COLLOQUIUM MATHEMATICUM

VOL. 113 2008 NO. 2

CHARACTERIZING METRIC SPACES WHOSE HYPERSPACES ARE HOMEOMORPHIC TO ℓ_2

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Abstract. It is shown that the hyperspace $\mathrm{Cld}_{\mathrm{H}}(X)$ (resp. $\mathrm{Bdd}_{\mathrm{H}}(X)$) of non-empty closed (resp. closed and bounded) subsets of a metric space (X,d) is homeomorphic to ℓ_2 if and only if the completion $\overline{X}$ of X is connected and locally connected, X is topologically complete and nowhere locally compact, and each subset (resp. each bounded subset) of X is totally bounded.

1. Introduction. In this paper we characterize metric spaces X whose hyperspaces $\mathrm{Cld}_{\mathrm{H}}(X)$ and $\mathrm{Bdd}_{\mathrm{H}}(X)$ of closed and closed bounded subsets are homeomorphic to the separable Hilbert space ℓ_2. For a metric space (X,d), we denote by $\mathrm{Cld}_{\mathrm{H}}(X)$ the space of non-empty closed subsets of X endowed with the topology generated by the Hausdorff "metric"

$$d_{\mathrm{H}}(A,B)=\max\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(b,A)\}.$$

For an unbounded metric space (X,d) the "metric" d_{H} can take the infinite value but it still generates a topology on $\mathrm{Cld}_{\mathrm{H}}(X)$ called the Hausdorff topology. More precisely, this topology is generated by the metric $\min\{1,d_{\mathrm{H}}\}$. By $\mathrm{Bdd}_{\mathrm{H}}(X)$ we denote the subspace of $\mathrm{Cld}_{\mathrm{H}}(X)$ consisting of the non-empty closed bounded subsets of the metric space (X,d). The hyperspace $\mathrm{Cld}_{\mathrm{H}}(X)$ is a classical object in topology and has applications in set-valued analysis (see e.g., [1]). For a compact metric space X the Hausdorff topology on $\mathrm{Cld}_{\mathrm{H}}(X)$ coincides with the Vietoris topology, another classical topology on $\mathrm{Cld}(X)$ (see [9, 2.7.20]; cf. [3]). More generally, the Vietoris topology coincides with the Hausdorff topology on the subspace $\mathrm{Comp}(X)$ of $\mathrm{Cld}(X)$ consisting of the non-empty compact subsets of X (see [9, 8.5.16(c)]).

One of the finest results concerning the topology of hyperspaces is the famous Curtis–Shori theorem [8] characterizing non-degenerate Peano continua as metric spaces X whose hyperspace $\mathrm{Cld}_{\mathrm{H}}(X)$ is homeomorphic to the Hilbert cube $Q=[0,1]^\omega$. The next step in this direction was made by D. Curtis who proved in [5] that the hyperspace $\mathrm{Comp}(X)$ is homeomorphic to $Q\times[0,1)$ if and only if X is non-compact, locally compact, con-

2000 *Mathematics Subject Classification*: 54B20, 57N20.
Key words and phrases: hyperspace, Hausdorff metric, Hilbert space.

nected, and locally connected. Another result of D. Curtis [4] states that $\mathrm{Comp}(X)$ is homeomorphic to ℓ_2 if and only if X is connected, locally connected, topologically complete and nowhere locally compact. We recall that a space X is *topologically complete* if X is homeomorphic to a complete metric space.

In this paper we characterize the metric spaces X whose hyperspaces $\mathrm{Cld}_{\mathrm{H}}(X)$ and $\mathrm{Bdd}_{\mathrm{H}}(X)$ are homeomorphic to ℓ_2. We call a metric space X *proper* if each closed bounded subset of X is compact.

THEOREM 1. *The hyperspace* $\mathrm{Bdd}_{\mathrm{H}}(X)$ (*resp.* $\mathrm{Cld}_{\mathrm{H}}(X)$) *of a metric space* (X,d) *is homeomorphic to* ℓ_2 *if and only if* X *is a topologically complete nowhere locally compact space and its completion* $\overline{X}$ *is proper* (*resp. compact*), *connected*, *and locally connected.*

Applying this theorem to the metric spaces $\mathbb{R}\setminus\mathbb{Q}$ and $I\setminus\mathbb{Q}$ of irrational numbers on the real line and the interval $I=[0,1]$, we obtain the following

COROLLARY 1. *The hyperspaces* $\mathrm{Cld}_{\mathrm{H}}(I\setminus\mathbb{Q})$ *and* $\mathrm{Bdd}_{\mathrm{H}}(\mathbb{R}\setminus\mathbb{Q})$ *are homeomorphic to* ℓ_2.

