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ON THE LONG-TIME BEHAVIOUR OF SOLUTIONS OF THE p-LAPLACIAN PARABOLIC SYSTEM

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Abstract. Convergence of global solutions to stationary solutions for a class of degenerate parabolic systems related to the p-Laplacian operator is proved. A similar result is obtained for a variable exponent p. In the case of p constant, the convergence is proved to be $\mathcal{C}^1_{\mathrm{loc}}$, and in the variable exponent case, L^2 and $W^{1,p(x)}$-weak.

1. Notation and the main result. We study global solutions of the system

$$u_t - \Delta_p u + \partial_2 f(x,u) = 0, \qquad u:\ [0,\infty)\times\Omega\to\mathbb{R}^m, \tag{1}$$

in the following setting:

(A1) in the space $C^{0,\alpha}([0,\infty), w + W_0^{1,p}(\Omega,\mathbb{R}^m))$,
(A2) for $p\ge 2$,
(A3) with Ω a domain in $\mathbb{R}^N$ (i.e. Ω open, bounded and connected).

We assume additionally that

(A4) for all t we have $u_t(t,\cdot)\in L^q(\Omega,\mathbb{R}^m)$, $q=(p^*)'=Np/(Np-N+p)$,

and that the following assumptions on f, Ω, w hold:

(A5) f is a Carathéodory function such that $\partial_2 f(x,y)$ (the derivative with respect to the second variable) exists a.e. in Ω. Moreover,
- (weak convexity) $\forall_{x\in\Omega}\langle\partial_2 f(x,y_1)-\partial_2 f(x,y_2), y_1-y_2\rangle\ge 0$,
- (growth conditions)

$$|f(x,y)|\le C|y|^q, \qquad |\partial_2 f(x,y)|\le C|y|^{q-1},$$

(A6) $\partial\Omega\in\mathcal{C}^1$,
(A7) $w\in W^{1,p}(\Omega,\mathbb{R}^m)$.

In the following, the corresponding spacewise weak formulation of (1) is considered: for any fixed t,

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$$\langle u_t, \phi\rangle_{L^2} + M(u)(\phi) + \int_\Omega \langle \partial_2 f(x,u), \phi\rangle = 0 \quad \forall_{\phi\in W_0^{1,p}(\Omega,\mathbb{R}^m)}, \tag{2}$$

with

$$M(u)(\phi) = \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\phi\rangle.$$

We use the L^2-scalar product notation for simplicity. Note that $\langle u_t, \phi\rangle$ is finite and well-defined even though we do not assume $u_t \in L^2$. However, the L^2-estimates for u_t have been proved by Alt and Luckhaus ([2]).

We shall use the functionals

$$E_A(u) = \int_A |\nabla u|^p, \qquad E(u) = E_\Omega(u),$$

and the variational functional related to (1),

$$\mathcal{E}(u) = \frac{1}{p} E(u) + \int_\Omega f(x,u). \tag{3}$$

In order to ensure the existence of minima of $\mathcal{E}$, we suppose that

(A8) the functional $\mathcal{E}(u)$ is coercive, that is,

$$\lim_{\|u\|_{W^{1,p}(\Omega,\mathbb{R}^m)}\to\infty} \mathcal{E}(u) = \infty$$

(see e.g. [7, Theorem 4.6]). By convexity of $\mathcal{E}(\cdot)$, the minimum in $w + W_0^{1,p}$ is unique and we shall denote it by $u_0(x)$. We shall write

$$\Omega_0 = \{x \in \Omega : |\nabla u_0(x)| = 0\}, \qquad \Omega_\varepsilon = \{x\in\Omega : |\nabla u_0(x)| < \varepsilon\}.$$

Long-time behaviour of degenerate parabolic equations has been widely studied, the p-Laplacian equation being a model example. One should mention in particular the results of Lieberman [13], who proved that, for zero boundary data and in the scalar case ($m = 1$), the solutions of (1) are bounded in L^∞. The paper of S. Kamin and J. L. Vázquez [9] gives a detailed description of the asymptotics of positive solutions of the p-Laplacian equation; Del Pino and Dolbeault [4] establish, for non-negative initial data and under some assumptions on p, not only the convergence to stationary solutions, but also an estimate on the convergence rate. Other results on the asymptotics of solutions in unbounded domains or in the whole space are due to Lee, Petrosyan and Vázquez [12], and Iagar and Vázquez [8].

