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LINEAR DERIVATIONS WITH RINGS OF CONSTANTS GENERATED BY LINEAR FORMS

BY

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Abstract. Let k be a field. We describe all linear derivations d of the polynomial algebra $k[x_1,\dots,x_m]$ such that the algebra of constants with respect to d is generated by linear forms: (a) over k in the case of $\operatorname{char} k = 0$, (b) over $k[x_1^p,\dots,x_m^p]$ in the case of $\operatorname{char} k = p > 0$.

Introduction. Throughout this paper k is a field of characteristic $p \geq 0$. We denote by $k[X]$ the polynomial algebra $k[x_1,\dots,x_m]$ with the natural grading

$$k[X] = \bigoplus_{j=0}^{\infty} k[X]_j,$$

where $k[X]_j$ is the subspace of forms of degree j. We also denote by $k[X^p]$ the subalgebra $k[x_1^p,\dots,x_m^p]$, but in the case of $p=0$ we assume $x_i^p = 1$, $i = 1,\dots,m$, and $k[X^p] = k$. If $v_1,\dots,v_n \in k[X]$, then we denote by $\langle v_1,\dots,v_n\rangle_k$ the k-linear space spanned by $v_1,\dots,v_n$. Throughout this paper we denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{F}_p$ the prime subfield of k.

A k-linear mapping $d\colon k[X] \to k[X]$ is called a *k-derivation* of $k[X]$ if $d(fg) = fd(g) + gd(f)$ for all $f, g \in k[X]$. If d is a k-derivation of $k[X]$, then we denote by $k[X]^d$ the *ring of constants* of d, that is,

$$k[X]^d = \{f \in k[X] : d(f) = 0\}.$$

Note that $k[X^p] \subseteq k[X]^d$, so $k[X]^d$ is a $k[X^p]$-algebra.

A mapping $d\colon k[X] \to k[X]$ is called a *linear derivation* if d is a k-derivation of $k[X]$ and $d(k[X]_j) \subseteq k[X]_j$ for $j = 0,1,2,\dots$. It is clear that a k-derivation d of $k[X]$ is a linear derivation if and only if

$$d(x_j) = \sum_{i=1}^{m} a_{ij} x_i \quad \text{for } j = 1,\dots,m,$$

where $a_{ij} \in k$ for $i,j = 1,\dots,m$. A linear derivation d is uniquely determined by the matrix (a_{ij}).

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In the case of $\operatorname{char} k = 0$, Nowicki ([2]) described the linear derivations of $k[X]$ such that $k[X]^d = k$. He also described such derivations satisfying the condition $k(X)^d = k$, where $k(X)$ is the field of rational functions. In this paper we consider the following, more general problem, concerning polynomial constants of linear derivations. Let $0 \le r \le m$. The problem is to describe all linear derivations d of $k[X]$ such that

$$k[X]^d = k[y_1, \ldots, y_r, y_{r+1}^p, \ldots, y_m^p]$$

(i.e. $k[X]^d = k[y_1, \ldots, y_r]$ in the case of $p = 0$) for some k-linear basis $y_1, \ldots, y_m$ of $k[X]_1$.

1. The Jordan case. In this section we consider a special case when the matrix (a_{ij}) of a linear derivation d of $k[X] = k[x_1, \ldots, x_m]$ is already in the Jordan form

$$\begin{pmatrix} J_{m_1}(\varrho_1) & 0 & \cdots & 0 \\ 0 & J_{m_2}(\varrho_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{m_s}(\varrho_s) \end{pmatrix}, \quad J_{m_i}(\varrho_i) = \underbrace{\begin{pmatrix} \varrho_i & 1 & \cdots & 0 \\ 0 & \varrho_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \varrho_i \end{pmatrix}}_{m_i},$$

$i = 1, \ldots, s$, where $s \ge 1$, $m_1 \ge \cdots \ge m_s$, $m_1 + \cdots + m_s = m$, and where $\varrho_1, \ldots, \varrho_s \in k$.

Let $n_1 = 1$ and $n_i = m_1 + \cdots + m_{i-1} + 1$ for $i = 2, \ldots, s$. Then $d(x_{n_i}) = \varrho_i x_{n_i}$ and $d(x_{n_i+l}) = x_{n_i+l-1} + \varrho_i x_{n_i+l}$ for $l = 1, \ldots, m_i - 1$, whenever $m_i > 1$.

