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# MOVING AVERAGES

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**Abstract.** In ergodic theory, certain sequences of averages $\{A_k f\}$ may not converge almost everywhere for all $f \in L^1(X)$, but a sufficiently rapidly growing subsequence $\{A_{m_k} f\}$ of these averages will be well behaved for all $f$. The order of growth of this subsequence that is sufficient is often hyperexponential, but not necessarily so. For example, if the averages are

$$A_k f(x) = \frac{1}{2^k} \sum_{j=4^k+1}^{4^k+2^k} f(T^j x),$$

then the subsequence $A_{k^2} f$ will not be pointwise good even on $L^\infty$, but the subsequence $A_{2^k} f$ will be pointwise good on $L^1$. Understanding when the hyperexponential rate of growth of the subsequence is required, and giving simple criteria for this, is the subject that we want to address here. We give a fairly simple description of a wide class of averaging operators for which this rate of growth can be seen to be necessary.

**1. Introduction.** Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and $f$ a $\mu$-almost everywhere finite $\mathcal{B}$-measurable function. We denote by $1_B(x)$ the characteristic function of $B$. We will use the notation $n_k \nearrow \infty$ for a non-decreasing unbounded sequence $\{n_k\}$.

Let $\{(n_k, l_k)\}_{k=1}^\infty$ be a sequence of pairs of natural numbers. In a number of articles, the a.e. convergence of averages

$$A_k f(x) = \frac{1}{l_k} \sum_{j=n_k+1}^{n_k+l_k} f(T^j x) \tag{1}$$

is considered for specific sequences $\{(n_k, l_k)\}_{k=1}^\infty$. We need to describe the background for these types of averages because we will be giving some results that use them. Our results give examples of when convergence occurs and when it does not based on the growth rate of various parameters.

In [1] it is shown that for $n_k = k$ and $l_k = \sqrt{k}$ there exists an $f \in L^\infty$ for which a.e. convergence of $A_k f$ fails. From work in [3] one sees that if $n_k = 4^k$ and $l_k = 2^k$ then convergence fails. On the other hand, it can be shown that if $n_k = 2^{2^k}$ and $l_k = \sqrt{n_k}$ then a.e. convergence of $A_k f$ occurs for all

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$f \in L^1$ (see [2]). The above-mentioned results also follow, respectively, from Theorem 2.5 (with $u(k) = \log_2 \log_2 \sqrt{k}$), Theorem 2.8 (with $u(k) = \log_2 k$), and Theorem 2.2 in the present paper.

For a non-empty, finite set $I$ of non-negative integers let $|I|$ be the cardinality of $I$. We consider averaging operators

$$A_I f(x) = \frac{1}{|I|} \sum_{i \in I} f(T^i x). \tag{2}$$

If $I = \{n_k + 1, \dots, n_k + l_k\}$ then the averages (1) and (2) coincide.

For two sets of integers $A$ and $B$ we denote by $A - B$ the set of integers $j$ for which there is $b \in B$ so that $j + b \in A$; in other words, $(j + B) \cap A \neq \emptyset$. Given intervals $[a, b]$, $[c, d]$ we define the interval

$$[a, b] - [c, d] = [a - d, b - c].$$

Let $\{I_n\}$ be a sequence of finite sets of non-negative integers. Let

$$Q(n) = \Big| \bigcup_{i=1}^{n} (I_n - I_i) \Big|.$$

Notice that $Q(n) \leq \sum_{i=1}^{n} |I_n - I_i|$, and that equality occurs if and only if all sets $I_n - I_i$ are pairwise disjoint.

Definition 1.1. We say that the *strong sweeping out property* holds for operators $A_n$ if and only if for every $\varepsilon > 0$ there is a set $B \in \mathcal{B}$ with $\mu(B) < \varepsilon$ such that

$$\limsup_{n \to \infty} A_n 1_B(x) = 1 \quad \text{for a.e. } x,$$
$$\liminf_{n \to \infty} A_n 1_B(x) = 0 \quad \text{for a.e. } x.$$

We will need the following theorem and remark. They are Theorem 2.5 and Remark 2.6 respectively in [4].

Theorem 1.2.

(a) *Let $\{I_n\}$ be a sequence of finite sets of non-negative integers. If*

$$C = \sup_n \frac{Q(n)}{|I_n|} < \infty \tag{3}$$

*then for any measure-preserving system $(X, \mathcal{B}, \mu, T)$, $f \in L^1$ and $\lambda > 0$, we have*

$$\mu(\sup_n |A_{I_n} f| > \lambda) \leq \frac{C}{\lambda} \|f\|_{L^1(X)}.$$

(b) *Let $\{I_n\}$ be a sequence of finite intervals of non-negative integers. Suppose*

$$\lim_{n \to \infty} \frac{Q(n)}{|I_n|} = \infty. \tag{4}$$

*Then in every non-atomic ergodic probability measure-preserving system* $(X, \mathcal{B}, \mu, T)$ *the operators* $A_{I_n}$ *have the strong sweeping out property.*

REMARK 1.3. In general, assumption (4) cannot be replaced by the weaker

$$\sup_n \frac{Q(n)}{|I_n|} = \infty. \tag{5}$$

However, if the sequence $\{|I_n|\}$ is increasing then property (5) suffices to deduce the strong sweeping out property.

In Example 2.7 in [4] it is stated that if $I_n = [n^2, n^2 + n)$ then the operators $A_{I_n}$ have the strong sweeping out property (this result also appears in [2]). Considering *subsequences* of $A_{I_n}$, the example shows that the subsequence $A_{I_{n_s}} f$ converges a.e. for $f \in L^1$, where $n_s = [2^{(1+\delta)^s}]$ for a positive $\delta$, but $n_s = 2^{s^t}$ for a fixed positive integer $t$ gives the subsequence $A_{I_{n_s}}$ that has the strong sweeping out property. All these statements are clear from results in this paper. See Example 2.9 and Remark 2.3.