These results, however, are essentially of single-equation type, as they deal with non-negative solutions, or with non-negative initial data.

The aim of this paper is to show that, in the system ($m \geq 1$) case, very simple and elementary arguments allow us to establish C^1-convergence of solutions global (in time) to stationary solutions, at the expense, however, of rather strong conditions on the associated variational functional. The existence of solutions under our assumptions has been proved by Alt and

Luckhaus [2]; similar questions have also been addressed by DiBenedetto and Herrero [6], Kuusi and Parviainen [11] and others.

The main result of the paper is the following theorem:

THEOREM 1. *Let* u *be a global solution of* (2), *satisfying the assumptions* (A1)–(A8) *given above. Then the solution* u *and its gradient* ∇u *converge almost uniformly in* Ω *to* u_0 *and* ∇u_0, *respectively*, *i.e.*

$$\forall_{\Omega' \Subset \Omega} \quad \|u(t,\cdot) - u_0(\cdot)\|_{\mathcal{C}^1(\Omega',\mathbb{R}^m)} \xrightarrow{t\to\infty} 0, \tag{4}$$

where u_0 *is the* (*unique*) *minimum of the energy functional* $\mathcal{E}(\cdot)$ *given by* (3).

REMARK. From time to time we shall refer to somewhat stronger hypotheses: $\partial\Omega \in \mathcal{C}^{1,\tilde{\alpha}}$ and $w \in \mathcal{C}^{1,\tilde{\alpha}}(\Omega,\mathbb{R}^m)$ for some $\tilde{\alpha} \in (0,1)$ (*strong hypotheses*).

THEOREM 2. *If the* strong hypotheses *hold true*, *the convergence of* u *is uniform in* Ω.

The proofs of Theorems 1 and 2 are virtually the same, the difference lying in the known regularity results for the solution, which are stated below. Therefore we shall prove both theorems simultaneously, stating explicitly where the strong hypotheses are assumed.

We shall use the following known facts on regularity of u and u_0:

- Regularity of u_0 ([14]):

  $$\forall_{\Omega' \Subset \Omega}\ \exists_{\alpha\in(0,1)} \quad u_0 \in \mathcal{C}^{1,\alpha}(\Omega',\mathbb{R}^m)$$

  If the *strong hypotheses* hold, we have $u_0 \in \mathcal{C}^{1,\alpha}(\Omega,\mathbb{R}^m)$.
- Interior regularity of u ([5]):

  $$\forall_{\Omega' \Subset \Omega}\ \exists_{\alpha\in(0,1)} \quad u(t,\cdot) \in \mathcal{C}^{1,\alpha}(\Omega',\mathbb{R}^m)$$

  and the Hölder constant of ∇u is bounded independently of t for all $t \ge t_0 > 0$.

  In the case of the *strong hypotheses* we obtain full regularity of $u(t,\cdot)$ ([5]):

  $$\exists_{\alpha\in(0,1)} \quad u(t,\cdot) \in \mathcal{C}^{0,\alpha}(\Omega,\mathbb{R}^m)$$

  with the Hölder constant of u bounded independently of t for all $t \ge 0$.

REMARK. The system (2) defines a gradient flow of $\mathcal{E}$ in $L^2(\Omega,\mathbb{R}^m)$. In particular,

$$\frac{d\mathcal{E}(u)}{dt} = M(u)(u_t) + \int_\Omega \langle \partial_2 f(x,u), u_t\rangle = -\|u_t\|^2_{L^2} \le 0, \tag{5}$$

so the energy $\mathcal{E}(u)$ does not increase with t.

2. Case of $f \equiv 0$. Let us start with a simpler "toy" case of

$$u_t - \Delta_p u = 0. \tag{6}$$

In this case, we can easily trace the main ideas of the general proof.

