequences $\{x_n\}$ and $\{z_n\}$, where $z_n = y_{n+1}$. ∎

PROPOSITION 6.3. *Let λ be a convex mean and ν be a nonexpansive mean on a complete metric space X. Then both the iterated composition $\mu = \lambda * \nu$ and the skewed iterated composition $\mu = \lambda *_{\mathrm{s}} \nu$ exist and are nonexpansive.*

Proof. For $x, y \in X$, we set $x_n = \lambda_n(x, y)$ and $y_n = \nu_n(x, y)$ (see Definition 6.1). Then $x_{n+1} = \lambda(x_n, y_n)$ and $y_{n+1} = \nu(x_n, y_n)$. We observe that

$$d(x_{n+1}, y_{n+1}) = d(\nu(x_{n+1}, x_{n+1}), \nu(x_n, y_n)) \leq \max\{d(x_{n+1}, x_n), d(x_{n+1}, y_n)\},$$

where the last inequality follows from the fact that ν is nonexpansive. Since λ is a convex mean, $x_{n+1} = \lambda(x_n, y_n)$ is a midpoint for x_n and y_n, hence $d(x_{n+1}, x_n) = d(x_{n+1}, y_n)$, and thus $d(x_{n+1}, y_{n+1}) \leq d(x_{n+1}, y_n)$, i.e., condition (i) of Lemma 6.2 is satisfied. It thus follows that $\lim_n x_n = \lim_n y_n$ exists, and we define this limit to be $\mu(x, y)$. If $x = y$, then it is immediate that $x = x_n = y_n$ for all n, so μ is a mean. Thus the iterated composition $\mu = \lambda * \nu$ exists.

For the case of the skewed iterated mean, we set $x_1 = \lambda(x, y)$, $y_1 = \nu(x, y)$ and

$$y_{n+1} = \nu(x_n, y_n), \qquad x_{n+1} = \lambda(x_n, y_{n+1}).$$

Then

$$d(x_k, y_{k+1}) = d(\nu(x_k, x_k), \nu(x_k, y_k)) \leq \max\{d(x_k, x_k), d(x_k, y_k)\} = d(x_k, y_k),$$

where the inequality follows from the nonexpansive property. Thus condition (ii) of Lemma 6.2 is satisfied. That the skewed iterated composition $\mu = \lambda *_{\mathrm{s}} \nu$ exists now follows as in the preceding paragraph.

It follows from Proposition 4.3 that λ is Lipschitz with Lipschitz constant 1 and the same holds for ν since it is nonexpansive. In both cases the higher numbered means λ_n and ν_n are built up from these by products and compositions, so are also 1-Lipschitz (recall that product metrics are always the sup metric). Since μ is the pointwise limit of the sequence $\{\lambda_n\}$ (and $\{\nu_n\}$), it is also 1-Lipschitz, i.e., nonexpansive. ∎

PROPOSITION 6.4. *Suppose in a complete metric space X that λ is a convex mean and ν is coordinatewise ϱ'-contractive*, $0 < \varrho' < 1$, *and nonexpansive. Then both the iterated composition $\lambda * \nu$ and the skewed iterated composition $\lambda *_{\mathrm{s}} \nu$ exist*, *are coordinatewise ϱ-contractive*, $\varrho = \max\{1/2, \varrho'\}$, *and nonexpansive*, *and hence β-extend to all orders greater than two.*

Proof. By Proposition 6.3 the iterated composition and skewed iterated composition both exist and are nonexpansive.

We establish that $\mu = \lambda * \nu$ is coordinatewise ϱ-contractive. Let $a, b, c \in X$. To calculate $\mu(a, b)$ and $\mu(a, c)$ we define inductively

$$\begin{aligned} &b_1^- = \lambda(a,b), \quad b_1^+ = \nu(a,b), \quad b_{k+1}^- = \lambda(b_k^-, b_k^+), \quad b_{k+1}^+ = \nu(b_k^-, b_k^+),\\ &c_1^- = \lambda(a,c), \quad c_1^+ = \nu(a,c), \quad c_{k+1}^- = \lambda(c_k^-, c_k^+), \quad c_{k+1}^+ = \nu(c_k^-, c_k^+). \end{aligned}$$

Note that $b_k^- = \lambda_k(a,b)$, $b_k^+ = \nu_k(a,b)$, $c_k^- = \lambda_k(a,c)$, $c_k^+ = \nu_k(a,c)$. We have $d(b_1^-, c_1^-) = d(\lambda(a,b), \lambda(a,c)) \le \frac{1}{2} d(b,c)$ by convexity of λ, and similarly $d(b_1^+, c_1^+) \le \varrho d(b,c)$ by coordinatewise ϱ-contractivity of ν.

We claim that by induction

$$d(b_n^-, c_n^-) \le \varrho d(b,c) \quad \text{and} \quad d(b_n^+, c_n^+) \le \varrho d(b,c).$$

By the preceding paragraph this holds for $n = 1$. Assume that it is true for $n = k$. Then

$$\begin{aligned} d(b_{k+1}^-, c_{k+1}^-) &= d(\lambda(b_k^-, b_k^+), \lambda(c_k^-, c_k^+))\\ &\le d(\lambda(b_k^-, b_k^+), \lambda(c_k^-, b_k^+)) + d(\lambda(c_k^-, b_k^+), \lambda(c_k^-, c_k^+))\\ &\le \frac{1}{2}\, d(b_k^-, c_k^-) + \frac{1}{2}\, d(b_k^+, c_k^+) \le \frac{1}{2}\,(\varrho d(b,c) + \varrho d(b,c)) = \varrho d(b,c). \end{aligned}$$

Using the nonexpansivity of ν_{k+1}, we obtain

$$\begin{aligned} d(b_{k+1}^+, c_{k+1}^+) &= d(\nu(b_k^-, b_k^+), \nu(c_k^-, c_k^+)) \le \max\{d(b_k^-, c_k^-), d(b_k^+, c_k^+)\}\\ &\le \max\{\varrho d(b,c), \varrho d(b,c)\} = \varrho d(b,c). \end{aligned}$$

This completes the induction. Note that in the alternative notation we have shown that $d(\lambda_n(a,b), \lambda_n(a,c)) \le \varrho d(b,c)$ and $d(\nu_n(a,b), \nu_n(a,c)) \le \varrho d(b,c)$ for all $n \in \mathbb{N}$.

Since $\mu = \lambda * \nu$, $\lim_n b_n^- = \lim_n \lambda_n(a,b) = \mu(a,b)$, $\lim_n c_n^- = \lim_n \lambda_n(a,c)$ $= \mu(a,c)$. By continuity of $d(\cdot,\cdot)$ and by the preceding paragraph, it follows that $d(\mu(a,b), \mu(a,c))) \le \varrho d(b,c)$.

The proof that the skewed iterated composition $\mu = \lambda *_{\mathrm{s}} \nu$ is coordinatewise ϱ-contractive is similar, but contains a twist or two. To calculate $\mu(a,b)$ and $\mu(a,c)$ for $a,b,c \in X$, we define inductively

$$\begin{aligned}
b_1^+ &= \nu(a,b), \quad b_1^- = \lambda(a,b), \quad b_{k+1}^+ = \nu(b_k^-, b_k^+), \quad b_{k+1}^- = \lambda(b_k^-, b_{k+1}^+),\\
c_1^+ &= \nu(a,c), \quad c_1^- = \lambda(a,c), \quad c_{k+1}^+ = \nu(c_k^-, c_k^+), \quad c_{k+1}^- = \lambda(c_k^-, c_{k+1}^+).
\end{aligned}$$

We have $d(b_1^-, c_1^-) = d(\lambda(a,b), \lambda(a,c)) \le \frac{1}{2} d(b,c)$ by convexity of λ, and similarly $d(b_1^+, c_1^+) \le \varrho d(b,c)$ by coordinatewise ϱ-contractivity of ν.