There is an alternative approach to Theorem 1.2 that is different (at least formally) and which we will find useful here. Let $\Omega$ be an infinite collection of lattice points with positive second coordinates. Define

$$\Omega_\alpha = \{(z, s) : |z - y| \le \alpha(s - r) \text{ for some } (y, r) \in \Omega,\ (z, s) \text{ a lattice point}\}.$$

In other words, $\Omega_\alpha$ is the union of lattice points in cones with aperture $\alpha$ and vertex in $\Omega$. The cross section of $\Omega_\alpha$ at integer height $\lambda > 0$ is

$$\Omega_\alpha(\lambda) = \{k : (k, \lambda) \in \Omega_\alpha\}.$$

For an ergodic measure preserving point transformation $T$ on $(X, \mathcal{B}, \mu)$ define the maximal function associated with the set $\Omega$ by

$$M_\Omega f(x) = \sup_{(k,n)\in\Omega} \frac{1}{n} \sum_{j=k+1}^{k+n} |f(T^j x)|.$$

The next two theorems are from [2].

THEOREM 1.4.

(a) *Assume there exist constants* $B$ *and* $\alpha > 0$ *such that* $|\Omega_\alpha(\lambda)| \le B\lambda$ *for every integer* $\lambda > 0$; *then* $M_\Omega$ *is weak type* $(1, 1)$ *and strong type* $(p, p)$ *for* $1 < p \le \infty$.

(b) *If* $M_\Omega$ *is weak type* $(p, p)$ *for some* $p > 0$ *then for every* $\alpha > 0$ *there exists* $B_\alpha < \infty$ *such that for every integer* $\lambda > 0$ *we have* $|\Omega_\alpha(\lambda)| \le B_\alpha\lambda$.

The linear growth condition for $\Omega_\alpha(\cdot)$, i.e. the existence of $B_\alpha < \infty$ such that $|\Omega_\alpha(\lambda)| \le B_\alpha\lambda$ for every integer $\lambda$, is called the *cone condition.* It holds for a particular $\alpha > 0$ if and only if it holds for all $\alpha > 0$. So we may consider the aperture $\alpha = 1$. We say that $\Omega$ satisfies a *linear growth condition* if there exists a constant $B < \infty$ such that $|\Omega(\lambda)| \le B\lambda$ for every integer $\lambda$, where $\Omega(\lambda) = \Omega_1(\lambda)$.

THEOREM 1.5. *Let $\Omega = \{(n_k, l_k) : l_k \nearrow \infty\}$. If the linear growth condition on $|\Omega_\alpha(\lambda)|$ fails, then the operators*

$$A_k f = \frac{1}{l_k} \sum_{j=n_k+1}^{n_k+l_k} f \circ T^j$$

*have the strong sweeping out property.*

In all of the results above, there is a common feature that certain sequences of averages may not converge almost everywhere in general, but a sufficiently rapidly growing subsequence of these averages will be well behaved. The order of growth of this subsequence is typically hyperexponential. This is the phenomenon that we want to address here. We describe 1) classes of averaging operators for which this rate of growth can be seen to be necessary, and 2) other classes for which it is excessive despite all of the examples.

**2. The interval condition.** In this section we consider intervals $I_k = [m_k^s + 1, m_k^s + m_k^t]$, where $m_k$ is a non-decreasing sequence of integers and $s > t > 0$. The associated averages in this case have the form

$$A_k f = \frac{1}{m_k^t} \sum_{j=m_k^s+1}^{m_k^s+m_k^t} f \circ T^j.$$

They are the averages $A_k f$ in (1) with $(n_k, l_k) = (m_k^s, m_k^t)$. Notice that $|I_k| = m_k^t$,

$$I_k - I_j = [m_k^s - m_j^s - m_j^t + 1, m_k^s + m_k^t - m_j^s - 1],$$

and for $1 \le j \le k$,

$$m_k^t \le |I_k - I_j| = m_k^t + m_j^t - 1 \le 2m_k^t. \tag{6}$$

REMARK 2.1. The intervals $I_k - I_{j+1}$ and $I_k - I_j$ intersect if and only if

$$m_{j+1}^s - m_j^s - m_j^t \le m_k^t - 2.$$

We say that the intervals $I_k - I_l, \ldots, I_k - I_i$, $l > i$, are *linked* if the intervals $I_k - I_{j+1}$ and $I_k - I_j$ intersect for $j = i, \ldots, l-1$. The resulting interval

$$[m_k^s - m_l^s - m_l^t + 1, m_k^s + m_k^t - m_i^s - 1]$$

has length

$$L_{l,i} = m_k^t + m_l^s + m_l^t - m_i^s - 1.$$

In particular, if the intervals $I_k - I_1, \dots, I_k - I_l$ are linked then we obtain the interval

$$[m_k^s - m_l^s - m_l^t + 1, m_k^s + m_k^t - m_1^s - 1]$$

with the length

$$L_l = m_k^t + m_l^s + m_l^t - m_1^s - 1. \tag{7}$$

Here is a convergence result under a hyperexponential condition on the subsequence being chosen.