By Young's inequality, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u-u_0\|^2_{L^2} &= \langle u_t, u-u_0\rangle_{L^2} = -M(u)(u-u_0) \\ &= -M(u)(u) + M(u)(u_0) \\ &= -E(u) + \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_0\rangle \\ &\le -E(u) + \frac{p-1}{p}\,E(u) + \frac{1}{p}\,E(u_0) \\ &= -\frac{1}{p}\,(E(u)-E(u_0)) \le 0. \end{aligned} \tag{7}$$

Thus $\|u-u_0\|^2_{L^2}$ is decreasing with $t\to\infty$, as u_0 is a minimizer for E. On the other hand, $\|u-u_0\|^2_{L^2}$ is bounded from below, so there must be a sequence t_i such that

$$\frac{d}{dt}\bigg|_{t=t_i}\frac{1}{2}\|u-u_0\|^2_{L^2} \xrightarrow{i\to\infty} 0. \tag{8}$$

To simplify the notation we shall write $u_i(\cdot)$ for $u(t_i,\cdot)$.

Next, notice that, by Bernoulli's inequality,

$$\begin{aligned} E(u)-E(u_0) &= \int_\Omega |\nabla u|^p - \int_\Omega |\nabla u_0|^p \\ &= \int_{\Omega_0} |\nabla u|^p + \int_{\Omega\setminus\Omega_0} [(|\nabla u_0|^2 + 2\langle\nabla u_0, \nabla(u-u_0)\rangle + |\nabla(u-u_0)|^2)^{p/2} - |\nabla u_0|^p] \\ &= E_{\Omega_0}(u-u_0) \\ &\quad + \int_{\Omega\setminus\Omega_0} |\nabla u_0|^p\left[\left(1+2\,\frac{\langle\nabla u_0,\nabla(u-u_0)\rangle}{|\nabla u_0|^2} + \frac{|\nabla(u-u_0)|^2}{|\nabla u_0|^2}\right)^{p/2} - 1\right] \\ &\ge E_{\Omega_0}(u-u_0) + p\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}\langle\nabla u_0, \nabla(u-u_0)\rangle \\ &\quad + \frac{p}{2}\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}|\nabla(u-u_0)|^2 \\ &= E_{\Omega_0}(u-u_0) + pM(u_0)(u-u_0) + \frac{p}{2}\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}|\nabla(u-u_0)|^2. \end{aligned} \tag{9}$$

On the other hand, u_0 is a minimizer for E, so $M(u_0)(\cdot) \equiv 0$, and we get

$$E(u) - E(u_0) \geq E_{\Omega_0}(u - u_0) + \frac{p}{2} \int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2} |\nabla(u - u_0)|^2. \tag{10}$$

By (8) we know that $E(u_i) \to E(u_0)$ as $i \to \infty$. We have, however, proved that $E(u(t,\cdot))$ is non-increasing, and therefore

$$E(u(t,\cdot)) \xrightarrow{t\to\infty} E(u_0(\cdot)). \tag{11}$$

Let us denote the right hand side of (10) by $V(u - u_0)$. By (11), we have $V(u(t,\cdot) - u_0(\cdot)) \to 0$ as $t \to \infty$. In the next step we shall prove the following lemma.

Lemma 1.

$$V(u(t,\cdot) - u_0(\cdot)) \xrightarrow{t\to\infty} 0 \ \Rightarrow\ \forall_{\Omega' \Subset \Omega} \ \sup_{\Omega'} |\nabla(u - u_0)| \xrightarrow{t\to\infty} 0.$$

Proof. Suppose otherwise. After passing to a subsequence $t_i \to \infty$, we have, for a fixed Ω' and some $b > 0$,

$$\sup_{\Omega'} |\nabla(u_i - u_0)| > b.$$

Suppose that, for every i, $|\nabla(u_i(\xi_i) - u_0(\xi_i))| \geq b$. Again, by passing to a subsequence, we may suppose that $\xi_i \to \xi_\infty \in \overline{\Omega'}$ as $i \to \infty$.

The functions $u_i - u_0$ are all in $\mathcal{C}^{1,\alpha}$, with the Hölder constant of the gradient bounded by some G, independently of i. Thus, there exists $\varrho = \varrho(\alpha, G, b, N)$ such that $|\nabla(u_i - u_0)| > \frac{1}{2}b$ on $B_\varrho(\xi_i) \cap \Omega'$ for all i. For i sufficiently large we also have $|\nabla(u_i - u_0)| > \frac{1}{2}b$ on $S := B_{\varrho/2}(\xi_\infty) \cap \Omega'$.