We claim that by induction

$$d(b_n^-, c_n^-) \le \varrho d(b,c) \quad \text{and} \quad d(b_n^+, c_n^+) \le \varrho d(b,c).$$

By the preceding paragraph this holds for $n = 1$. Assume that it is true for $n = k$. Using the nonexpansivity of ν, we obtain

$$\begin{aligned}
d(b_{k+1}^+, c_{k+1}^+) &= d(\nu(b_k^-, b_k^+), \nu(c_k^-, c_k^+)) \le \max\{d(b_k^-, c_k^-), d(b_k^+, c_k^+)\}\\
&\le \max\{\varrho d(b,c), \varrho d(b,c)\} = \varrho d(b,c).
\end{aligned}$$

It then follows that

$$\begin{aligned}
d(b_{k+1}^-, c_{k+1}^-) &= d(\lambda(b_k^-, b_{k+1}^+), \lambda(c_k^-, c_{k+1}^+))\\
&\le d(\lambda(b_k^-, b_{k+1}^+), \lambda(c_k^-, b_{k+1}^+)) + d(\lambda(c_k^-, b_{k+1}^+), \lambda(c_k^-, c_{k+1}^+))\\
&\le \frac{1}{2} d(b_k^-, c_k^-) + \frac{1}{2} d(b_{k+1}^+, c_{k+1}^+)\\
&\le \frac{1}{2} (\varrho d(b,c) + \varrho d(b,c)) = \varrho d(b,c).
\end{aligned}$$

This completes the induction.

By hypothesis $\lim_n b_n^- = \lim_n \lambda_n(a,b) = \mu(a,b)$, $\lim_n c_n^- = \lim_n \lambda_n(a,c) = \mu(a,c)$. By continuity of $d(\cdot,\cdot)$ and the preceding paragraph, it follows that $d(\mu(a,b), \mu(a,c))) \le \varrho d(b,c)$.

The last assertion of the proposition now follows from Theorem 3.14. ∎

7. Categorical constructions. In this section we consider the behavior of mean extensions with respect to standard constructions such as continuous images, products, and subspaces.

Definition 7.1. A function $g : (X,\mu) \to (Y,\nu)$, where μ, ν are k-means on X and Y respectively, is called a *k-mean homomorphism*, or *homomorphism* for short, if $g \circ \mu = \nu \circ g_k$, that is, the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{g} & Y\\
\mu \big\uparrow & & \big\uparrow \nu\\
X^k & \xrightarrow{g_k} & Y^k
\end{array}$$

where $g_k : X^k \to Y^k$, $g_k(x_1, \dots, x_k) := (g(x_1), \dots, g(x_k))$.

PROPOSITION 7.2. *Let (X,μ) and (Y,ν) be topological k-means, and let $g : X \to Y$ be a continuous k-mean homomorphism.*

(i) *If each of μ and ν β-extends to $(k+1)$-means $\widetilde{\mu}$ and $\widetilde{\nu}$ resp., then $g : X \to Y$ is a $(k+1)$-mean homomorphism.*
(ii) *If g is surjective and μ β-extends to a $(k+1)$-mean $\widetilde{\mu}$, then ν β-extends to a $(k+1)$-mean $\widetilde{\nu}$, and g is then a $(k+1)$-mean homomorphism.*

Proof. (i) It follows directly from the fact that g is k-mean homomorphism that $\beta_Y g_{k+1} = g_{k+1}\beta_X : X^{k+1} \to Y^{k+1}$ (indeed, commutation of g_k with β is an equivalence). By induction

$$\beta_Y^n g_{k+1} = g_{k+1}\beta_X^n : X^{k+1} \to Y^{k+1}$$

for all $n > 0$. Taking the limit of both sides as $n \to \infty$ and projecting onto the first coordinate yields (i).

(ii) For $\mathbf{y} \in Y^{k+1}$, there exists $\mathbf{x} \in X^{k+1}$ such that $g_{k+1}(\mathbf{x}) = \mathbf{y}$. Again we have

$$\beta_Y^n(\mathbf{y}) = \beta_Y^n g_{k+1}(\mathbf{x}) = g_{k+1}\beta_X^n(\mathbf{x})$$

for $n > 0$. By hypothesis the right-hand side converges to a diagonal element with entries $g(\widetilde{\mu}(\mathbf{x}))$ as $n \to \infty$, so that the left-hand side also converges to a diagonal element. Thus ν β-extends to $\widetilde{\nu}$. The last assertion follows from (i). ∎

EXAMPLE 7.3. Let

$$\mu_f(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$$

be a *quasi-arithmetic mean* defined on $\mathbb{R}^+$ by a continuous strictly monotonic function f (see [24]). By the preceding proposition applied to $g = f^{-1}$ and by Example 4.5,

$$\mu_k(x_1,\ldots,x_k) = f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} f(x_i)\right).$$

Note that the arithmetic, geometric, and harmonic means belong to this class by taking the identity map, the logarithmic map, and the inversion map (on the positive reals) respectively. More generally, one can take on the positive reals the generalized or power mean $m(x,y) = ((x^\alpha + y^\alpha)2)^{1/\alpha}$ for $\alpha \neq 0$ with $f(x) = x^\alpha$.

An analogous construction and characterization of the higher order means remains valid for the power means on the space of positive definite matrices. Note that the case $\alpha = 1$ gives the arithmetic mean and the case $\alpha = -1$ gives the harmonic mean.

Definition 7.4. Let μ and ν be k-means on X and Y, respectively. Define $\mu\times\nu : (X\times Y)^k \to X\times Y$ by

$$(\mu\times\nu)(x_1,y_1,x_2,y_2,\ldots,x_k,y_k) = (\mu(x_1,x_2,\ldots,x_k),\nu(y_1,y_2,\ldots,y_k)).$$

Then $\mu\times\nu$ is a k-mean on $X\times Y$, called the *product mean* of μ and ν. Indeed, if $\mathbf{z}=(x,y,x,y,\ldots,x,y)\in(X\times Y)^k$ for fixed $(x,y)\in X\times Y$, then $(\mu\times\nu)(\mathbf{z}) = (\mu(x,x,\ldots,x),\nu(y,y,\ldots,y)) = (x,y)$.