THEOREM 2.2. *Let $s > t$ be integers. If $m_k = 2^{2^k}$ then for every $f \in L^1(X)$ the sequence of moving averages*

$$A_k f = \frac{1}{m_k^t} \sum_{j=m_k^s+1}^{m_k^s+m_k^t} f \circ T^j$$

*satisfies the weak $(1,1)$ maximal inequality*

$$\mu(\sup_k |A_k f| > \lambda) \le \frac{C}{\lambda} \|f\|_{L^1(X)}.$$

*Proof.* The theorem will follow if we show that condition (3) of Theorem 1.2 is satisfied. Let $n$ be a natural number such that $s/t \le 2^n$. This is equivalent to saying that $m_{k-n}^s \le m_k^t$, where $k > n$. Then for $j = 1, \dots, k-n-1$,

$$m_{j+1}^s - m_j^s - m_j^t \le m_{j+1}^s - 2 \le m_{k-n}^s - 2 \le m_k^t - 2.$$

By Remark 2.1 the intervals $I_k - I_{k-n}, I_k - I_{k-n-1}, \dots, I_k - I_1$ are linked, and since $m_{k-n}^s \le m_k^t$, the resulting interval has the length

$$L_{k-n} = m_k^t + m_{k-n}^s + m_{k-n}^t - m_1^s - 1 \le 3m_k^t.$$

Now we estimate $Q(k)$ using (6):

$$\begin{aligned} Q(k) &\le |I_k - I_k| + \dots + |I_k - I_{k-n+1}| + L_{k-n} \\ &\le (n-1)|I_k - I_k| + L_{k-n} \le (2n+1)m_k^t. \end{aligned}$$

It is clear that condition (3) of Theorem 1.2 is satisfied. ∎

REMARK 2.3. It is easy to see that the previous theorem holds for $m_k = [a^{b^{p(k)}}]$ where $a, b > 1$ and $p(k)$ is a polynomial of degree $\ge 1$. The remaining results in this section are stated for $m_k = [2^{2^{u(k)}}]$, but they are also valid for $m_k = [a^{b^{u(k)}}]$ where $a, b > 1$.

LEMMA 2.4. *Suppose that $m_k = [2^{2^{u(k)}}]$, where $u(k)$ is non-decreasing, $u(k) = o(k)$, $u(k) \to \infty$. Let $p < 1$. Then for any integer $M$,*

$$\sup_k \frac{m_{k-M}}{m_k^p} = \infty. \tag{8}$$

*Proof.* Assume that there exists an integer $M$ such that for all $k$,

$$m_{k-M} \leq D m_k^p$$

for some constant $D$. For simplicity, we will use $C$ to denote different constants. Let $w(k) = 2^{u(k)}$. Then

$$2^{w(k-M)} - 1 \leq m_{k-M} \leq (D\, 2^{w(k)})^p = 2^{pw(k)+C}.$$

Consequently,

$$2^{w(k-M)}(1 - 2^{-w(k-M)}) \leq 2^{pw(k)+C}.$$

With $\gamma(k) = \log_2(1 - 2^{-w(k-M)})$ we have

$$w(k-M) + \gamma(k) \leq pw(k) + C.$$

For $a = \log_2(1/p) > 0$ this gives

$$2^{u(k-M)} + \gamma(k) \leq 2^{u(k)-a} + C,$$

$$2^{u(k-M)}(1 + \gamma(k) \cdot 2^{-u(k-M)}) \leq 2^{u(k)-a}(1 + C \cdot 2^{-u(k)+a}).$$

We rewrite it as

$$2^{u(k-M)+\alpha(k)} \leq 2^{u(k)-a+\beta(k)},$$

where

$$\alpha(k) = \log_2(1 + \gamma(k) \cdot 2^{-u(k-M)}), \qquad \beta(k) = \log_2(1 + C \cdot 2^{-u(k)+a}).$$

Now

$$u(k-M) + \alpha(k) \leq u(k) - a + \beta(k), \qquad u(k) \geq u(k-M) + a + \alpha(k) - \beta(k).$$

Note that $\alpha(k) \to 0,\ \beta(k) \to 0$ as $k \to \infty$. Choose $k'$ such that for all $k \geq k'$ we have $a + \alpha(k) - \beta(k) > a/2$. Then for all $k \geq k'$,

$$u(k) > u(k-M) + \frac{a}{2}.$$

Also,

$$u(k+M) > u(k) + \frac{a}{2}.$$

Then

$$u(k+2M) > u(k+M) + \frac{a}{2} > u(k) + 2\,\frac{a}{2} > 2\,\frac{a}{2}.$$

In general,

$$u(k+jM) > j\,\frac{a}{2}.$$

Taking $k = k' + jM$, $j = 1, 2, \ldots,$ we see that

$$\frac{u(k)}{k} \geq \frac{ja}{2(k'+jM)} \to \frac{a}{2M} > 0$$

as $j \to \infty$, i.e. as $k \to \infty$ along the arithmetic progression $k = k' + jM$. This contradicts our assumption that $u(k) = o(k)$. Therefore, (8) holds. ∎

Here is a companion result to Theorem 2.2 which shows that under suitable regularity the hyperexponential condition of Theorem 2.2 is necessary. The regularity assumption here addresses cases where $\{m_k\}$ is increasing rather slowly; other cases will be treated later.

THEOREM 2.5. *Let $t$ be an integer, $s = t + 1$. Let $m_k = [2^{2^{u(k)}}]$, where $u(k)$ is non-decreasing, $u(k) = o(k)$, $u(k) \to \infty$ and $m_{k+1} - m_k \le B$ for some constant $B$. Then the operators $A_k$ where*

$$A_k f = \frac{1}{m_k^t} \sum_{j=m_k^s+1}^{m_k^s+m_k^t} f \circ T^j$$

*have the strong sweeping out property.*

*Proof.* We need to show that

$$\sup_k \frac{Q(k)}{m_k^t} = \infty. \tag{9}$$

The statement will then follow from Remark 1.3.