Let

$$I = \{1, \ldots, s\}, \qquad I_0 = \{i \in I : \varrho_i = 0\}.$$

We denote by $d|_{k[X]_1}$ the restriction of d to $k[X]_1$. The kernel of $d|_{k[X]_1}$ is k-linearly spanned by all the elements of the form x_{n_i}, where $i \in I_0$, that is,

$$k[X]^d \cap k[X]_1 = \langle x_{n_i};\, i \in I_0 \rangle_k.$$

This implies the following fact.

PROPOSITION 1.1. *Assume that $k[X]^d = k[y_1, \ldots, y_r, y_{r+1}^p, \ldots, y_m^p]$ for some k-linear basis $y_1, \ldots, y_m$ of $k[X]_1$. Then $\langle y_1, \ldots, y_r \rangle_k = \langle x_{n_i};\, i \in I_0 \rangle_k$ and $k[X]^d$ is generated over $k[X^p]$ by the elements x_{n_i}, where $i \in I_0$, that is,*

$$k[X]^d = k[X^p][x_{n_i};\, i \in I_0]. \ \blacksquare$$

The aim of this section is to prove the following theorems.

THEOREM 1.2. *Let $p = 0$. The equality $k[X]^d = k[x_{n_i};\, i \in I_0]$ holds if and only if the following three conditions are satisfied*:

(1) *the system $(\varrho_i;\, i \in I \setminus I_0)$ is linearly independent over $\mathbb{N}$,*

(2) $m_i \le 2$ *for every* $i \in I_0$,
(3) $m_i = 2$ *for at most one* $i \in I_0$. ∎

THEOREM 1.3. *Let* $p > 0$. *The equality* $k[X]^d = k[X^p][x_{n_i}; i \in I_0]$ *holds if and only if the following three conditions are satisfied*:

(1) *the system* $(\varrho_i;\ i \in I \setminus I_0)$ *is linearly independent over* $\mathbb{F}_p$,
(2) $m_1 \le 2$ *or* $m_1 = 3$, $p = 2$,
(3) $m_2 = 1$. ∎

Let $\lambda_1, \ldots, \lambda_m$ be the diagonal elements of the matrix of d, that is, $\lambda_{n_i} = \cdots = \lambda_{n_i+m_i-1} = \varrho_i$ for $i = 1, \ldots, s$. Obviously $d = d_D + d_N$, where d_D and d_N are the linear derivations defined by:

$$d_D(x_j) = \lambda_j x_j \qquad \text{for } j = 1, \ldots, m,$$
$$d_N(x_j) = \begin{cases} 0 & \text{for } j = n_1, \ldots, n_s, \\ x_{j-1} & \text{for } j \neq n_1, \ldots, n_s. \end{cases}$$

We see that

$$d_D(x_1^{l_1} \ldots x_m^{l_m}) = (l_1\lambda_1 + \cdots + l_m\lambda_m) x_1^{l_1} \ldots x_m^{l_m},$$
$$d_N(x_1^{l_1} \ldots x_m^{l_m}) = \sum_{j \neq n_1, \ldots, n_s} l_j x_1^{l_1} \ldots x_{j-1}^{l_{j-1}+1} x_j^{l_j - 1} \ldots x_m^{l_m}.$$

PROPOSITION 1.4. $k[X]^d = k[X]^{d_N} \cap k[X]^{d_D}$.

Proof. In the case of $p = 0$ this fact is well known ([4, Corollary 2.3] or [3, Corollary 9.4.4]). Assume that $p > 0$.

The inclusion $k[X]^{d_N} \cap k[X]^{d_D} \subseteq k[X]^d$ is clear. To prove the reverse inclusion, suppose that $d(f) = 0$ for some $f \in k[X]$. Let l be a positive integer such that $p^l \ge m$, where $m = \dim_k k[X]_1$. Then $(d_N|_{k[X]_1})^{p^l} = 0$, so $d_N^{p^l} = 0$, and we have $d_D^{p^l}(f) = d^{p^l}(f) = 0$.