We may always suppose (possibly after enlarging Ω') that $\overline{\Omega'} = \overline{\operatorname{int} \Omega'}$. Moreover, we may take ε small enough to have $\mu(\Omega_\varepsilon \setminus \Omega_0) < \frac{1}{2}\mu(S)$. Then either $S \cap \Omega_0$ or $S \cap \Omega \setminus \Omega_\varepsilon$ is of non-zero Lebesgue measure. If $\mu(S \cap \Omega_0) > 0$, then $E_{\Omega_0}(u_i - u_0)$ cannot tend to zero (the integrand is bounded from below, independently of i), and if $\mu(S \cap \Omega \setminus \Omega_\varepsilon) > 0$, the second term in $V(u_i - u_0)$ cannot tend to zero. ■

Note that $\{u(t,\cdot)\}$ is a bounded set in $w + W_0^{1,p}(\Omega, \mathbb{R}^m)$, and thus, by the Alaoglu and Rellich–Kondrashov theorems, one can find a sequence $\{u_i(\cdot) = u(t_i,\cdot)\}$ with $t_i \to \infty$ such that

$$\begin{aligned} u_i &\rightharpoonup u_\infty && \text{weakly in } w + W_0^{1,p}(\Omega, \mathbb{R}^m), \\ u_i &\to u_\infty && \text{strongly in } L^2(\Omega, \mathbb{R}^m). \end{aligned}$$

By the weak convergence in $w + W_0^{1,p}(\Omega, \mathbb{R}^m)$,

$$M(u_\infty)(u_\infty - u_i) \xrightarrow{i\to\infty} 0. \tag{12}$$

But, using Young's inequality as in (7) and (11), we obtain

$$M(u_\infty)(u_\infty - u_i) \geq E(u_\infty) - E(u_i) \xrightarrow{i\to\infty} E(u_\infty) - E(u_0) \geq 0.$$

If $u_\infty \not\equiv u_0$, then, by the uniqueness of minimum of E, $E(u_\infty) - E(u_0) > 0$, which contradicts the convergence in (12).

This shows that $u(t_i,\cdot) \to u_0(\cdot)$ in $L^2(\Omega,\mathbb{R}^m)$ as $i \to \infty$, and the monotonicity of $\|u-u_0\|_{L^2}$ implies that this convergence holds, in fact, for $t \to \infty$:

$$\|u - u_0\|_{L^2} \xrightarrow{t\to\infty} 0.$$

Moreover, $u(t,\cdot)$ also converges weakly to $u_0(\cdot)$ in $w + W_0^{1,p}(\Omega,\mathbb{R}^m)$. Indeed, suppose otherwise. Then there exist a (weak-topology) neighbourhood U of u_0 and a sequence $t_i \to \infty$ such that $\{u(t_i,\cdot)\}_{i\in\mathbb{N}} \cap U = \emptyset$. On the other hand, $\{u(t_i,\cdot)\}$ is bounded in $w+W_0^{1,p}(\Omega,\mathbb{R}^m)$, hence we may choose a subsequence weakly convergent to some u_∞. The above reasoning shows that $u_\infty = u_0$, which is a contradiction.

Having established the L^2-convergence of u to u_0, we may now repeat the argument from Lemma 1: for a fixed $\Omega' \Subset \Omega$, the functions $u(t,\cdot) - u_0(\cdot)$ are Lipschitz, with the Lipschitz constant bounded independently of t (as $\nabla(u - u_0)$ converges uniformly to 0 in Ω', it is bounded independently of t). If $u - u_0$ does not converge uniformly to 0 in Ω', we may choose a sequence $u_i - u_0$ such that

$$\forall_i \quad \sup_{\Omega'} |u_i(\cdot) - u_0(\cdot)| \geq b > 0,$$

and thus a set $A \subset \Omega'$ of positive measure on which

$$\forall_{i>i_0} \forall_{x\in A} \quad |u_i(x) - u_0(x)| > b/2,$$

which contradicts the already proved L^2-convergence of u to u_0.