Theorem 7.5. *Let (X,d_1) and (Y,d_2) be complete metric spaces equipped with nonexpansive, coordinatewise ϱ-contractive resp. ϱ'-contractive $(0<\varrho,\varrho'<1)$ k-means $\mu: X^k\to X$ resp. $\nu: Y^k\to Y$. Then the mean $\mu\times\nu$ is a nonexpansive, coordinatewise* $\max\{\varrho,\varrho'\}$*-contractive k-mean on $X\times Y$. Furthermore, for $n\ge k$, its n-mean β-extension $(\mu\times\nu)_n$ coincides with the product mean $\mu_n\times\nu_n$ of the individual β-extension n-means*:

$$(\mu\times\nu)_n = \mu_n\times\nu_n.$$

Proof. For $\mathbf{z}=(x_1,y_1,\ldots,x_k,y_k)\in(X\times Y)^k$, we define $\mathbf{z}_x=(x_1,\ldots,x_k)\in X^k$, $\mathbf{z}_y=(y_1,\ldots,y_k)\in Y^k$. Then $(\mu\times\nu)(\mathbf{z}) = (\mu(\mathbf{z}_x),\nu(\mathbf{z}_y))$.

Let $\mathbf{z}=(x_{11},y_{11},x_{12},y_{12},\ldots,x_{1k},y_{1k})$, $\mathbf{w}=(x_{21},y_{21},x_{22},y_{22},\ldots,x_{2k},y_{2k})\in(X\times Y)^k$. Then by nonexpansive property of μ and ν,

$$\begin{aligned}
d((\mu\times\nu)(\mathbf{z}),(\mu\times\nu)(\mathbf{w})) &= d((\mu(\mathbf{z}_x),\nu(\mathbf{z}_y)),(\mu(\mathbf{w}_x),\nu(\mathbf{w}_y)))\\
&= \max\{d_1(\mu(\mathbf{z}_x),\mu(\mathbf{w}_x)),d_2(\nu(\mathbf{z}_y),\nu(\mathbf{w}_y))\}\\
&\le \max\{\max\{d_1(x_{1j},x_{2j})\},\max\{d_2(y_{1j},y_{2j})\} :\\
&\qquad\qquad 1\le j\le k\}\\
&= \max\{d_1(x_{1j},x_{2j}),d_2(y_{1j},y_{2j}) : 1\le j\le k\}\\
&= \max\{d((x_{1j},y_{1j}),(x_{2j},y_{2j})) : 1\le j\le k\},
\end{aligned}$$

which implies that $\mu\times\nu$ is a nonexpansive k-mean on $X\times Y$ equipped with the supmetric.

If $\mathbf{z}$ and $\mathbf{w}$ differ only in one coordinate of $(X\times Y)^k$, say $(x_{1j},y_{1j})\ne(x_{2j},y_{2j})$, but $(x_{1i},y_{1i})=(x_{2i},y_{2i})$, $1\le i\ne j\le k$, then the inequality in the preceding argument turns into

$$\begin{aligned}
d((\mu\times\nu)(\mathbf{z}),(\mu\times\nu)(\mathbf{w})) &= d((\mu(\mathbf{z}_x),\nu(\mathbf{z}_y)),(\mu(\mathbf{w}_x),\nu(\mathbf{w}_y)))\\
&= \max\{d_1(\mu(\mathbf{z}_x),\mu(\mathbf{w}_x)),d_2(\nu(\mathbf{z}_y),\nu(\mathbf{w}_y))\}\\
&\le \max\{\varrho d_1(x_{1j},x_{2j}),\varrho' d_2(y_{1j},y_{2j})\}\\
&\le \max\{\varrho,\varrho'\}\max\{d_1(x_{1j},x_{2j}),d_2(y_{1j},y_{2j})\}\\
&\le \max\{\varrho,\varrho'\}d((x_{1j},y_{1j}),(x_{2j},y_{2j})).
\end{aligned}$$

Therefore $\mu\times\nu$ is a coordinatewise $\max\{\varrho,\varrho'\}$-contractive k-mean on $X\times Y$.

Next, we will prove $(\mu\times\nu)_n(\mathbf{z}) = (\mu_n(\mathbf{z}_x),\,\nu(\mathbf{z}_y))$, $\mathbf{z}\in(X\times Y)^n$, $n\ge k$, by induction. The case $n=k$ follows by the definition of $\mu\times\nu$. Suppose that the assertion holds true for $n-1$. Let $\gamma: (X\times Y)^n\to X\times Y$ be defined by

$\gamma(\mathbf{z}) = (\mu_n(\mathbf{z}_x), \nu_n(\mathbf{z}_y))$. Then γ is continuous and hence it suffices to show $\gamma \circ \beta_n = \gamma$ by the uniqueness of mean extension (Proposition 2.4), where β_n is the barycentric operator on $(X \times Y)^n$ obtained from the $(n-1)$-mean $(\mu \times \nu)_{n-1}$.

For $\mathbf{z} = (z_1, \ldots, z_n) \in (X \times Y)^n$, $z_i = (x_i, y_i) \in X \times Y$, we have from induction

$$(\mu \times \nu)_{n-1}(\pi_{\neq i}\mathbf{z}) = (\mu_{n-1}(\pi_{\neq i}\mathbf{z}_x), \nu_{n-1}(\pi_{\neq i}\mathbf{z}_y)), \qquad 1 \leq i \leq n,$$

and then from

$$\beta_n(\mathbf{z}) = ((\mu \times \nu)_{n-1}(\pi_{\neq 1}\mathbf{z}), (\mu \times \nu)_{n-1}(\pi_{\neq 2}\mathbf{z}), \ldots, (\mu \times \nu)_{n-1}(\pi_{\neq n}\mathbf{z}))$$

we get

$$\beta_n(\mathbf{z})_x = (\mu_{n-1}(\pi_{\neq 1}\mathbf{z}_x), \mu_{n-1}(\pi_{\neq 2}\mathbf{z}_x), \ldots, \mu_{n-1}(\pi_{\neq n}\mathbf{z}_x)) = \beta_\mu(\mathbf{z}_x),$$
$$\beta_n(\mathbf{z})_y = (\nu_{n-1}(\pi_{\neq 1}\mathbf{z}_y), \nu_{n-1}(\pi_{\neq 2}\mathbf{z}_y), \ldots, \nu_{n-1}(\pi_{\neq n}\mathbf{z}_y)) = \beta_\nu(\mathbf{z}_y),$$

where β_μ, β_ν denote the barycentric operators of the n-means μ_n, ν_n; moreover, $\mu_n \circ \beta_\mu = \mu_n$ and $\nu_n \circ \beta_\nu = \nu_n$. Now,

$$\begin{aligned}(\gamma \circ \beta_n)(\mathbf{z}) &= \gamma(\beta_n(\mathbf{z})) = (\mu_n(\beta_n(\mathbf{z})_x), \nu_n(\beta_n(\mathbf{z})_y)) \\ &= (\mu_n(\beta_\mu(\mathbf{z}_x)), \nu_n(\beta_\nu(\mathbf{z}_y))) = (\mu_n(\mathbf{z}_x), \nu_n(\mathbf{z}_y)) = \gamma(\mathbf{z}),\end{aligned}$$

which completes the proof. ∎

REMARK 7.6. The product mean satisfies the associative law

$$\mu \times (\nu \times \omega) = (\mu \times \nu) \times \omega$$

for any k-means μ, ν, and ω on X, Y, Z. If these are nonexpansive and coordinatewise contractive, then their n-mean extension satisfies

$$(\mu \times \nu \times \omega)_n = \mu_n \times \nu_n \times \omega_n.$$

The next result of the section is quite straightforward and hence the proof is omitted.