For a given $k$, let $J(k) \subseteq \{1, \dots, k-1\}$ be the set such that the consecutive intervals $I_k - I_{j+1}$ and $I_k - I_j$ are disjoint for $j \in J(k)$. If $\{|J(k)| : k \in \mathbb{N}\}$ is unbounded then, by (6),

$$Q(k) \ge \sum_{j \in J(k)} |I_k - I_j| > |J(k)| m_k^t,$$

so (9) is satisfied.

Now suppose that $\{|J(k)| : k \in \mathbb{N}\}$ is bounded, i.e. there exists an integer $M$ such that $|J(k)| \le M$ for all $k$. This means that for each $k$ the intervals $I_k - I_j$, $j = 1, \dots, k$, form no more than $M + 1$ linked pieces. Fix $k$. Let

$$k \ge l_N \ge r_N > l_{N-1} \ge r_{N-1} > \cdots > l_1 \ge r_1 \ge 1$$

so that we have $N$ (where $N \le M + 1$) linked parts as follows: the intervals $I_k - I_{l_1}, \dots, I_k - I_{r_1}$ are linked with the total length

$$L_1 = m_k^t + m_{l_1}^s + m_{l_1}^t - m_{r_1}^s - 1;$$

the intervals $I_k - I_{l_2}, \dots, I_k - I_{r_2}$ are linked with the total length

$$L_2 = m_k^t + m_{l_2}^s + m_{l_2}^t - m_{r_2}^s - 1;$$

and so on, the intervals $I_k - I_{l_{N-1}}, \dots, I_k - I_{r_{N-1}}$ are linked with the total length

$$L_{N-1} = m_k^t + m_{l_{N-1}}^s + m_{l_{N-1}}^t - m_{r_{N-1}}^s - 1;$$

and finally, $I_k - I_{l_N}, \dots, I_k - I_{r_N}$ are linked with the total length

$$L_N = m_k^t + m_{l_N}^s + m_{l_N}^t - m_{r_N}^s - 1. \tag{10}$$

Now

$$\begin{aligned}Q(k) &\ge L_N + L_{N-1} + \cdots + L_2 + L_1\\ &= m_k^t + m_{l_N}^s + m_{l_N}^t - m_{r_N}^s\\ &\quad + m_k^t + m_{l_{N-1}}^s + m_{l_{N-1}}^t - m_{r_{N-1}}^s + \cdots\\ &\quad + m_k^t + m_{l_2}^s + m_{l_2}^t - m_{r_2}^s\\ &\quad + m_k^t + m_{l_1}^s + m_{l_1}^t - m_{r_1}^s - N.\end{aligned}$$

Note that $r_1 \le M$ and $k - M \le l_N \le k$ since $|J(k)| \le M$. We see that

$$Q(k) > m_{l_N}^s - (m_{r_N}^s - m_{l_{N-1}}^s) - (m_{r_{N-1}}^s - m_{l_{N-2}}^s) - \cdots - (m_{r_2}^s - m_{l_1}^s).$$

Then

$$\begin{aligned}(11)\qquad &\frac{Q(k)}{m_k^t} - \frac{m_{l_N}^s}{m_k^t}\\ &\qquad > -\frac{m_{r_N}^s - m_{l_{N-1}}^s}{m_k^t} - \frac{m_{r_{N-1}}^s - m_{l_{N-2}}^s}{m_k^t} - \cdots - \frac{m_{r_2}^s - m_{l_1}^s}{m_k^t}.\end{aligned}$$

To estimate the terms on the right hand side of (11) suppose we have $r, l \in \mathbb{N},\ l \le r \le l + M \le k$. By the assumption of the theorem, $m_r - m_l \le MB$, so by the binomial formula,

$$m_r^s - m_l^s \le (m_r - m_l) s m_r^{s-1} = s(m_r - m_l) m_r^t \le sMBm_r^t.$$

Therefore,

$$\frac{m_r^s - m_l^s}{m_k^t} < sMB\left(\frac{m_r}{m_k}\right)^t \le sMB.$$

There are at most $M$ terms in (11). So all these terms are bounded by the same constant that does not depend on $k$. Then

$$\frac{Q(k)}{m_k^t} > \frac{m_{l_N}^s}{m_k^t} - C.$$

The integer $l_N$ depends on $k$, and we write $l_N = l_{N(k)}$. Notice that $m_{k-M} \le m_{l_N(k)}$ for every $k$. Using Lemma 2.4 we see that

$$\sup_k \frac{Q(k)}{m_k^t} \ge \sup_k \frac{m_{l_{N(k)}}^s}{m_k^t} - C \ge \sup_k \frac{m_{k-M}^s}{m_k^t} - C = \infty. \ \blacksquare$$

Here is a more general version of the need for hyperexponential growth on the subsequence. We need this simple observation. Suppose we have two sets of cones, one with the set of vertices $\Omega = \{(n_k, l_k)\}$ and the other with the set of vertices $\Omega' = \{(n'_k, l_k)\}$. Suppose $n_{k+1} - n_k \le n'_{k+1} - n'_k$. Then at each level $\lambda$ the cross section $\Omega'(\lambda)$ is not smaller than the cross section $\Omega(\lambda)$. So if $\Omega$ fails the linear growth condition then $\Omega'$ also fails the linear growth condition. If $\Omega'$ satisfies the linear growth condition then so does $\Omega$. See also Lemma 3 in [2].