If the *strong hypotheses* hold true, we may use the fact that the functions $u(t,\cdot) - u_0(\cdot)$ are in fact Hölder continuous in Ω, with the Hölder constant independent of t, and apply the above argument for $\Omega' = \Omega$. This yields the uniform convergence of u to u_0.

This concludes the proof of Theorems 1 and 2 for $f \equiv 0$.

3. Non-homogeneous case. In this section we shall prove Theorems 1 and 2.

Proof. In this case, $u_0(\cdot)$ is a minimizer for $\mathcal{E}$, therefore

$$M(u_0)(\phi) + \int_\Omega \langle \partial_2 f(x,u_0), \phi\rangle = 0 \quad \forall_{\phi\in W_0^{1,p}(\Omega,\mathbb{R}^m)}. \tag{13}$$

Let us now calculate a counterpart of (7) in this case:

$$
\begin{aligned}
(14)\qquad \frac{d}{dt}\frac{1}{2}\|u-u_0\|_{L^2}^2 &= \langle u_t, u-u_0\rangle_{L^2}\\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle\\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle + M(u_0)(u-u_0)\\
&\quad + \int_\Omega \langle \partial_2 f(x,u_0), u-u_0\rangle\\
&= -E(u) + M(u)(u_0) + M(u_0)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u) - \partial_2 f(x,u_0), u-u_0\rangle\\
&\le -E(u) + M(u)(u_0) + M(u_0)(u-u_0).
\end{aligned}
$$

We shall continue this calculation in two ways. In the first one, we use Hölder's inequality, obtaining

$$
\begin{aligned}
(15)\qquad \frac{d}{dt}\frac{1}{2}\|u-u_0\|_{L^2}^2 &\le -E(u) + M(u)(u_0) + M(u_0)(u-u_0)\\
&= -E(u) + \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_0\rangle + \int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0, \nabla u\rangle - E(u_0)\\
&\le -E(u) + E(u)^{(p-1)/p}E(u_0)^{1/p} + E(u)^{1/p}E(u_0)^{(p-1)/p}\\
&= -(E(u)^{1/p} - E(u_0)^{1/p})(E(u)^{(p-1)/p} - E(u_0)^{(p-1)/p}) \le 0.
\end{aligned}
$$

Notice that in this case we do not necessarily have $E(u) \ge E(u_0)$, because u_0 is a minimizer for $\mathcal{E}$, not for E.

The second continuation of (14) uses Young's inequality:

$$
\begin{aligned}
(16)\qquad \frac{d}{dt}\frac{1}{2}\|u-u_0\|_{L^2}^2 &\le -E(u) + M(u)(u_0) + M(u_0)(u-u_0)\\
&= -E(u) + \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_0\rangle + M(u_0)(u-u_0)\\
&\le -E(u) + \int_\Omega |\nabla u|^{p-1}|\nabla u_0| + M(u_0)(u-u_0)\\
&\le -E(u) + \int_\Omega \left(\frac{p-1}{p}\,|\nabla u|^p + \frac{1}{p}\,|\nabla u_0|^p\right) + M(u_0)(u-u_0)\\
&= -\left(\frac{1}{p}\,E(u) - \frac{1}{p}\,E(u_0) - M(u_0)(u-u_0)\right).
\end{aligned}
$$

One may show that the above quantity is still non-positive by decomposing $M(u_0)(u-u_0)$ and using Young's inequality one more time.

Just as for $f \equiv 0$, there exists a sequence t_i such that $\frac{d}{dt}\big|_{t=t_i}\|u-u_0\|^2_{L^2} \to 0$ as $i \to \infty$. As before, we write $u_i(\cdot) := u(t_i, \cdot)$.

The inequality (15) implies that $E(u_i) \to E(u_0)$ as $i \to \infty$, and (16) together with (9) gives $V(u_i - u_0) \to 0$ as $i \to \infty$. Lemma 1 then shows that ∇u_i is almost uniformly convergent to ∇u_0 in Ω.

Next, we need to pass from the convergence of the sequence ∇u_i to the convergence for all t. First, we need to establish the convergence of $\mathcal{E}(u)$ to $\mathcal{E}(u_0)$ as $t \to \infty$. By monotonicity of $\mathcal{E}(u(t,\cdot))$ it is enough to prove it for the sequence t_i.