PROPOSITION 7.7. *If Z is a nonempty closed k-submean of a topological k-mean (X, μ) that β-extends, then Z also β-extends and $\widetilde{\mu}|_{Z^{k+1}} = \widetilde{\mu|_{Z^k}}$.*

COROLLARY 7.8. *Suppose that a k-mean (X, μ) restricts to a nonexpansive, coordinatewise ϱ_n-contractive mean on A_n, where $0 < \varrho_n < 1$ for each n and A_n is an increasing sequence of closed convex sets with $X = \bigcup_n A_n$. Then μ inductively β-extends to a j-mean for each $j > k$ in such a way that the restriction to A_n is the appropriate β-extension of the restriction of μ to A_n.*

Proof. Given any j-tuple in X^j for $j > k$, there exists some A_m that contains all entries of the tuple. Applying Theorem 3.14 to the restriction of μ to A_m, we conclude that $\mu|_{A_m}$ β-extends for every index greater than k.

Thus in particular the appropriate extension exists to evaluate the given j-tuple. It is clear that if a larger A_{m+n} is chosen, one obtains the same calculation. Thus the β-extension is independent of the containing A_m, and hence we obtain a β-extension of μ on all of X. ∎

8. Stability and reductions

DEFINITION 8.1. A k-mean μ on a set X is called *β-stable* if the graph of μ is invariant under β, that is,

$$\forall \mathbf{x} \in X^{k+1}, \qquad \pi_{k+1}(\mathbf{x}) = \mu(\pi_{\neq k+1}(\mathbf{x})) \Rightarrow \pi_{k+1}(\beta(\mathbf{x})) = \mu(\pi_{\neq k+1}(\beta(\mathbf{x}))).$$

A $(k+1)$-mean ν on X is called a *stable extension* of μ if

$$\nu(x_1, \dots, x_k, \mu(x_1, \dots, x_k)) = \mu(x_1, \dots, x_k) \quad \text{for all } x_1, \dots, x_k \in X.$$

Reciprocally, μ is called a *stable reduction* of ν.

In [16] Horwitz says that ν is type 2 invariant with respect to μ if ν is a stable extension of μ. He only considers the case of a 2-mean μ and a 3-mean ν.

PROPOSITION 8.2. *If a topological k-mean μ is β-stable and admits a β-extension $\widetilde{\mu}$, then $\widetilde{\mu}$ is a stable extension of μ.*

Proof. Note that β does not change the last coordinate of any $(k+1)$-tuple $\mathbf{x} = (x_1, \dots, x_k, \mu(x_1, \dots, x_k))$ in the graph of μ. Hence if μ is β-stable, it follows that $\beta^n(\mathbf{x})$ has the same last coordinate for all n. Since β power converges to $\widetilde{\mu}$, it follows that the diagonal limiting value has entries the last coordinate of $\mathbf{x}$, namely $\mu(x_1, \dots, x_k)$. Thus $\widetilde{\mu}(\mathbf{x}) = \mu(x_1, \dots, x_k)$, so $\widetilde{\mu}$ is a stable extension of μ. ∎

PROPOSITION 8.3. *If X is a complete metric space equipped with a coordinatewise ϱ-contractive* (*symmetric*) *$(k+1)$-mean ν, $k \geq 2$, then ν admits exactly one stable reduction* (*which is also symmetric*).

Proof. Suppose that X is a complete metric space equipped with a coordinatewise ϱ-contractive $(k+1)$-mean ν. The map $g : X \to X$ defined by $g(x) = \nu(x_1, \dots, x_k, x)$ is by hypothesis ϱ-contractive, and hence has a unique fixed point. Define $\mu(x_1, \dots, x_k)$ to be this fixed point. It follows immediately that μ is a stable reduction of ν, and uniqueness of the fixed point guarantees that it is the unique reduction. Note that the fact that ν is a mean and the definition of μ imply that μ is also a mean. The symmetry of μ follows directly from that of ν. ∎

The following corollary is an immediate consequence of Propositions 3.13, 8.2, and 8.3.

COROLLARY 8.4. *Let X be a complete metric space equipped with a nonexpansive*, *coordinatewise ϱ-contractive k-mean μ. If μ is β-stable*, *then the* *unique β-extension $\widetilde{\mu}$ is a stable extension of μ, and μ is the unique stable reduction of $\widetilde{\mu}$.*

DEFINITION 8.5. A 2-mean μ on a set X satisfies the *limited medial property* if $\mu(a,b) = \mu(x,y) =: m$ implies that $\mu(\mu(a,x),\mu(b,y)) = m$. For $m \in X$,

$$X_m := \{(x,y) \in X^2 : \mu(x,y) = m\}.$$

REMARK 8.6. (1) If a 2-mean $\mu(x,y) = x\#y$ satisfies the limited medial property, then it is β-stable since

$$\beta(x,y,x\#y) = (y\#(x\#y), x\#(x\#y), x\#y).$$

The latter is in the graph of μ, since by limited mediality $x\#y = (x\#y)\#(x\#y)$ implies $(x\#(x\#y))\#(y\#(x\#y)) = x\#y$.

(2) It was shown in [1] that the matrix geometric mean for positive definite matrices satisfies the limited medial property. This was extended to very general notions of geometric mean in [22], in particular for the geometric mean of positive operators on a Hilbert space or, more generally, for the positive elements of a C^*-algebra. Hence by part (1) and the earlier results each k-extension of the geometric mean yields both the higher order ones by stable extension and the lower order ones by stable reduction.

LEMMA 8.7. *The mean μ satisfies the limited medial property if and only if $(\mu \times \mu)(X_m \times X_m) \subset X_m$ for each $m \in X$.*

Proof. Let $(a,b),(x,y) \in X_m$, that is, $\mu(a,b) = \mu(x,y) = m$. Then $(\mu(a,x),\mu(b,y)) \in X_m$ (limited medial property) if and only if

$$\mu(\mu(a,x),\mu(b,y)) = (\mu \times \mu)((a,b),(x,y)) = m. \blacksquare$$

PROPOSITION 8.8. *Let X be complete metric space equipped with nonexpansive, coordinatewise ϱ-contractive 2-mean $\mu(x,y) = x\#y$ satisfying the limited medial property. If $x_i\#y_i = m$ for all $1 \le i \le n$, then $\mu_n(\mathbf{x})\#\mu_n(\mathbf{y}) = m$, where $\mathbf{x} = (x_1,\dots,x_n), \mathbf{y} = (y_1,\dots,y_n) \in X^n$.*

Proof. By the previous lemma $(\mu \times \mu)(X_m \times X_m) \subset X_m$, and X_m is a nonempty closed subset of X^2. Let $\omega := (\mu \times \mu)|_{X_m^2}$. By Theorem 7.5, the n-mean extension $\omega_n : X_m^n \to X_m$ of ω is given by

$$\omega_n = (\mu_n \times \mu_n)|_{X_m^n}.$$

Suppose that $x_i\#y_i = m$, $i = 1,\dots,n$. Then $\mathbf{z} := (x_1,y_1,\dots,x_n,y_n) \in X_m^n$ and so $\omega_n(\mathbf{z}) = (\mu_n(\mathbf{x}),\mu_n(\mathbf{y})) \in X_m$, which implies that $\mu_n(\mathbf{x})\#\mu_n(\mathbf{y}) = m$. $\blacksquare$

REMARK 8.9. Under the assumption of the preceding proposition, if $a\#b = x\#y = m$ then $X_m \subseteq \{(z,w) \in X \times X : \mu_3(a,b,z)\#\mu_3(x,y,w) = m\}$.