THEOREM 2.6. *Let $p$ and $t$ be integers, $p > t$. Let $m_k = [2^{2^{u(k)}}]$, where $u(k)$ is non-decreasing, $u(k) = o(k)$, $u(k) \to \infty$, $m_{k+1} - m_k \le B$ for some constant $B$. Then the operators*

$$A_k f = \frac{1}{m_k^t} \sum_{j=m_k^p+1}^{m_k^p+m_k^t} f \circ T^j$$

*have the strong sweeping out property.*

*Proof.* For $s = t+1$, $m_k = [2^{2^{u(k)}}]$, the averages $A_k$ have the strong sweeping out property by Theorem 2.5. Let $\Omega = \{(m_k^s, m_k^t) : k \in \mathbb{N}\}$ and $\Omega' = \{(m_k^p, m_k^t) : k \in \mathbb{N}\}$. By Theorem 1.4, $\Omega$ does not satisfy a linear growth condition. Taking $a = m_{k+1}$, $b = m_k$ and observing that $a^p - b^p \ge a^s - b^s$ we see from the comment above that for $\Omega'$ the linear growth condition also fails. Theorem 1.5 then gives the strong sweeping property for the averages $A_k$. ∎

EXAMPLE 2.7. The basic example of this result is when $u(k) = \log_2 \log_2 k$.

We can also establish the need for hyperexponential growth of the subsequence of alternative regularity assumptions as in the following. This result is meant to handle cases where $\{m_k\}$ is increasing rather quickly, in contrast with the previous results.

THEOREM 2.8. *Let $s$ and $t$ be integers, $s > t$. Let $m_k = [2^{2^{u(k)}}]$, where $u(k)$ is non-decreasing, $u(k) = o(k)$, $u(k) \to \infty$. If $2^{u(k+1)} - 2^{u(k)} \ge B$ eventually for some constant $B > 0$ then the operators*

$$A_k f = \frac{1}{m_k^t} \sum_{j=m_k^s+1}^{m_k^s+m_k^t} f \circ T^j$$

*have the strong sweeping out property.*

*Proof.* The theorem will follow from Remark 1.3 if we show that

$$\sup_k \frac{Q(k)}{m_k^t} = \infty. \tag{12}$$

We will use notations from the proof of Theorem 2.5. As in that proof, (12) holds if $\{|J(k)| : k \in \mathbb{N}\}$ is unbounded. So assume that $|J(k)| \le M$ for each $k$. Then from (10) it follows that for each $k$,

$$Q(k) \ge L_N = m_k^t + m_{l_N}^s + m_{l_N}^t - m_{r_N}^s - 1 \ge m_{l_N}^s - m_{r_N}^s.$$

Assume that for all $k$,

$$Q(k) \le C m_k^t$$

for some positive constant $C$. Then

$$m_{l_N}^s \le m_{r_N}^s + Cm_k^t.$$

Here $l_N = l_{N(k)}$, $r_N = r_{N(k)}$ depend on $k$. We will write $l(k) = l_{N(k)}$, $r(k) = r_{N(k)}$ and $w(k) = 2^{u(k)}$. There is no harm in assuming that $l(k) > r(k)$ and $r(k) \to \infty$ as $k \to \infty$. We have

$$m_{l(k)} \le m_{r(k)} + Cm_k^{t/s}.$$

Choose $E > 1$ such that $\log_2 E < B/2$. Let $\alpha = 1/E$, $\beta = 1 - \alpha$. Then

$$\alpha m_{l(k)} + \beta m_{l(k)} \le m_{r(k)} + Cm_k^{t/s}.$$

Note that $k - l(k) \le M$, so $m_{k-M} \le m_{l(k)}$. By Lemma 2.4 we must have $\beta m_{l(k)} \ge Cm_k^{t/s}$, and hence $\alpha m_{l(k)} \le m_{r(k)}$, for all $k$ in some infinite set $K$. For all $k \in K$ we have $m_{l(k)} \le Em_{r(k)}$. Then

$$2^{w(l(k))} - 1 \le E \cdot 2^{w(r(k))}.$$

So

$$2^{w(l(k))} \le 2^{w(r(k))}(E + 2^{-w(r(k))}).$$

Then

$$w(l(k)) - w(r(k)) \le \log_2(E + 2^{-w(r(k))}) \le B/2,$$

which contradicts the assumption $w(j+1) - w(j) \ge B$ of the theorem. ∎

Example 2.9. Suppose $m_k = [2^{p(k)}]$ or $m_k = [p(k)]$, where $p$ is a polynomial of degree at least 1. Then Theorem 2.8 (applied with $u(k) = \log_2 p(k)$ and $u(k) = \log_2 \log_2 p(k)$ respectively) implies that pointwise convergence of the averages $A_k f$ fails for $f$ the characteristic function of a measurable set. Notice that Theorem 2.5 does not apply here unless $m_k = [p(k)]$ and the degree of $p$ is one.

At the expense of some additional technicalities, the previous theorem can be strengthened.

Theorem 2.10. *Let $s$ and $t$ be integers, $s > t$. Let $m_k = [2^{2^{u(k)}}]$, where $u(k) = o(k), u(k)$ is non-decreasing, $u(k) \to \infty$. Suppose $2^{u(k)} - 2^{u(k-1)} \ge 10f(u(k))$ where $f(z) > 0$, $f(z) \searrow 0$ as $z \to \infty$. If there exists $0 < a < 1 - t/s$ such that $f(u(k)) \ge m_k^{-a}$ then the operators*

$$A_k f = \frac{1}{m_k^t} \sum_{j=m_k^s+1}^{m_k^s+m_k^t} f \circ T^j$$

*have the strong sweeping out property.*

*Proof.* The theorem will follow from Remark 1.3 if we show that

$$\sup_k \frac{Q(k)}{m_k^t} = \infty. \tag{13}$$

We will use notations from the proof of Theorem 2.5. As in that proof, (13) holds if $\{|J(k)| : k \in \mathbb{N}\}$ is unbounded. So assume that $|J(k)| \le M$ for each $k$. Then from (10) it follows that for each $k$,

$$Q(k) \ge L_N = m_k^t + m_{l_N}^s + m_{l_N}^t - m_{r_N}^s - 1 \ge m_{l_N}^s - m_{r_N}^s.$$