Let us remark that in contrast the hyperspace $\mathrm{Cld}_{\mathrm{H}}(\mathbb{R}\setminus\mathbb{Q})$ is not homeomorphic to ℓ_2 since it is neither connected nor separable.

Applying Theorem 1 to a dense G_δ-subset $X\subset\mathbb{R}^n$ we obtain the following corollary partly improving Theorem 5.1 of W. Kubiś and K. Sakai [10].

COROLLARY 2. *For any dense nowhere locally compact* G_δ*-subset* $X\subset\mathbb{R}^m$ *the hyperspace* $\mathrm{Bdd}_{\mathrm{H}}(X)$ *is homeomorphic to* ℓ_2.

As a by-product of the proof of Theorem 1 we obtain the following characterizations of metric spaces whose hyperspaces are separable absolute retracts.

THEOREM 2. *The hyperspace* $\mathrm{Bdd}_{\mathrm{H}}(X)$ (*resp.* $\mathrm{Cld}_{\mathrm{H}}(X)$) *of a metric space* X *is a separable AR if and only if the completion* $\overline{X}$ *of* X *is proper* (*resp. compact*), *connected and locally connected.*

2. Homotopy dense subsets in the Hilbert cube. A subset Y of a topological space X is *homotopy dense* in X if there is a homotopy $(h_t)_{t\in I}: X\to X$ such that $h_0=\mathrm{id}$ and $h_t(X)\subset Y$ for every $t>0$. The following lemma detecting topological copies of ℓ_2 in the Hilbert cube Q is due to D. Curtis [6] and is our main tool in the proof of Theorem 1.

LEMMA 1. *A homotopy dense* G_δ*-subset* $X\subset Q$ *with homotopy dense complement in the Hilbert cube* Q *is homeomorphic to* ℓ_2.

3. Topology of Lawson semilattices. Theorem 2 will be derived from a more general result concerning Lawson semilattices. By a *topological semilattice* we understand a pair $(L,\vee)$ consisting of a topological space L and a continuous associative commutative idempotent operation $\vee : L\times L\to L$. A topological semilattice $(L,\vee)$ is a *Lawson semilattice* if the open subsemilattices form a base of the topology of L. A typical example of a Lawson semilattice is the hyperspace $\mathrm{Cld}_{\mathrm{H}}(X)$ endowed with the union operation (see [11, 5.4]).

Each semilattice $(L,\vee)$ carries a natural partial order: $x\le y$ iff $x\vee y=y$. A semilattice $(L,\vee)$ is called *complete* if each subset $A\subset L$ has the smallest upper bound $\sup A\in L$. It is well-known (and can be easily proved) that each compact topological semilattice is complete.

LEMMA 2. *If L is a locally compact Lawson semilattice, then each compact subset $K\subset L$ has the smallest upper bound* $\sup K\in L$. *Moreover, the map* $\sup : \mathrm{Comp}(L)\to L$, $K\mapsto \sup K$, *is a continuous semilattice homomorphism. Also for every subset $A\subset L$ with compact closure $\bar A$ we have* $\sup A=\sup \bar A$.

This lemma easily follows from its compact version proved by J. Lawson in [13].

In Lawson semilattices many geometric questions reduce to one dimension. The following fact illustrating this phenomenon is proved in [11].

LEMMA 3. *Let X be a dense subsemilattice of a metrizable Lawson semilattice L. If X is relatively LC^0 in L (resp. X is relatively LC^0 in L and path-connected), then X and L are ANRs (resp. ARs) and X is homotopy dense in L.*

A subset $Y\subset X$ is defined to be *relatively LC^0* in X if for every $x\in X$, each neighborhood U of x in X contains a smaller neighborhood V of x such that any two points of $V\cap Y$ can be joined by a path in $U\cap Y$.

Under a suitable completeness condition, the density of a subsemilattice is equivalent to its homotopical density.