9. Ordered convex metric spaces. Throughout this section we assume that X is a complete metric space equipped with a nonexpansive, coordinatewise ϱ-contractive k-mean $\mu : X^k \to X$, $k \geq 2$, and μ_n $(n > k)$ denotes the nonexpansive, coordinatewise ϱ-contractive n-mean obtained inductively. We further assume that X is equipped with a closed partial order $\leq$. We let $\leq_k$ be the product order on X^k defined by

$$(x_1, \ldots, x_k) \leq_k (y_1, \ldots, y_k) \quad \text{if and only if} \quad x_i \leq y_i,\ 1 \leq i \leq k.$$

Recall that for a map $g : X \to Y$, we let $g_k : X^k \to Y^k, (x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$.

Definition 9.1. A k-mean ν on X is said to be *monotone* for the partial order $\leq$ if $\nu(\mathbf{x}) \leq \nu(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in X^k$ with $\mathbf{x} \leq \mathbf{y}$.

Theorem 9.2.

(1) *If a nonexpansive, coordinatewise ϱ-contractive k-mean μ is monotone for the closed partial order $\leq$, then μ_n is also monotone for any $n \geq k$:*

$$\mathbf{x} \leq_n \mathbf{y} \Rightarrow \mu_n(\mathbf{x}) \leq \mu_n(\mathbf{y}).$$

(2) *Let (X, δ) be another complete metric space equipped with a nonexpansive, coordinatewise ϱ-contractive k-mean $\nu : X^k \to X$. Suppose that $\leq$ is closed in the product topology $(X, d) \times (X, \delta)$ and either μ or ν is monotone with respect to $\leq$. Then $\mu \leq \nu$ implies $\mu_n \leq \nu_n$ for any $n \geq k$.*

Proof. (1) We induct on m beginning at k. Suppose that μ_m is monotone. Let $\mathbf{x}, \mathbf{y} \in X^{m+1}$ with $\mathbf{x} \leq_{m+1} \mathbf{y}$. Then $\pi_{\neq i}\mathbf{x} \leq_m \pi_{\neq i}\mathbf{y}$ for all $1 \leq i \leq m+1$. By the induction hypothesis, $\mu_m(\pi_{\neq i}\mathbf{x}) \leq \mu_m(\pi_{\neq i}\mathbf{y})$ for each i, and hence $\beta_{m+1}(\mathbf{x}) \leq_{m+1} \beta_{m+1}(\mathbf{y})$. By repeated application, we have $\beta^n_{m+1}(\mathbf{x}) \leq_{m+1} \beta^n_{m+1}(\mathbf{y})$, $n = 1, 2, \ldots$. By the closedness of the order, $\lim_{n\to\infty} \beta^n_{n+1}(\mathbf{x}) \leq_{n+1} \lim_{n\to\infty} \beta^n_{n+1}(\mathbf{y})$. In particular, $\mu_{n+1}(\mathbf{x}) \leq \mu_{n+1}(\mathbf{y})$.

(2) Assume that $\mu_m(\mathbf{x}) \leq \nu_m(\mathbf{x})$ for all $\mathbf{x} \in X^m$ for some $m \geq k$. Let $\mathbf{x} \in X^{m+1}$. Then $\mu_m(\pi_{\neq j}\mathbf{x}) \leq \nu_m(\pi_{\neq j}\mathbf{x})$ for $1 \leq j \leq m+1$ and hence

$$\beta(\mathbf{x}) \leq_{m+1} \alpha(\mathbf{x}) \tag{9.1}$$

where β and α are the barycentric operators with respect to the means μ and ν, respectively. We will show by induction that $\beta^n(\mathbf{x}) \leq_{m+1} \alpha^n(\mathbf{x})$ for all n. Suppose that this holds true for n. Then $\pi_{\neq j}\beta^n(\mathbf{x}) \leq \pi_{\neq j}\alpha^n(\mathbf{x})$ for all $1 \leq j \leq m+1$. If ν is monotone, then ν_m is monotone by (1) and

$$\nu_m(\pi_{\neq j}\beta^n(\mathbf{x})) \leq \nu_m(\pi_{\neq j}\alpha^n(\mathbf{x})), \quad 1 \leq j \leq m+1. \tag{9.2}$$

Therefore,

$$\begin{aligned}\beta^{n+1}(\mathbf{x}) = \beta(\beta^n(\mathbf{x})) &\overset{(9.1)}{\leq}_{m+1} \alpha(\beta^n(\mathbf{x})) \\ &= (\nu_m(\pi_{\neq 1}\beta^n(\mathbf{x})), \ldots, \nu_m(\pi_{\neq m+1}\beta^n(\mathbf{x}))) \\ &\overset{(9.2)}{\leq}_{m+1} (\nu_m(\pi_{\neq 1}\alpha^n(\mathbf{x})), \ldots, \nu_m(\pi_{\neq k+1}\alpha^n(\mathbf{x}))) \\ &= \alpha(\alpha^n(\mathbf{x})) = \alpha^{n+1}(\mathbf{x}).\end{aligned}$$

In the case where μ is monotone, (9.2) changes to

$$\mu_m(\pi_{\neq j}\beta^n(\mathbf{x})) \leq \mu_m(\pi_{\neq j}\alpha^n(\mathbf{x})), \quad 1 \leq j \leq k+1, \tag{9.3}$$

and $\beta^{n+1}(\mathbf{x}) \leq_{m+1} \alpha^{n+1}(\mathbf{x})$ from

$$\begin{aligned}\beta^{n+1}(\mathbf{x}) = \beta(\beta^n(\mathbf{x})) &= (\mu_m(\pi_{\neq 1}\beta^n(\mathbf{x})), \ldots, \mu_m(\pi_{\neq m+1}\beta^n(\mathbf{x}))) \\ &\overset{(9.3)}{\leq}_{m+1} (\mu_m(\pi_{\neq 1}\alpha^n(\mathbf{x})), \ldots, \mu_m(\pi_{\neq m+1}\alpha^n(\mathbf{x}))) \\ &\overset{\text{induction}}{\leq}_{m+1} (\nu_m(\pi_{\neq 1}\alpha^n(\mathbf{x})), \ldots, \nu_m(\pi_{\neq m+1}\alpha^n(\mathbf{x}))) \\ &= \alpha(\alpha^n(\mathbf{x})) = \alpha^{n+1}(\mathbf{x}).\end{aligned}$$

Since the order is closed in the product topology, the inequality holds for limits:

$$\lim_{n\to\infty} \beta^n(\mathbf{x}) \leq_{m+1} \lim_{n\to\infty} \alpha^n(\mathbf{x}).$$

In particular, $\mu_{m+1}(\mathbf{x}) \leq \nu_{m+1}(\mathbf{x})$. ∎

THEOREM 9.3. *Let X and Y be complete metric spaces equipped with nonexpansive, coordinatewise ϱ (resp. ϱ')-contractive k-means μ and ν, respectively. Let $\leq$ be a closed partial order on Y. Suppose that the mean ν is monotone for the partial order $\leq$ on Y. If $g : X \to Y$ is continuous and satisfies $g \circ \mu \leq \nu \circ g_k$ (resp. $g \circ \mu \geq \nu \circ g_k$), then for any $n \geq k$,*

$$g \circ \mu_n \leq \nu_n \circ g_n \quad (\textit{resp. } g \circ \mu_n \geq \nu_n \circ g_n).$$