Here (as in Theorem 2.5) $l_N = l_{N(k)}$, $r_N = r_{N(k)}$ depend on $k$. We will write $l(k) = l_{N(k)}$, $r(k) = r_{N(k)}$ and $w(k) = 2^{u(k)}$. We may assume that $l(k) > r(k)$. Suppose that for all $k$,

$$Q(k) \le Cm_k^t$$

for some positive constant $C$. Then

$$m_{l(k)}^s \le m_{r(k)}^s + Cm_k^t. \tag{14}$$

Let $E_k = e^{3f(u(k))}$, $\alpha_k = 1/E_k$, $\beta_k = 1 - \alpha_k$, $p = t/s + a < 1$.

1) We shall show that eventually

$$\frac{1}{\beta_k} = \frac{E_k}{E_k - 1} \le m_k^a. \tag{15}$$

Indeed, $m_k^a \le 2m_k^a - 2$, so using the fact that

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1, \tag{$*$}$$

and hence eventually $x \ge \frac{2}{3}\log(1+x)$, we see that

$$f(u(k)) \ge m_k^{-a} \ge \frac{1}{2} \cdot \frac{1}{m_k^a - 1} \ge \frac{1}{2} \cdot \frac{2}{3} \log\left(1 + \frac{1}{m_k^a - 1}\right).$$

Since $3f(u(k)) \ge \log(1 + 1/(m_k^a - 1))$, we have $E_k - 1 \ge 1/(m_k^a - 1)$, and then

$$\frac{1}{\beta_k} = \frac{E_k}{E_k - 1} = 1 + \frac{1}{E_k - 1} \le m_k^a.$$

2) By Lemma 2.4, $m_{k-M} > 3m_k^p > 3m_k^a$ eventually. Therefore,

$$2^{w(l(k))} \ge m_{l(k)} \ge m_{k-M} > 3m_k^a > 2m_k^a + 1.$$

Then $2^{w(l(k))} - 1 > 2m_k^a$, so

$$\frac{2}{2^{w(l(k))} - 1} < \frac{1}{m_k^a}.$$

Let $\gamma(k) = (2^{w(l(k))} - 1)2^{-w(l(k))}$. Using again $(*)$ we have

$$-\log_2 \gamma(k) = \log_2\left(1 + \frac{1}{2^{w(l(k))} - 1}\right) < \frac{2}{2^{w(l(k))} - 1} < m_k^{-a} \le f(u(k)).$$

So

$$-\log_2 \gamma(k) < f(u(k)). \tag{16}$$

3) Note that $p = t/s + a < 1$. Using Lemma 2.4 and taking an appropriate subsequence if needed, we may assume that

$$\lim_{k\to\infty} \frac{m_{k-M}}{m_k^p} = \infty.$$

Recall that $k - M \le l(k) \le k$. So we also have

$$\lim_{k\to\infty} \frac{m_{l(k)}}{m_k^p} = \infty.$$

Then for all $k \ge k_1$ (for some $k_1$), $m_{l(k)} \ge C m_k^p$, i.e. by (15),

$$\beta_k m_{l(k)} \ge C m_k^{t/s}.$$

Since $\alpha_k + \beta_k = 1$, from (14) we see that for all $k \ge k_1$,

$$\alpha_k m_{l(k)} \le m_{r(k)}.$$

Then

$$2^{w(l(k))} - 1 \le m_{l(k)} \le 2^{w(r(k)) - \log_2 \alpha_k},$$

that is,

$$w(l(k)) + \log_2 \gamma(k) \le w(r(k)) - \log_2 \alpha_k.$$

Then by (16) and the definition of $\alpha_k$,

$$w(l(k)) - w(r(k)) \le -\log_2 \gamma(k) - \log_2 \alpha_k < 7f(u(k)).$$

On the other hand, by the assumptions of the theorem,

$$w(l(k)) - w(r(k)) \ge 10 f(u(l(k))) \ge 10 f(u(k)).$$

The contradiction shows that we must have (13). ∎

EXAMPLE 2.11.

(a) Consider $u(k) = o(k)$ where $u(k) = \log_2(\log_2 k + \sqrt{k})$. Then $m_k = k[2^{\sqrt{k}}]$. Note that $2^{u(k)} - 2^{u(k-1)} \to 0$ and $m_{k+1} - m_k \to \infty$ as $k \to \infty$. We cannot apply Theorem 2.5 or 2.8, but it is easy to see that the conditions of Theorem 2.10 are satisfied. Hence, the averages $A_k$ have the strong sweeping out property.

(b) Let $u(k) = \log_2 \sqrt{k}$, or more generally, $u(k) = \log_2 k^\alpha$ for some $0 < \alpha < 1$. In this case $m_k = [2^{k^\alpha}]$. Note that $2^{u(k)} - 2^{u(k-1)} \to 0$ and $m_{k+1} - m_k \to \infty$ as $k \to \infty$. Again, while Theorems 2.5 and 2.8 are not applicable, we may use Theorem 2.10 to conclude that the averages $A_k$ have the strong sweeping out property.