A subsemilattice X of semilattice L is defined to be *relatively complete* in L if for any subset $A\subset X$ having the smallest upper bound $\sup A$ in L this bound belongs to X.

PROPOSITION 1. *Let L be a metrizable locally compact locally connected Lawson semilattice. Each dense relatively complete subsemilattice $X\subset L$ is homotopy dense in L.*

Proof. According to Lemma 3 it suffices to check that X is relatively LC^0 in L. Given a point $x_0\in L$ and a neighborhood $U\subset L$ of x_0, consider the canonical retraction $\sup : \mathrm{Comp}(L)\to L$. The space L, being locally compact and locally connected, is locally path-connected (see [12, §50.II]). By Lemma 3, the Lawson semilattice L is an ANR. Using the continuity of sup, find a path-connected neighborhood $V \subset L$ of x_0 such that $\sup(\mathrm{Comp}(\overline{V})) \subset U$. We claim that any two points $x, y \in X \cap V$ can be connected by a path in $X \cap U$. First we construct a path $\gamma : [0,1] \to \overline{V}$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma^{-1}(X)$ is dense in $[0,1]$. Let $\{q_n : n \in \omega\}$ be a countable dense subset in $[0,1]$ with $q_0 = 0$ and $q_1 = 1$. The space L, being locally compact, admits a complete metric ϱ. The path-connectedness of V implies the existence of a continuous map $\gamma_0 : [0,1] \to V$ such that $\gamma_0(0) = x$ and $\gamma_0(1) = y$. Using the local path-connectedness of L we can construct inductively a sequence of functions $\gamma_n : [0,1] \to V$ such that

- $\gamma_n(q_k) = \gamma_{n-1}(q_k)$ for all $k \le n$;
- $\gamma_n(q_{n+1}) \in X$;
- $\sup_{t\in[0,1]} \varrho(\gamma_n(t), \gamma_{n-1}(t)) < 2^{-n}$.

Then the map $\gamma = \lim_{n\to\infty} \gamma_n : [0,1] \to \overline{V}$ is continuous and has the desired properties: $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(q_n) \in X$ for all $n \in \omega$.

For every $t \in [0,1]$ set $\Gamma(t) = \{\gamma(s) : |t - s| \le \mathrm{dist}(t, \{0,1\})\}$. It is clear that the map $\Gamma : [0,1] \to \mathrm{Comp}(L)$ is continuous and so is the composition $\sup \circ \Gamma : [0,1] \to L$. Observe that $\sup \circ \Gamma(0) = \sup\{\gamma(0)\} = \gamma(0) = x$, $\sup \circ \Gamma(1) = y$, and $\sup \circ \Gamma([0,1]) \subset \sup(\mathrm{Comp}(\overline{V})) \subset U$. Since for every $t \in (0,1)$ the set $\Gamma(t)$ equals $\overline{\Gamma(t) \cap X}$, we get $\sup \Gamma(t) = \sup(\Gamma(t) \cap X) \in X$ by the relative completeness of X in L. Thus $\sup \circ \Gamma : [0,1] \to U \cap X$ is a path connecting x and y in U. ∎

4. Some topological properties of hyperspaces. In this section we collect some easy (and known) lemmas that will be used in the subsequent proofs.

LEMMA 4. *For a metric space X the following conditions are equivalent*:

(1) *X is topologically complete*;
(2) $\mathrm{Cld}_{\mathrm{H}}(X)$ *is topologically complete*;
(3) $\mathrm{Bdd}_{\mathrm{H}}(X)$ *is topologically complete.*

LEMMA 5. *For a metric space X the following conditions are equivalent*:

(1) *X is nowhere locally compact*;
(2) $\mathrm{Cld}_{\mathrm{H}}(X)$ *is nowhere locally compact*;
(3) $\mathrm{Bdd}_{\mathrm{H}}(X)$ *is nowhere locally compact.*

LEMMA 6. *Let X be a metric space. The hyperspace* $\mathrm{Cld}_{\mathrm{H}}(X)$ (*resp.* $\mathrm{Bdd}_{\mathrm{H}}(X)$) *is separable if and only if each subset* (*resp. each bounded subset*) *of X is totally bounded.*

The following lemma is not trivial and can be found in [2, 3.7].