Proof. Suppose that $g \circ \mu \leq \nu \circ g_k$; the other case is similar. Suppose that $g \circ \mu_m \leq \nu_m \circ g_m$ holds true for some $m \geq k$. Let α be the barycentric operator with respect to the mean ν on Y. Then $g(\mu_m(\pi_{\neq j}\mathbf{x})) \leq \nu_m g_m(\pi_{\neq j}\mathbf{x})$ for $\mathbf{x} \in X^{m+1}$ and $1 \leq j \leq m+1$, and thus

$$\begin{aligned}g_{m+1}(\beta(\mathbf{x})) &= (g(\mu_m(\pi_{\neq 1}\mathbf{x})), \ldots, g(\mu_m(\pi_{\neq m+1}\mathbf{x}))) \\ &\leq_{m+1} (\nu_m(g_m(\pi_{\neq 1}\mathbf{x})), \ldots, \nu_m(g_m(\pi_{\neq m+1}\mathbf{x}))) \\ &= (\nu_m(\pi_{\neq 1}g_{m+1}(\mathbf{x})), \ldots, \nu_m(\pi_{\neq m+1}g_{m+1}(\mathbf{x}))) = \alpha(g_{m+1}(\mathbf{x}))\end{aligned}$$

and therefore $g_{m+1} \circ \beta \leq_{m+1} \alpha \circ g_{m+1}$. Since ν is monotone, α is monotone for $\leq_m$ (Theorem 9.2), and therefore

$$g_{m+1}(\beta^2(\mathbf{x})) = (g_{m+1}\beta)(\beta(\mathbf{x})) \leq_{m+1} \alpha(g_{m+1}(\beta(\mathbf{x}))) \leq \alpha^2(g_{m+1}(\mathbf{x}))$$

and inductively $g_{m+1}(\beta^n(\mathbf{x})) \leq_{m+1} \alpha^n(g_{m+1}(\mathbf{x})))$ for all n. Since the order is closed,

$$g_{m+1}(\lim_{n\to\infty} \beta^n(\mathbf{x})) \leq_{m+1} \lim_{n\to\infty} \alpha^n(g_{m+1}(\mathbf{x})).$$

In particular, $g(\mu_{m+1}(\mathbf{x})) \leq \nu_{m+1}(g_{m+1}(\mathbf{x}))$. Induction on m yields the theorem. ∎

COROLLARY 9.4. *Let μ, ν and ω be nonexpansive, coordinatewise contractive k-means on complete metric spaces X, Y, Z respectively. Let $\leq$ be a closed partial order on Z and let $g : X \times Y \to Z$ be a continuous function. If $g \circ (\mu \times \nu) \leq \omega \circ g_k$, then $g \circ (\mu \times \nu)_n \leq \omega_n \circ g_n$ for any $n \geq k$:*

$$g(\mu_n(x_1, \dots, x_n), \nu_n(y_1, \dots, y_n)) \leq \omega_n(g(x_1, y_1), \dots, g(x_n, y_n)).$$

Proof. By Theorem 7.5, the product mean $\mu \times \nu$ is a nonexpansive, coordinatewise contractive k-mean on $X \times Y$. By Theorem 9.3 the inequality $g \circ (\mu \times \nu) \leq \omega \circ g_k$ can be extended to n-means: $g \circ (\mu \times \nu)_n \leq \omega_n \circ g_n$. From

$$(\mu \times \nu)_n(x_1, y_1, x_2, y_2, \dots, x_n, y_n) = (\mu_n(x_1, x_2, \dots, x_n), \nu_n(y_1, y_2, \dots, y_n))$$
$$g_n(x_1, y_1, x_2, y_2, \dots, x_n, y_n) = (g(x_1, y_1), g(x_2, y_2), \dots, g(x_n, y_n)),$$

the assertion follows. ∎

10. Means on Hilbert space operators. In this and the next section we apply and extend our preceding results to the special setting of positive definite operators on a Hilbert space, in particular to positive definite Hermitian matrices (in the case where the Hilbert space is finite-dimensional).

For a Hilbert space E, let $\mathcal{B}(E)$ denote the set of bounded linear operators, $\mathcal{S}(E) \subseteq \mathcal{B}(E)$ the symmetric operators, and $\Omega \subseteq \mathcal{S}(E)$ the positive definite operators on E. We define a closed positive order on $\mathcal{S}(E)$ by $A \leq B$ if $B - A$ is positive semidefinite. Note that the identity operator I (and indeed any positive definite operator) is an order unit for $\mathcal{S}(E)$ (that is, $\mathcal{S}(E) = \bigcup_{n=1}^{\infty}[-nI, nI]$, where in general $[A, B]$ denotes the order interval $\{X \in \mathcal{S}(E) : A \leq X \leq B\}$). There is a corresponding order unit norm given by

$$\|A\| = \inf\{t \geq 0 : A \in [-tI, tI]\}.$$

This norm generates the same topology on $\mathcal{S}(E)$ as does the operator norm.

We primarily employ the *Thompson* (or *part*) *metric* on Ω given by

$$d(A, B) = \max\{\log M(A/B), \log M(B/A)\} \quad \text{where}$$
$$M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\}.$$

A. C. Thompson [25] has shown that Ω is a complete metric space with respect to this metric and the corresponding metric topology on Ω agrees with the relative norm topology. We list some additional elementary properties of the Thompson metric.

Lemma 10.1. *The Thompson metric on the set Ω of positive definite Hilbert space operators satisfies*:

(i) $d(A+B, A+C) \le d(B,C)$.
(ii) $A_1 \le A_2$ *implies* $d(A_1+B, A_1+C) \ge d(A_2+B, A_2+C)$.
(iii) *For* $r>0$, $d(rA, rB) = d(A,B)$.
(iv) $d(A+B, C+D) \le \max\{d(A,C), d(B,D)\}$.
(v) $d(A^{-1}, B^{-1}) = d(A,B)$.

Proof. (i) There exists $r \ge 1$ such that $\log r = d(B,C)$. Then $B \le rC$, and thus $A+B \le A+rC \le rA+rC = r(A+C)$, and similarly $C \le rB$ implies $A+C \le r(A+B)$. Hence $d(A+B, A+C) \le \log r = d(B,C)$.

(ii) The proof of (i) remains valid for $A \ge 0$, and then (ii) follows by rewriting A_2 as $A_1 + (A_2 - A_1)$.

(iii) For $r>0$, $d(rA, rB) = d(A,B)$ since scalar multiplication by r is an order isomorphism.

(iv) Suppose that $d(A,C) \le d(B,D) = \log r$. Then $B \le rD$, $D \le rB$, $A \le rC$, $C \le rA$, and thus $A+B \le rC+rD = r(C+D)$, $C+D \le rA+rB = r(A+B)$. Hence $d(A+B, C+D) \le \log r = d(B,D)$.

(v) This follows from the order reversing property of operator inversion. ∎

The results of Section 6 require forming iterated means from means that are on the one hand convex, and on the other coordinatewise ϱ-contractive and nonexpansive. The convex mean we employ is the geometric mean on Ω defined by $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ (see [19] for a variety of other characterizations). It is known (see, for example, [12], [21]) that:

Lemma 10.2. *The geometric mean $A\#B$ is a convex mean on Ω with respect to the Thompson metric.*

We close this section by establishing that the arithmetic mean and the harmonic mean are nonexpansive and coordinatewise ϱ-contractive with respect to the Thompson metric when we restrict to order intervals. This takes some computation.