Of the two terms in $\mathcal{E}(u_i)$ we already know that $E(u_i) \to E(u_0)$ as $i \to \infty$. It remains to prove that

$$\int_\Omega [f(x, u(t_i, x)) - f(x, u_0(x))] \xrightarrow{i\to\infty} 0.$$

By convexity of f we have, for some $\theta \in (0,1)$ and $u_\theta = \theta u_i + (1-\theta)u_0$,

$$\begin{aligned}(17)\qquad \int_\Omega [f(x,u_i) - f(x,u_0)] &= \int_\Omega \langle \partial_2 f(x,u_\theta), u_i - u_0\rangle \\ &= \frac{1}{\theta}\int_\Omega \langle \partial_2 f(x,u_\theta), u_\theta - u_0\rangle \geq \frac{1}{\theta}\int_\Omega \langle \partial_2 f(x,u_0), u_\theta - u_0\rangle \\ &= \int_\Omega \langle \partial_2 f(x,u_0), u_i - u_0\rangle = -M(u_0)(u_i - u_0) \geq E(u_0) - E(u_i).\end{aligned}$$

Similarly

$$\begin{aligned}(18)\qquad \int_\Omega [f(x,u_i) - f(x,u_0)] &= \int_\Omega \langle \partial_2 f(x,u_\theta), u_i - u_0\rangle \\ &= \frac{1}{1-\theta}\int_\Omega \langle \partial_2 f(x,u_\theta), u_i - u_\theta\rangle \leq \frac{1}{1-\theta}\int_\Omega \langle \partial_2 f(x,u_i), u_i - u_\theta\rangle \\ &= \int_\Omega \langle \partial_2 f(x,u_i), u_i - u_0\rangle = -M(u_i)(u_i - u_0) - \langle u_t|_{t=t_i}, u_i - u_0\rangle_{L^2} \\ &\leq E(u_0) - E(u_i) - \frac{d}{dt}\bigg|_{t=t_i} \frac{1}{2}\|u-u_0\|^2_{L^2}.\end{aligned}$$

We see that both the lower (17) and upper (18) estimate for the integral $\int_\Omega [f(x,u_i)) - f(x,u_0)]$ tend, for this particular sequence t_i, to 0. This concludes the proof of the energy convergence.

Next, note that by (17) and (9),

$$\begin{aligned}\mathcal{E}(u) - \mathcal{E}(u_0) &= \frac{1}{p}E(u) - \frac{1}{p}E(u_0) + \int_\Omega [f(x,u) - f(x,u_0)] \\ &\geq \frac{1}{p}E(u) - \frac{1}{p}E(u_0) - M(u_0)(u - u_0) \\ &\geq V(u - u_0) \geq 0,\end{aligned}$$

and, applying Lemma 1, we obtain the almost uniform convergence of ∇u to ∇u_0.

We proceed as in the previous section. By the coercivity and the convergence of $\mathcal{E}$ (which implies the boundedness of $\{u(t,\cdot)\}_{t>0}$ in $W^{1,p}(\Omega,\mathbb{R}^m)$) we can choose a sequence $\{t_i\}$, $t_i\to\infty$, such that

$$\begin{aligned} u_i &\rightharpoonup u_\infty \quad \text{weakly in } w+W_0^{1,p}(\Omega,\mathbb{R}^m),\\ u_i &\to u_\infty \quad \text{strongly in } L^2(\Omega,\mathbb{R}^m); \end{aligned}$$

as before, $u_i(\cdot)=u(t_i,\cdot)$. By this weak convergence, the convexity of f (see (18)), and Young's inequality (as in (16)),

$$\begin{aligned} \mathcal{E}(u_\infty)-\mathcal{E}(u_i) &= E(u_\infty)-E(u_i)+\int_\Omega [f(x,u_\infty)-f(x,u_i)]\\ &\le M(u_\infty)(u_\infty-u_i)\\ &\quad+\int_\Omega \langle \partial_2 f(x,u_\infty),u_\infty-u_i\rangle \xrightarrow{i\to\infty} 0. \end{aligned} \tag{19}$$

On the other hand, $\mathcal{E}(u_\infty)-\mathcal{E}(u_i)\to\mathcal{E}(u_\infty)-\mathcal{E}(u_0)$ as $i\to\infty$.