LEMMA 7. *Let X be a dense subspace of a metric space M. The hyperspace* $\mathrm{Cld}_{\mathrm{H}}(X)$ (*resp.* $\mathrm{Bdd}_{\mathrm{H}}(X)$) *is an absolute retract if and only if so is* $\mathrm{Cld}_{\mathrm{H}}(M)$ (*resp.* $\mathrm{Bdd}_{\mathrm{H}}(M)$).

For a metric space X we denote by $\mathrm{Fin}(X)$ the subspace of $\mathrm{Comp}(X)$ consisting of non-empty finite subspaces of X.

LEMMA 8. *If Y is a subset of a locally path-connected space X, then the subset* $L = \mathrm{Fin}(X) \setminus \mathrm{Fin}(Y)$ *is relatively LC^0 in* $\mathrm{Comp}(X)$.

Proof. By the argument of [7] we can show that $\mathrm{Fin}(X)$ is relatively LC^0 in $\mathrm{Comp}(X)$. Consequently, for every $K \in \mathrm{Comp}(X)$ and a neighborhood $U \subset \mathrm{Comp}(X)$ of K there is a neighborhood $V \subset \mathrm{Comp}(X)$ of K such that any two points $A, B \in \mathrm{Fin}(X) \cap V$ can be joined by a path in $\mathrm{Fin}(X) \cap U$. Since $\mathrm{Comp}(X)$ is a Lawson semilattice, we may assume that U and V are subsemilattices of $\mathrm{Comp}(X)$. We claim that any two points $A, B \in L \cap V$ can be connected by a path in $L \cap U$. Since $L \subset \mathrm{Fin}(X)$, there is a path $\gamma : [0, 1] \to U \cap \mathrm{Fin}(X)$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Define a new path $\gamma' : [0, 1] \to U \cap \mathrm{Fin}(X)$ by letting $\gamma'(t) = \gamma(\max\{0, 2t - 1\}) \cup \gamma(\min\{2t, 1\})$. Observe that $A \subset \gamma'(t)$ if $t \le 1/2$ and $B \subset \gamma'(t)$ if $t \ge 1/2$. Since $A, B \notin \mathrm{Fin}(Y)$, we conclude that $\gamma'([0, 1]) \subset L \cap U$. ∎

5. Proof of Theorem 2. Let X be a metric space and $\overline{X}$ be its completion. First we prove that $\mathrm{Bdd}_{\mathrm{H}}(X)$ is a separable AR if and only if $\overline{X}$ is proper, connected and locally connected.

To prove the "only if" part, assume that $\mathrm{Bdd}_{\mathrm{H}}(X)$ is a separable absolute retract. By Lemma 7, so is $\mathrm{Bdd}_{\mathrm{H}}(\overline{X})$. By Lemma 6, the separability of $\mathrm{Bdd}_{\mathrm{H}}(X)$ implies that each bounded subset of X is totally bounded, which is equivalent to the properness of $\overline{X}$. In this case $\mathrm{Comp}(\overline{X}) = \mathrm{Bdd}_{\mathrm{H}}(\overline{X})$ is an absolute retract and we can apply the Curtis theorem [5] to conclude that the locally compact space $\overline{X}$ is connected and locally connected.

Next, we prove the "if" part. Assume that $\overline{X}$ is proper, connected, and locally connected. Then $\mathrm{Bdd}_{\mathrm{H}}(\overline{X}) = \mathrm{Comp}(\overline{X})$ is a separable locally compact absolute retract by [5]. The subsemilattice $\mathrm{Bdd}_{\mathrm{H}}(X)$, being relatively complete in $\mathrm{Bdd}_{\mathrm{H}}(\overline{X})$, is homotopy dense in $\mathrm{Bdd}_{\mathrm{H}}(\overline{X})$ by Proposition 1.

Now we prove that $\mathrm{Cld}_{\mathrm{H}}(X)$ is a separable AR if and only if $\overline{X}$ is compact, connected and locally connected.