It is easy to see that all the monomials of the form $x_1^{l_1} \ldots x_m^{l_m}$ such that $l_1\lambda_1 + \cdots + l_m\lambda_m = 0$ form a k-linear basis of $k[X]^{d_D}$, and all the monomials of the form $x_1^{l_1} \ldots x_m^{l_m}$ such that $l_1\lambda_1^{p^l} + \cdots + l_m\lambda_m^{p^l} = 0$ form a k-linear basis of $k[X]^{d_D^{p^l}}$. Since $l_1\lambda_1^{p^l} + \cdots + l_m\lambda_m^{p^l} = (l_1\lambda_1 + \cdots + l_m\lambda_m)^{p^l}$ for every $l_1, \ldots, l_m \in \mathbb{Z}$, we have $k[X]^{d_D^{p^l}} = k[X]^{d_D}$. This implies that $d_D(f) = 0$, so $d_N(f) = d(f) - d_D(f) = 0$, and finally $f \in k[X]^{d_N} \cap k[X]^{d_D}$. ∎

Note the following useful proposition.

PROPOSITION 1.5. *Let* K *be a domain of characteristic* $p \ge 0$. *Let* δ *be a* K*-derivation of* $K[x_1, \ldots, x_m]$ *such that* $\delta(x_i) = 0$ *for* $i \le r$ *and* $\delta(x_i) = \mu_i x_i$ *for* $i > r$, *where* $\mu_{r+1}, \ldots, \mu_m \in K \setminus \{0\}$ *are linearly independent* (*over* $\mathbb{F}_p$ *in the case of* $p > 0$, *over* $\mathbb{N}$ *in the case of* $p = 0$). *Then* $K[x_1, \ldots, x_m]^\delta = K[x_1, \ldots, x_r, x_{r+1}^p, \ldots, x_m^p]$.

Proof. It is enough to observe that $d(x_1^{l_1}\dots x_m^{l_m}) = (l_{r+1}\mu_{r+1} + \dots + l_m\mu_m)x_1^{l_1}\dots x_m^{l_m}$ for every $l_1,\dots,l_m \geq 0$, so $K[x_1,\dots,x_m]^\delta$ is a free K-module and the monomials $x_1^{l_1}\dots x_m^{l_m}$ such that $l_{r+1}\mu_{r+1} + \dots + l_m\mu_m = 0$ form a basis of this module. ∎

Recall that $I = \{1,\dots,s\}$ and $I_0 = \{i \in I : \varrho_i = 0\}$. Let $J = \{1,\dots,m\}$ and $J_0 = \{j \in J : \lambda_j = 0\}$.

Proof of Theorem 1.2. ($\Rightarrow$) (1) Assume that the system $(\varrho_i;\, i \in I\setminus I_0)$ is linearly dependent over $\mathbb{N}$. Then there exist $l_1,\dots,l_s \in \mathbb{N}$ such that $l_1\varrho_1 + \dots + l_s\varrho_s = 0$ and $l_j > 0$ for some $j \in I \setminus I_0$. In this case $x_{n_1}^{l_1}\dots x_{n_s}^{l_s} \in k[X]^d \setminus k[x_{n_i};\, i \in I_0]$.

(2) The condition $m_i \geq 3$ for some $i \in I_0$ means that $d(x_{n_i}) = 0$, $d(x_{n_i+1}) = x_{n_i}$ and $d(x_{n_i+2}) = x_{n_i+1}$. Then $x_{n_i+1}^2 - 2x_{n_i}x_{n_i+2} \in k[X]^d \setminus k[x_{n_i};\, i \in I_0]$.

(3) The condition $m_i, m_j \geq 2$ for some $i, j \in I_0$, $i \neq j$ means that $d(x_{n_i}) = 0$, $d(x_{n_i+1}) = x_{n_i}$, $d(x_{n_j}) = 0$ and $d(x_{n_j+1}) = x_{n_j}$. Then $x_{n_i}x_{n_j+1} - x_{n_i+1}x_{n_j} \in k[X]^d \setminus k[x_{n_i};\, i \in I_0]$.

($\Leftarrow$) Assume that conditions (1)–(3) hold.