Lemma 10.3. *For each $n \in \mathbb{N}$, there exists ϱ_n, $0 < \varrho_n < 1$, such that the arithmetic mean $A(X,Y) = (X+Y)/2$ and the harmonic mean $H(X,Y) = 2(X^{-1}+Y^{-1})^{-1}$ are coordinatewise ϱ_n-contractive and nonexpansive on the order interval $[(1/n)I, nI] \subseteq \Omega$, where I is the identity operator.*

Proof. By Lemma 10.1(iv), the arithmetic mean is nonexpansive. We seek $\varrho = \varrho_n$ such that for any $A, B, C \in [(1/n)I, nI]$, $d(A+B, A+C) \le \varrho d(B,C)$ (note that we can drop the factor of $1/2$ by Lemma 10.1(iii)). This is equivalent to

$$\max\{\log M((A+B)/(A+C)), \log M((A+C)/(A+B))\} \le \varrho \max\{\log M(B/C), \log M(C/B)\}.$$

Note that if $B \neq C$ (the desired inequality is trivially true if $B = C$), then either $B \nleq C$ or $C \nleq B$, and then there exists $r > 1$ such that $C \leq rB$, $r = M(C/B)$, and $\log r = d(B, C)$ (or vice versa with the roles of B and C interchanged).

Suppose now that we can find ϱ, $0 < \varrho < 1$, such that for all $A, B, C \in [(1/n)I, nI]$, we have $A + C \leq r^{\varrho}(A + B)$, where $r = M(C/B) = \max\{M(B/C), M(C/B)\}$. Then

$$d(A + B, A + C) \leq \log r^{\varrho} = \varrho \log r = \varrho d(B, C),$$

which is the coordinatewise ϱ-contractive property. We conclude that to establish the coordinatewise ϱ-contractive property, it suffices to show the existence of some ϱ, $0 < \varrho < 1$, such that for all $A, B, C \in [(1/n)I, nI]$, if $C \leq rB$ for $1 < r \leq n^2$, then $A + C \leq r^{\varrho}(A + B)$. (Note that we can restrict to $r \leq n^2$ since $C \leq nI = n^2(1/n)I \leq n^2 B$.) We establish that this is indeed the case by means of the following two claims.

Claim 1. *For given $0 < \varrho < 1$ and $r > 1$, assume that $C + (1/n)I \leq r^{\varrho}(B + (1/n)I)$ whenever $C \leq rB$. Then $A + C \leq r^{\varrho}(A + B)$ for all $A \geq (1/n)I$.*

Indeed,

$$\begin{aligned} A + C &= \left(A - \frac{1}{n}I\right) + \frac{1}{n}I + C \leq \left(A - \frac{1}{n}I\right) + r^{\varrho}\left(\frac{1}{n}I + B\right) \\ &\leq r^{\varrho}\left(A - \frac{1}{n}I\right) + r^{\varrho}\left(\frac{1}{n}I + B\right) = r^{\varrho}(A + B). \end{aligned}$$

Claim 2. *There exists ϱ, $0 < \varrho < 1$, such that for all $1 < r \leq n^2$, $C + (1/n)I \leq r^{\varrho}(B + (1/n)I)$ whenever $C \leq rB$, $B, C \in [(1/n)I, nI]$.*

Indeed, for any $0 < \varrho < 1$,

$$C + \frac{1}{n}I \leq rB + \frac{1}{n}I = r^{\varrho}B + (r - r^{\varrho})B + \frac{1}{n}I.$$

For the last two terms we have

$$(r - r^{\varrho})B + \frac{1}{n}I \leq (r - r^{\varrho})(nI) + \frac{1}{n}I = ((r - r^{\varrho})n^2 + 1)\frac{1}{n}I.$$

To complete the proof of Claim 2, we need to choose $\varrho < 1$, but large enough so that $(r - r^{\varrho})n^2 + 1 \leq r^{\varrho}$ for $1 \leq r \leq n^2$. The function $f(x) = x^{\varrho} - n^2(x - x^{\varrho}) - 1$ has derivative

$$f'(x) = \varrho x^{\varrho - 1} - n^2(1 - \varrho x^{\varrho - 1}) = (1 + n^2)\varrho x^{\varrho - 1} - n^2.$$

Since the limit of the right-hand expression is 1 as $\varrho \to 1^-$, we conclude that the derivative is positive for all $x \in [1, n^2]$ for large enough ϱ below 1. Thus f is increasing on $[1, n^2]$ for ϱ near, but below, 1, and hence

$$(r - r^{\varrho})n^2 + 1 \leq r^{\varrho} \quad \text{for any } 1 \leq r \leq n^2,\ |1 - \varrho| < \varepsilon$$

for some $\varepsilon > 0$.

The case of the harmonic mean follows from the fact that the inversion is an isometry (Lemma 10.1(v)) and by applying the preceding result on the arithmetic mean. ■

11. Extending means on Hilbert space operators. We summarize fundamental results of Kubo and Ando [18] (see also [5], [24]) concerning operator means and their relationship to means on the positive reals. We consider continuous means on the positive reals, $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, satisfying

(i) if $x \le x'$ and $y \le y'$, then $\mu(x, y) \le \mu(x', y')$ (monotonicity),
(ii) $\mu(tx, ty) = t\mu(x, y)$ for $t, x, y > 0$ (homogeneity).

A homogeneous two-variable function μ can be reduced to a one-variable function $f(x) = \mu(1, x)$. This reduction defines a one-to-one correspondence between the continuous means satisfying (i) and (ii) and the continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(O) $f(1) = 1$,
(I) f is nondecreasing.

We consider continuous operator means μ on Ω, the set of positive operators on a Hilbert space, satisfying

(a) if $A \le A'$, $B \le B'$, then $\mu(A, B) \le \mu(A', B')$ (monotonicity),
(b) $C\mu(A, B)C^* = \mu(CAC^*, CBC^*)$ for all C invertible (the transformer equality),

where, as usual, $A \le B$ means that $B - A$ is positive semidefinite.

The key result of the theory is that the continuous operator means on Ω satisfying (a) and (b) are in one-to-one correspondence with the operator monotone functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (O)–(I), where the correspondence $\mu \leftrightarrow f$ is given by

$$\mu(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

Recall that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *operator monotone* if its extension to Ω via the functional calculus is monotone. It is this extension that appears in the displayed equality. The scalar function f is called the *representing function* of μ.

Basic examples of the preceding theory include: (i) the arithmetic mean on $\mathbb{R}^+$ with representing function $f(x) = \frac{1}{2}(1 + x)$ and operator mean $\frac{1}{2}(A + B)$, (ii) the geometric mean $\sqrt{ab}$ on $\mathbb{R}^+$ with representing function $f(x) = \sqrt{x}$ and operator geometric mean $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, and (iii) the harmonic mean $2/(a^{-1}+b^{-1})$ with representing function $f(x) = 2x/(1 + x)$ and operator harmonic mean $2(A^{-1} + B^{-1})^{-1}$.