In the previous theorems we needed some extra regularity requirements in addition to the conditions that $u(k) = o(k)$, $u(k)$ is non-decreasing, and $u(k) \to \infty$ in order to get the strong sweeping out property for $A_k$. The next example shows that in general we do not get the strong sweeping out property without additional requirements of some sort.

EXAMPLE 2.12. Let $s = 2$, $t = 1$. Define $u(k) = n$ whenever $2^n+1 \le k \le 2^{n+1}$, $n = 1, 2, \ldots$. Then $u(k) = o(k)$, $u(k)$ is non-decreasing, $u(k) \to \infty$.

Fix $k$. Say, $2^n + 1 \le k \le 2^{n+1}$, so $u(k) = n$. For any $1 \le i \le n$ the intervals $I_k - I_j$ and $I_k - I_t$ are equal for $2^i + 1 \le j,\ t \le 2^{i+1}$. Let $1 \le i < n$. If $j = 2^i$, i.e. $j + 1 = 2^i + 1$, then $u(j + 1) = i$ and

$$m_{j+1}^2 - m_j^2 - m_j \le m_{j+1}^2 - 2 = 2^{2^{i+1}} - 2 \le 2^{2^n} - 2 = m_k - 2.$$

By Remark 2.1 the intervals $I_k - I_{j+1}$ and $I_k - I_j$ intersect. Let $l = 2^n$, so $m_l^2 = m_k$. Notice that all intervals $I_k - I_l, \ldots, I_k - I_1$ are linked with the total length, given by (7),

$$L = m_k + m_l^2 + m_l - m_1^2 - 1 < 3m_k.$$

Then

$$\begin{aligned} Q(k) = \Big|\bigcup_{j=1}^{k}(I_k - I_j)\Big| &\le \Big|\bigcup_{j=1}^{l}(I_k - I_j)\Big| + \Big|\bigcup_{j=l+1}^{k}(I_k - I_j)\Big| \\ &\le 3m_k + |I_k - I_k| \le 3m_k + 2m_k = 5m_k. \end{aligned}$$

Hence,

$$\sup_k \frac{Q(k)}{|I_k|} = \sup_k \frac{Q(k)}{m_k} \le 5,$$

and by Theorem 1.2 the sequence $\{A_k f\}$ of moving averages satisfies the weak $(1, 1)$ maximal inequality.

**3. The cone condition.** There are several basic properties of subsequence results for moving averages that need to be put in the record. These all relate to the use of Theorem 1.4; they give a general framework in which to view the technical estimates in the previous section. The arguments given here also show how to work effectively with the cone condition. See [2] for additional information and related results. According to the discussion in the introduction, we may consider cones with aperture 1, and we denote $\Omega_1(\lambda)$ by $\Omega(\lambda)$.

Two cones with vertices $(v_i, l_i)$ and $(v_j, l_j)$ where $v_i \le v_j$ are *disjoint at the level* $\lambda$ if and only if

$$v_j - v_i + l_j + l_i \ge 2\lambda. \tag{17}$$

The cross section of the cone with vertex $(v_i, l_i)$ at the level $\lambda > l_i$ is $2(\lambda - l_i)$.

We consider the averages

$$A_k f = \frac{1}{l_k} \sum_{i=v_k+1}^{v_k+l_k} f \circ T^i.$$

PROPOSITION 3.1. *Given a sequence $l_1 < l_2 < \cdots$ there exists $\{v_n\}$ such that the linear growth condition for $\Omega = \{(v_k, l_k) : k \in \mathbb{N}\}$ fails.*

*Proof.* We take any positive $v_1 \le v_2$ and construct $v_n$'s in dyadic blocks. Namely, for each $k \in \mathbb{N}$ let

$$v_i = v_{i-1} + 2\lambda_k$$

for $i = 2^k + 1, \dots, 2^{k+1}$ where

$$\lambda_k = l^2_{2^{k+1}}.$$

Without loss of generality we may assume that $l_1 \ge 2$, so $l_i^2 - l_i \ge \frac{1}{2} l_i^2$ for all $i$. Since $v_i - v_{i-1} \ge 2\lambda_k$ for $i = 2^k + 1, \dots, 2^{k+1}$, from (17) we see that the cones with vertices $(v_i, l_i)$ are all disjoint for $i = 2^k + 1, \dots, 2^{k+1}$ at the level $\lambda_k$. Note that $\lambda_k > l_i$ for $i = 2^k + 1, \dots, 2^{k+1}$. Then

$$\begin{aligned}\frac{\Omega(\lambda_k)}{2} &\ge \sum_{i=2^k+1}^{2^{k+1}} (\lambda_k - l_i) \ge 2^k(\lambda_k - l_{2^{k+1}}) = 2^k(l^2_{2^{k+1}} - l_{2^{k+1}}) \\ &\ge 2^{k-1} l^2_{2^{k+1}} = 2^{k-1}\lambda_k.\end{aligned}$$

Hence,

$$\frac{\Omega(\lambda_k)}{\lambda_k} \ge 2^k.$$

It is clear that the linear growth condition fails for $\Omega$. ∎

REMARK 3.2. This result says that no matter how fast $\{l_k\}$ grows, there can be $\{v_k\}$ such that $\{(v_k, l_k)\}$ does not satisfy the cone condition.

PROPOSITION 3.3. *Given $v_k \to \infty$ there exists $l_k \to \infty$ (not necessarily strictly increasing) such that the linear growth condition fails for*

$$\Omega = \{(v_k, l_k) : k \in \mathbb{N}\}.$$

*Proof.* Let $M_j \nearrow \infty$ be a sequence. We construct the sequence $l_k$ in blocks that correspond to the sequence $M_j$.