If $u_\infty\not\equiv u_0$, then, by the uniqueness of minimum for $\mathcal{E}$, $\mathcal{E}(u_\infty)-\mathcal{E}(u_0)>0$, which gives a contradiction. Therefore $u_\infty\equiv u_0$ in $w+W_0^{1,p}(\Omega,\mathbb{R}^m)$.

The fact that $u(t_i,\cdot)$ converges strongly in $L^2(\Omega,\mathbb{R}^m)$ to $u_0(\cdot)$ together with the monotonicity of $\|u-u_0\|$ yields

$$u(t,\cdot)\xrightarrow{t\to\infty}u_0(\cdot)\quad \text{in } L^2(\Omega,\mathbb{R}^m). \tag{20}$$

This convergence, together with the fact that $u(t,\cdot)$ are Lipschitz with time-independent Lipschitz constant, gives us the almost uniform convergence of $u(t,\cdot)$ to $u_0(\cdot)$ (see previous section). If the *strong hypotheses* hold true, we may use the fact that $u(t,\cdot)-u_0(\cdot)$ are Hölder continuous on the whole Ω to prove that the convergence of $u(t,\cdot)$ to $u_0(\cdot)$ is, in fact, uniform. This ends the proof of Theorems 1 and 2. ∎

REMARK. The same argument as in the previous section gives us also the convergence

$$u(t,\cdot)\xrightarrow{t\to\infty}u_0(\cdot)\quad \text{weakly in } w+W_0^{1,p}(\Omega,\mathbb{R}^m).$$

4. Variable exponent case. In the last few years a counterpart of (1) for a variable exponent $p=p(x)$ has attracted more and more attention (see e.g. [1], [3]). A question arises: how much of the results of the previous sections can be proved in this case? If we assume that the exponent function $p(x)$ stays in the range of exponents dealt with in the previous section, most of the arguments, with some care, can be repeated.

Let $p(x)$ be a measurable, bounded function, $p_1>p(x)\ge 2$ (note that no regularity conditions on $p(x)$ are imposed), and

$$\langle u_t, \phi\rangle_{L^2} + M(u)(\phi) + \int_\Omega \langle \partial_2 f(x,u), \phi\rangle = 0 \quad \forall_{\phi\in W_0^{1,p(x)}(\Omega,\mathbb{R}^m)}. \tag{21}$$

This time

$$E(u) = \int_\Omega |\nabla u|^{p(x)}, \qquad M(u)(\phi) = \int_\Omega p(x)|\nabla u|^{p(x)-2}\langle \nabla u, \nabla\phi\rangle, \tag{22}$$

$$\mathcal{E}(u) = E(u) + \int_\Omega f(x,u). \tag{23}$$

The main difference between the variable exponent case and the preceding ones is that no partial regularity result for time-dependent solutions of (21) is known. Therefore the methods used for $p = \text{const}$ cannot yield any pointwise convergence. However, this regularity result (widely believed to be true, at least under sufficient continuity assumptions on $p(x)$) is the only missing detail, and were it proved, one could apply the same technique as for p constant, obtaining the same convergence result.

From now on, we assume that for a.e. x,

$$|f(x,y)| \le C|y|^{q(x)}, \qquad |\partial_2 f(x,y)| \le C|y|^{q(x)-1},$$
$$\langle \partial_2 f(x,y_1) - \partial_2 f(x,y_2), y_1 - y_2\rangle \ge 0$$

where $q(x) = (p(x)^*)' = p(x)N/(p(x)N - N + p(x))$. Moreover, as before, we suppose that $\mathcal{E}(\cdot)$ is coercive.