We have $d_D(x_j) = 0$ for $j \in J_0$ and $d_D(x_j) = \lambda_j x_j$ for $j \in J \setminus J_0$, where $\lambda_j = \varrho_i \neq 0$, $n_i \leq j < n_i + m_i$, $i \in I\setminus I_0$. The system $(\lambda_j;\, j \in J\setminus J_0)$ is linearly independent over $\mathbb{N}$, because $(\varrho_i;\, i \in I \setminus I_0)$ is, so $k[X]^{d_D} = k[x_j;\, j \in J_0]$ by Proposition 1.5.

Let d'_N be the restriction of d_N to $k[x_j;\, j \in J_0]$. Then, by Proposition 1.4, $k[X]^d = (k[X]^{d_D})^{d'_N} = k[x_j;\, j \in J_0]^{d'_N}$. If $m_{i_0} = 2$ for some $i_0 \in I_0$, then it is easy to see that $k[x_j;\, j \in J_0]^{d'_N} = k[x_j;\, j \in J_0 \setminus \{n_{i_0+1}\}] = k[x_{n_i};\, i \in I_0]$. If $m_i = 1$ for every $i \in I_0$, then $d'_N = 0$, so $k[x_j;\, j \in J_0]^{d'_N} = k[x_j;\, j \in J_0] = k[x_{n_i};\, i \in I_0]$. ∎

Proof of Theorem 1.3. ($\Rightarrow$) (1) Assume that the system $(\varrho_i;\, i \in I\setminus I_0)$ is linearly dependent over $\mathbb{F}_p$. Then there exist nonnegative integers $l_1,\dots,l_s < p$ such that $l_1\varrho_1 + \dots + l_s\varrho_s = 0$ and $l_j > 0$ for some $j \in I \setminus I_0$. In this case $x_{n_1}^{l_1}\dots x_{n_s}^{l_s} \in k[X]^d \setminus k[X^p][x_{n_i};\, i \in I_0]$.

(2) The condition $m_1 \geq 3$ means that $d(x_1) = \varrho_1 x_1$, $d(x_2) = x_1 + \varrho_1 x_2$ and $d(x_3) = x_2 + \varrho_1 x_3$. Then for $p > 2$ we have $x_1^{p-2}x_2^2 - 2x_1^{p-1}x_3 \in k[X]^d \setminus k[X^p][x_{n_i};\, i \in I_0]$.

The condition $m_1 \geq 4$ means that $d(x_1) = \varrho_1 x_1$, $d(x_2) = x_1 + \varrho_1 x_2$, $d(x_3) = x_2 + \varrho_1 x_3$ and $d(x_4) = x_3 + \varrho_1 x_4$. Then for $p = 2$ we have $x_1^3x_4 + x_1^2x_2x_3 + x_1x_2^3 \in k[X]^d \setminus k[X^p][x_{n_i};\, i \in I_0]$.

(3) The condition $m_2 \geq 2$ means that $d(x_1) = \varrho_1 x_1$, $d(x_2) = x_1 + \varrho_1 x_2$, $d(x_{m_1+1}) = \varrho_2 x_{m_1+1}$ and $d(x_{m_1+2}) = x_{m_1+1} + \varrho_2 x_{m_1+2}$. Then $x_1^{p-1}x_2x_{m_1+1}^p - x_1^p x_{m_1+1}^{p-1}x_{m_1+2} \in k[X]^d \setminus k[X^p][x_{n_i};\, i \in I_0]$.

($\Leftarrow$) Assume that conditions (1)–(3) hold.

Let d'_D be the restriction of d_D to $k[X^p][x_{n_1},\dots,x_{n_s}]$. Recall that $J=\{1,\dots,m\}$. Consider the set $J'=J\setminus\{n_1,\dots,n_s\}$. Let $K=k[x_j^p;\ j\in J']$. We see that d'_D is a K-derivation of $K[x_{n_1},\dots,x_{n_s}]=k[X^p][x_{n_1},\dots,x_{n_s}]$ such that $d'_D(x_{n_i})=\varrho_i x_{n_i}$, where $\varrho_i=0$ for $i\in I_0$, $\varrho_i\neq 0$ for $i\in I\setminus I_0$ and the system $(\varrho_i;\ i\in I\setminus I_0)$ is linearly independent over $\mathbb{F}_p$. Proposition 1.5 implies that $k[X^p][x_{n_1},\dots,x_{n_s}]^{d'_D}=k[X^p][x_{n_i};\ i\in I_0]$. This ends the proof if $m_1=1$.