Pick $v_{k_0} > 0$. For $i = 1, \dots, M_1$ choose $v_{k_i}$ such that $v_{k_i} - v_{k_{i-1}} \ge 4M_1$. Define $l_n$'s in the block corresponding to $M_1$ by $l_n = M_1$ for $n = 1, \dots, k_{M_1}$.

For $i = M_1 + 1, \dots, M_1 + M_2$ choose $v_{k_i}$ such that $v_{k_i} - v_{k_{i-1}} \ge 4M_2$. Define $l_n$'s in the block corresponding to $M_2$ by $l_n = M_2$ for $n = k_{M_1+1}, k_{M_1+1} + 1, \dots, k_{M_1+M_2}$.

In general, let $j \ge 3$. For $i = M_1 + \dots + M_{j-1} + 1, \dots, M_1 + \dots + M_j$ choose $v_{k_i}$ such that $v_{k_i} - v_{k_{i-1}} \ge 4M_j$. Define $l_n = M_j$ for $n = k_{M_1+\dots+M_{j-1}+1}$, $k_{M_1+\dots+M_{j-1}+1} + 1, \dots, k_{M_1+\dots+M_j}$.

For $\lambda = 2M_j$ we have, for $i = M_1 + \dots + M_{j-1} + 1, \dots, M_1 + \dots + M_j$,

$$v_{k_i} - v_{k_{i-1}} \ge 4M_j > 2\lambda - l_{k_i} - l_{k_{i-1}},$$

which by (17) means that the cones with vertices $(v_{k_i}, l_{k_i})$ are disjoint for $i = M_1 + \dots + M_{j-1} + 1, \dots, M_1 + \dots + M_j$. At the level $\lambda = 2M_j$ we have

$M_j$ disjoint cones, each contributing $2(\lambda - l_{k_i}) = 2M_j$ to $\Omega(\lambda)$. Then

$$\frac{\Omega(\lambda)}{2} \geq M_j^2.$$

So

$$\frac{\Omega(\lambda)}{\lambda} \geq \frac{2M_j^2}{2M_j} = M_j = \frac{\lambda}{2}.$$

It is obvious that the linear growth condition for $\Omega = \{(v_k, l_k) : k \in \mathbb{N}\}$ fails. ■

Remark 3.4. This result says that once one moves away from the best scenario where $v_k = 0$ for all $k$, then one cannot guarantee that $\{(v_k, l_k)\}$ satisfies the cone condition by just assuming that $l_k \to \infty$.

Here is an obvious fact, but one that is useful nonetheless.

Lemma 3.5. *Suppose we are given $\{v_k\}$. Then for any $\{l_k\}$ such that $v_k/l_k$ is bounded, if $l_k \to \infty$, then the averages*

$$A_k f = \frac{1}{l_k} \sum_{i=v_k+1}^{v_k+l_k} f \circ T^i$$

*converge a.e. for every $f \in L^1$ to*

$$I(x) = \lim_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} f \circ T^i.$$

*Proof.* Since $\{v_k/l_k\}$ is bounded, the maximal function of $\{A_k f\}$ is finite a.e. and this is sufficient for the result to follow since $l_k \to \infty$. More directly,

$$\left|\frac{1}{l_k} \sum_{i=v_k+1}^{v_k+l_k} f \circ T^i - I(x)\right|$$

$$= \left|\frac{v_k + l_k}{l_k} \cdot \frac{1}{v_k + l_k} \sum_{i=1}^{v_k+l_k} f \circ T^i - \frac{v_k}{l_k} \cdot \frac{1}{v_k} \sum_{i=1}^{v_k} f \circ T^i - \frac{v_k + l_k}{l_k} I(x) + \frac{v_k}{l_k} I(x)\right|$$

$$\leq \frac{v_k + l_k}{l_k}\left|\frac{1}{v_k + l_k} \sum_{i=1}^{v_k+l_k} f \circ T^i - I(x)\right| + \frac{v_k}{l_k}\left|\frac{1}{v_k} \sum_{i=1}^{v_k} f \circ T^i - I(x)\right|.$$

Say $v_k/l_k \leq b$. For every $\varepsilon > 0$ the first term in the last line is no larger than $(b+1)\varepsilon$ if $l_k$ is large enough. Also, the second term is no larger than $b\varepsilon$ for large enough $k$. This follows fairly easily from $v_k/l_k$ being bounded and $l_k \to \infty$ even though $\{v_k\}$ is not assumed to be bounded nor itself going to infinity. ■

Now we can easily see that for any $v_k \to \infty$, even though there may be $\{l_k\}$ for which $\{(v_k, l_k)\}$ do not satisfy the cone condition, there will be some $\{l_k\}$ for which they do.

COROLLARY 3.6. *Given $\{v_k\}$ there exists a rate $L_k$ such that for any $\{l_k\}$ with $l_k \geq L_k$ the averages*

$$A_k f = \frac{1}{l_k} \sum_{i=v_k+1}^{v_k+l_k} f \circ T^i$$

*converge a.e. for every $f \in L^1$.*

*Proof.* Take $L_k = \max\{k, v_k\}$. Then if $l_k \geq L_k$, we have $v_k/l_k$ bounded and $l_k \to \infty$. Now apply Lemma 3.5. ∎

EXAMPLE 3.7. Let $v_k = k$. If $l_k$ is strictly increasing then

$$l_k \geq (k-1) + l_1 \geq k - 1 \geq \frac{1}{2}\, k.$$

By Lemma 3.5 the averages

$$A_k f = \frac{1}{l_k} \sum_{i=v_k+1}^{v_k+l_k} f \circ T^i$$

converge a.e. for every $f \in L^1$. This shows why in Proposition 3.3 it may not be possible to have $\{l_k\}$ strictly increasing as well.

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