THEOREM 3. *With the above assumptions on $p(x)$ and $f(x,y)$, a global solution $u(t,x)$ of (21) in $w + W_0^{1,p(x)}(\Omega,\mathbb{R}^m)$ converges to a stationary solution $u_0(x)$ strongly in $L^2(\Omega,\mathbb{R}^m)$ and weakly in $W^{1,p(x)}(\Omega,\mathbb{R}^m)$.*

We repeat, with slight alterations, the calculation (14), using Young's inequality (note that only pointwise inequalities, and not the Hölder inequality, can be safely used in the $p(x)$ case):

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u-u_0\|_{L^2}^2 &= \langle u_t, u-u_0\rangle_{L^2} = -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle \\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle + M(u_0)(u-u_0) \\
&\quad + \int_\Omega \langle \partial_2 f(x,u_0), u-u_0\rangle \\
&= -M(u)(u-u_0) + M(u_0)(u) - \int_\Omega p(x)|\nabla u_0|^{p(x)} \\
&\quad - \int_\Omega \langle \partial_2 f(x,u) - \partial_2 f(x,u_0), u-u_0\rangle \\
&\le -M(u)(u-u_0) + \int_\Omega p(x)|\nabla u_0|^{p(x)-1}|\nabla u| - \int_\Omega p(x)|\nabla u_0|^{p(x)}
\end{aligned} \tag{24}$$

$$\leq -M(u)(u-u_0) + \int_\Omega (p(x)-1)|\nabla u_0|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} - \int_\Omega p(x)|\nabla u_0|^{p(x)}$$
$$= E(u) - E(u_0) - M(u)(u-u_0).$$

Applying Young's inequality once more, this time to the second term in $M(u)(u-u_0) = M(u)(u) - M(u)(u_0)$, proves that the right hand side of (24) is still non-positive. We thus get a sequence $t_i \to \infty$ such that

$$\frac{d}{dt}\bigg|_{t=t_i} \|u-u_0\|_{L^2} \xrightarrow{i\to\infty} 0,$$

in particular

$$E(u_i) - E(u_0) - M(u_i)(u_i-u_0) \xrightarrow{i\to\infty} 0,$$

where, as before, $u_i(\cdot) = u(t_i,\cdot)$.

This allows us to prove the energy convergence, by estimates similar to those in (18):

$$\begin{aligned}(25)\quad \mathcal{E}(u) - \mathcal{E}(u_0) &= E(u) - E(u_0) + \int_\Omega f(x,u) - \int_\Omega f(x,u_0)\\ &\leq E(u) - E(u_0) + \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle\\ &= E(u) - E(u_0) - M(u)(u-u_0) - \langle u_t, u-u_0\rangle_{L^2}\\ &= E(u) - E(u_0) - M(u)(u-u_0) - \frac{d}{dt}\|u-u_0\|_{L^2}.\end{aligned}$$

For the sequence $t = t_i$ the right hand side of (25) tends to 0 as $i \to \infty$, and $\mathcal{E}(u) - \mathcal{E}(u_0) \geq 0$. However, as in the previous sections, the energy $\mathcal{E}(u)$ is decreasing with t:

$$\frac{d}{dt}\mathcal{E}(u) = -M(u)(u_t) - \int_\Omega \langle \partial_2 f(x,u), u_t\rangle = -\|u_t\|_{L^2}^2.$$

Therefore, we have $\mathcal{E}(u) \to \mathcal{E}(u_0)$ as $t \to \infty$.

Now, we proceed as in the previous section. By coercivity of $\mathcal{E}$ and the convergence of $\mathcal{E}$ proved above, the set $\{u(t)\}_{t>0}$ is bounded in $w + W_0^{1,p(x)}(\Omega,\mathbb{R}^m)$, and thus in $W^{1,2}(\Omega,\mathbb{R}^m)$. By the Alaoglu and Rellich–Kondrashov theorems (see [10] for appropriate properties of variable exponent Sobolev spaces) we may choose a sequence $t_i \to \infty$ ($u_i(\cdot) = u(t_i,\cdot)$) such that

$$\begin{aligned} u_i &\rightharpoonup u_\infty \quad \text{weakly in } w + W_0^{1,p(x)}(\Omega,\mathbb{R}^m),\\ u_i &\to u_\infty \quad \text{strongly in } L^2(\Omega,\mathbb{R}^m).\end{aligned}$$

Using exactly the same calculation as in (19) we show that $u_\infty = u_0$. Monotonicity of $\|u-u_0\|_{L^2(\Omega)}$ ensures that the L^2-convergence holds for all t, while the argument from Section 2 ($\{u(t,\cdot)\}_{t>0}$ is precompact and every $W^{1,p(x)}$-weakly convergent sequence with $t \to \infty$ converges to u_0) yields the $w + W_0^{1,p(x)}$-weak convergence.

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