If $m_1=2$, then it is easy to see that $k[X]^{d_N}=k[x_1,x_2^p,x_3,\dots,x_m]=k[X^p][x_{n_1},\dots,x_{n_s}]$. If $p=2$ and $m_1=3$, then it is easy to see that $k[X]^{d_N}=k[x_1,x_2^p,x_3^p,x_4,\dots,x_m]=k[X^p][x_{n_1},\dots,x_{n_s}]$. In both cases, by Proposition 1.4, $k[X]^d=(k[X]^{d_N})^{d'_D}=k[X^p][x_{n_i};\ i\in I_0]$. ∎

2. Some facts about graded algebras. In this section by a *graded k-algebra* we mean a k-algebra with a $\mathbb{Z}$-grading $A=\bigoplus_{j=0}^{\infty}A_j$. Nonzero elements of A_j are called *homogeneous of degree j*.

Note the following well known fact.

LEMMA 2.1. *Let $B=\bigoplus_{j=0}^{\infty}B_j$ be a graded commutative k-algebra, $B_0=k$, and $M=\bigoplus_{j>0}B_j$. Let $f_1,\dots,f_n\in M$.*

(a) *If $B=k[f_1,\dots,f_n]$, then $M/M^2=\langle f_1+M^2,\dots,f_n+M^2\rangle_k$.*
(b) *If $f_1,\dots,f_n$ are homogeneous elements and $M/M^2=\langle f_1+M^2,\dots\dots,f_n+M^2\rangle_k$, then $B=k[f_1,\dots,f_n]$.* ∎

The original version of this lemma ([1, II.3.2]) was formulated as an equivalence of three conditions under the assumptions that $k=\mathbb{C}$ and $f_1,\dots,f_n$ are homogeneous elements. However, the proof is valid for an arbitrary field k and the implication in (a) is true for arbitrary $f_1,\dots,f_n\in M$.

If elements $f_1,\dots,f_n$ generate the k-algebra B, with n smallest possible, then we say that $f_1,\dots,f_n$ form a *minimal system of generators* of B. Using the previous lemma we can easily establish the following proposition.

PROPOSITION 2.2. *Let $B=\bigoplus_{j=0}^{\infty}B_j$ be a graded commutative k-algebra with $B_0=k$ and let $C_j=\sum_{l=1}^{j-1}B_l\cdot B_{j-l}$ for $j>1$, $C_1=0$, $C_0=k$.*

(a) *Homogeneous elements $f_1,\dots,f_n$ form a minimal system of generators of B if and only if for every j the residue classes modulo C_j of all the elements f_i of degree j form a basis of the k-linear space B_j/C_j.*
(b) *Let $k\subseteq k'$ be a field extension. Denote by B' the graded k'-algebra $k'\otimes_k B$. Let $C'_j=\sum_{l=1}^{j-1}B'_l\cdot B'_{j-l}$ for $j>1$, $C'_1=0$, $C'_0=k'$. Then $\dim_k B_j/C_j=\dim_{k'}B'_j/C'_j$ for every j. Moreover, if homogeneous*

elements $f_1, \dots, f_n$ form a minimal system of generators of the k-algebra B, then the elements $1 \otimes f_1, \dots, 1 \otimes f_n$ form a minimal system of generators of the k'-algebra B'.

Proof. (a) Let $M = \bigoplus_{j>0} B_j$. Then $M^2 = \bigoplus_{j>0} C_j$. Lemma 2.1 implies that the elements $f_1, \dots, f_n$ generate the k-algebra B if and only if their residue classes modulo M generate the linear space $M/M^2 \simeq \bigoplus_j B_j/C_j$. So $f_1, \dots, f_n$ form a minimal system of generators of B if and only if for every j the residue classes modulo C_j of all the elements f_i of degree j form a basis of B_j/C_j.

(b) For every $j > 1$ we have a canonical k'-linear isomorphism

$$\sum_{l=1}^{j-1} (k' \otimes_k B_l) \cdot (k' \otimes_k B_{j-l}) \simeq k' \otimes_k \sum_{l=1}^{j-1} B_l \cdot B_{j-l},$$

that is, $C'_j \simeq k' \otimes_k C_j$. This implies that $\dim_k B_j/C_j = \dim_{k'} B'_j/C'_j$.