We recall from Section 6 and from [18] the notion of the iterated composition $\sigma * \tau$ of two operator means σ and τ on Ω. Starting with $\sigma_1 = \sigma$ and $\tau_1 = \tau$, we define inductively the sequences of means $\{\sigma_n\}$ and $\{\tau_n\}$ by

$$\sigma_{n+1}(A,B) = \sigma(\sigma_n(A,B), \tau_n(A,B)), \quad \tau_{n+1}(A,B) = \tau(\sigma_n(A,b), \tau_n(A,B)).$$

By Theorem 6.2 of [18], if σ and τ are continuous, monotonic means satisfying the transformer equality and if at least one is neither the left- nor the right-trivial mean, then $\{\sigma_n\}$ and $\{\tau_n\}$ converge to the same limiting mean σ_∞, which means that for all $A, B \in \Omega$, $\lim_n \sigma_n(A,B) = \sigma_\infty(A,B) = \lim_n \tau_n(A,B)$, where the limit is taken in the weak operator topology.

THEOREM 11.1. *Let Ω denote the set of positive operators on a Hilbert space, let $\lambda(A,B) = A\#B$ denote the geometric mean of A, B, and let ν be a continuous, monotonic mean on Ω satisfying the transformer equality that is also coordinatewise ϱ_n-contractive for $0 < \varrho_n < 1$ and nonexpansive on the order interval $[(1/n)I, nI]$ for each n with respect to the Thompson metric. Then the iterated composition $\mu = \lambda * \nu$ (resp. skewed iterated composition $\mu = \lambda *_{\mathrm{s}} \nu$) exists, is a coordinatewise ϱ_n-contractive, nonexpansive mean when restricted to $[(1/n)I, nI]$ for each n, and hence inductively β-converges to a β-extension for each $n > 2$.*

Proof. We consider some fixed order interval $\Gamma_n = [(1/n)I, nI]$ and $A, B \in [(1/n)I, nI]$; note that $[(1/n)I, nI]$ is closed and convex with respect to any monotonic mean, in particular with respect to λ and ν. By Proposition 6.4 there exists an iterated composition $\mu_n = \lambda|_{\Gamma_n} * \nu|_{\Gamma_n}$ that is nonexpansive and coordinatewise ϱ_n-contractive. Clearly, if $m < n$, then μ_n is an extension of μ_m. Thus there exists a unique mean μ that extends all of them. Since any A, B belong to the domain of some μ_n, μ is the iterated composition $\lambda * \nu$.

Since μ_n is nonexpansive and coordinatewise ϱ_n-contractive on Γ_n, it inductively admits a β-extension for each $n > 2$. It then follows from Corollary 7.8 that a β-extension of μ exists inductively for each $n > 2$.

The case of the skewed iterated composition is analogous. ∎

The iterated composition of the operator arithmetic mean and operator geometric mean yields the arithmetic-geometric operator mean [15]. Similarly we have the harmonic-geometric operator mean.

DEFINITION 11.2. For two positive definite operators $A, B \in \Omega$ on the Hilbert space E, we define the *arithmetic-geometric mean* or *Gauss mean* $\mathrm{AGM}(A,B)$ to be the iterated composition of the geometric and arithmetic means, that is, the limit $\lim_n \lambda_n(A,B) = \lim_n \nu_n(A,B)$, where we define $\lambda_1(A,B) = A\#B$, the geometric mean, $\nu_1(A,B) = (A+B)/2$, the arithmetic mean, and inductively $\lambda_{n+1}(A,B) = \lambda_n(A,B)\#\nu_n(A,B)$ and $\nu_{n+1}(A,B) = \frac{1}{2}(\lambda_n(A,B) + \nu_n(A,B))$.

It is standard that the preceding iteration defining the arithmetic-geometric mean of two positive operators converges in the weak operator topology and agrees with the one arising from the representing function of arithmetic-geometric mean on the positive real numbers [18, Section 6].

COROLLARY 11.3. *On any order interval* $[(1/n)I, nI]$ *the arithmetic-geometric mean* (*resp. the harmonic-geometric mean*) *is nonexpansive and coordinatewise* ϱ_n*-contractive for some* ϱ_n, $0 < \varrho_n < 1$. *Hence on* Ω *the arithmetic-geometric mean* (*resp. the harmonic-geometric mean*) *inductively* β*-extends to an* n*-mean for each* $n > 2$.

Proof. Fix some positive integer m. By Lemma 10.3 the arithmetic mean is nonexpansive and coordinatewise ϱ_m-contractive on $[(1/m)I, mI]$ for some $0 < \varrho_m < 1$. We have already remarked that the geometric mean is a convex mean with respect to the Thompson metric. The result now follows from Theorem 11.1. ■

We briefly recall the operator logarithmic mean as discussed in [24]. The logarithmic mean is defined on $\mathbb{R}^+$ by $L(a, b) = (b - a)/(\log b - \log a)$. Its representing function is $f(x) = L(1, x) = (x - 1)/\log x$, which is an operator monotone function. Hence there exists a corresponding operator logarithmic mean. B. C. Carlson [10] has shown that the logarithmic mean on $\mathbb{R}^+$ is the skewed iterated composition of the geometric and arithmetic means. It then follows from the theory of operator means as developed in Section 6 of [18], particularly Lemma 6.1 and Theorem 6.2 there, that the operator logarithmic mean is the corresponding skewed iterated composition of the operator geometric and arithmetic means on Ω, the set of positive operators on a Hilbert space, where the limits are taken in the weak operator topology. However, the same arguments applied in the previous corollary to the arithmetic-geometric mean viewed as the iterated composition of the geometric and arithmetic means apply equally well to the logarithmic mean viewed as the skewed iterated composition of the geometric and arithmetic means. We thus analogously obtain the following

COROLLARY 11.4. *On any order interval* $[(1/n)I, nI]$ *the logarithmic mean is nonexpansive and coordinatewise* ϱ_n*-contractive for some* ϱ_n, $0 < \varrho_n < 1$. *Hence on* Ω *the logarithmic mean inductively* β*-extends to an* n*-mean for each* $n > 2$.

The preceding corollary provides a positive solution to a problem raised by Petz and Temesi ([24], [23]) as to whether the logarithmic mean β-converges and hence admits higher-dimensional extensions.

It is easy to obtain the order relation $L(A, B) \le \mathrm{AGM}(A, B)$ between the logarithmic mean and the arithmetic-geometric mean, which are monotone (Definition 9.1). Applying Theorem 9.2 we have the following

Corollary 11.5. *The order relation*

$$L_n(A_1, \ldots, A_n) \leq \mathrm{AGM}_n(A_1, \ldots, A_n)$$

holds for the extended logarithmic and arithmetic-geometric n-means for each $n > 2$.

Remark 11.6. In a similar way one can show that 2-means μ and ν that satisfy the inequality $\mu \leq \nu$ and that can both be derived by some iteration or skew iteration of the arithmetic and geometric resp. harmonic and geometric means satisfy $\mu_n \leq \nu_n$ for all $n > 2$. In this way, for example, one derives the principal results of [17] as a corollary to our preceding results.

We remark in closing that a number of ideas in this paper can be carried over to the study of means on the set of positive elements of a C^*-algebra, particularly by viewing the C^*-algebra as a closed subalgebra of the algebra of bounded operators on a Hilbert space. For example, one could define the logarithmic mean to be the skewed iterated composition of the geometric and arithmetic means and show that it inductively β-extends to all higher dimensions.

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