If $\overline{X}$ is compact, connected, and locally connected, then $\mathrm{Cld}_{\mathrm{H}}(X) = \mathrm{Bdd}_{\mathrm{H}}(X)$ is a separable AR by the preceding case. Conversely, if $\mathrm{Cld}_{\mathrm{H}}(X)$ is a separable AR, then Lemma 6 guarantees that X is totally bounded, and hence $\mathrm{Cld}_{\mathrm{H}}(X) = \mathrm{Bdd}_{\mathrm{H}}(X)$ and we can apply the preceding case to conclude that $\overline{X}$ is connected and locally connected. It is also compact, being the completion of a totally bounded metric space X.

6. Proof of Theorem 1. Let X be a metric space. If $\mathrm{Bdd}_{\mathrm{H}}(X)$ (resp. $\mathrm{Cld}_{\mathrm{H}}(X)$) is homeomorphic to ℓ_2, then X is topologically complete and nowhere locally compact by Lemmas 4 and 5. Since ℓ_2 is a separable AR, we may apply Theorem 2 to conclude that the completion $\overline{X}$ of X is connected, locally connected, and proper (resp. compact). This proves the "only if" part of Theorem 1.

To prove the "if" part, assume that X is topologically complete and nowhere locally compact, and $\overline{X}$ is proper, connected and locally connected. First we consider the case of $\overline{X}$ compact. By the Curtis–Shori theorem [8], the hyperspace $\mathrm{Cld}_{\mathrm{H}}(\overline{X}) = \mathrm{Comp}(\overline{X})$ is homeomorphic to Q. Now consider the map $e : \mathrm{Cld}_{\mathrm{H}}(X) \to \mathrm{Cld}_{\mathrm{H}}(\overline{X})$ assigning to each closed subset $F \subset X$ its closure $\overline{F}$ in $\overline{X}$. As this is an isometric embedding, we can identify $\mathrm{Cld}_{\mathrm{H}}(X)$ with the subspace $\{F \in \mathrm{Cld}_{\mathrm{H}}(\overline{X}) : F = \mathrm{cl}(F \cap X)\}$ of $\mathrm{Cld}_{\mathrm{H}}(\overline{X})$. It is easy to check that this subspace is dense and relatively complete in the Lawson semilattice $\mathrm{Cld}_{\mathrm{H}}(\overline{X})$. Hence it is homotopically dense in $\mathrm{Cld}_{\mathrm{H}}(\overline{X})$ by Proposition 1 and Lemma 3. By Lemma 4, the subset $\mathrm{Cld}_{\mathrm{H}}(X)$, being topologically complete, is a G_δ-set in $\mathrm{Cld}_{\mathrm{H}}(\overline{X})$. Since X is nowhere locally compact, $\overline{X} \setminus X$ is dense in $\overline{X}$. By Lemmas 4 and 8, the dense subsemilattice $L = \mathrm{Fin}(\overline{X}) \setminus \mathrm{Fin}(X)$ is homotopy dense in $\mathrm{Cld}_{\mathrm{H}}(\overline{X})$. Since $L \cap \mathrm{Cld}_{\mathrm{H}}(X) = \emptyset$, we find that $\mathrm{Cld}_{\mathrm{H}}(X)$ is a homotopy dense G_δ-subset in $\mathrm{Cld}_{\mathrm{H}}(\overline{X})$ with homotopy dense complement. Applying Lemma 1 we conclude that $\mathrm{Cld}_{\mathrm{H}}(X)$ is homeomorphic to ℓ_2.

Next, we consider the case of $\overline{X}$ non-compact. It follows from the properness of $\overline{X}$ that $\mathrm{Bdd}_{\mathrm{H}}(\overline{X}) = \mathrm{Comp}_{\mathrm{H}}(\overline{X})$ and hence $\mathrm{Bdd}_{\mathrm{H}}(\overline{X})$ is homeomorphic to $Q \setminus \{\mathrm{pt}\}$ by the Curtis theorem [5]. Repeating the preceding argument, we can prove that $\mathrm{Bdd}_{\mathrm{H}}(X)$ can be identified with a homotopy dense G_δ-set with homotopy negligible complement in $\mathrm{Bdd}_{\mathrm{H}}(\overline{X})$. Since the one-point compactification of $\mathrm{Bdd}_{\mathrm{H}}(\overline{X})$ is homeomorphic to the Hilbert cube, we can apply Lemma 1 to conclude that $\mathrm{Bdd}_{\mathrm{H}}(X)$ is homeomorphic to ℓ_2.

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Received 18 April 2007;
revised 25 January 2008 (4902)