Let $f_{i_1}, \dots, f_{i_s}$ be all the elements f_i of degree j. By (a), the residue classes modulo C_j of $f_{i_1}, \dots, f_{i_s}$ form a k-linear basis of B_j/C_j. Then the residue classes of $1 \otimes f_{i_1}, \dots, 1 \otimes f_{i_s}$ form a k'-linear basis of B'_j/C'_j. Again by (a), the elements $1 \otimes f_1, \dots, 1 \otimes f_n$ form a minimal system of generators of the k'-algebra B'. ∎

Note the following immediate consequence of Lemma 2.1 and Proposition 2.2(a).

COROLLARY 2.3. *If B is generated by n elements (not necessarily homogeneous), then B is generated by some n homogeneous elements.*

Proof. Let M and C_j be defined as in Lemma 2.1 and Proposition 2.2. It is enough to observe that $M/M^2 \simeq \bigoplus_{j=0}^{\infty} B_j/C_j$, so $\sum_{j=0}^{\infty} \dim_k B_j/C_j \le n$. ∎

Now we will prove the following proposition.

PROPOSITION 2.4. *Let $k \subseteq k'$ be an extension of fields of arbitrary characteristic $p \ge 0$, let B be a graded subalgebra of $k[X]$ and $B' = k' \otimes_k B$ the corresponding subalgebra of $k'[X]$. Let $r \in \{0, 1, \dots, m\}$. Then the following conditions are equivalent:*

(i) *$B = k[y_1, \dots, y_r, y_{r+1}^p, \dots, y_m^p]$ for some k-linear basis $y_1, \dots, y_m$ of $k[X]_1$,*

(ii) *$B' = k'[z_1, \dots, z_r, z_{r+1}^p, \dots, z_m^p]$ for some k'-linear basis $z_1, \dots, z_m$ of $k'[X]_1$.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(i). Assume that $B' = k'[z_1, \dots, z_r, z_{r+1}^p, \dots, z_m^p]$ for some k'-linear basis $z_1, \dots, z_m$ of $k'[X]_1$. Let C_j and C'_j be defined as in Proposition 2.2.

Let $p=0$. The elements $z_1,\dots,z_r$ form a minimal system of generators of the k'-algebra B', so $\dim_{k'} B'_1 = r$ and $\dim_{k'} B'_j/C'_j = 0$ for $j>1$ by Proposition 2.2(a). Proposition 2.2(b) implies that $\dim_k B_1 = r$ and $\dim_k B_j/C_j = 0$ for $j>1$. Let $y_1,\dots,y_r$ be a k-linear basis of B_1. Then $y_1,\dots,y_r$ form a minimal system of generators of the k-algebra B (Proposition 2.2(a)), so $B = k[y_1,\dots,y_r]$.

Now let $p>0$. Using similar arguments to those for $p=0$, we show that the elements $y_1,\dots,y_r$ of a k-linear basis of B_1 together with some elements $t_{r+1},\dots,t_m \in B_p$ form a minimal system of generators of the k-algebra B. We can enlarge $\{y_1,\dots,y_r\}$ to a basis $\{y_1,\dots,y_m\}$ of $k[X]_1$. Let $V = k[y_1,\dots,y_r]_p + \langle y_{r+1}^p,\dots,y_m^p\rangle_k$. Then $V \subseteq B_p$, but we see that $\dim_k V = \dim_k B_p$, so $V = B_p$, that is, $k[y_1,\dots,y_r]_p + \langle y_{r+1}^p,\dots,y_m^p\rangle_k = k[y_1,\dots,y_r]_p + \langle t_{r+1},\dots,t_m\rangle_k$. This implies that $B = k[y_1,\dots,y_r,y_{r+1}^p,\dots,y_m^p]$. ∎

Recall the following fact.

Proposition ([3, 5.1.1], [2, 2.1]). *Let $k \subseteq k'$ be a field extension and let d be a k-derivation of a k-algebra A. Denote by i the inclusion $A^d \hookrightarrow A$. Then $d' = 1 \otimes d$ is a k'-derivation of the k'-algebra $A' = k' \otimes_k A$ and $(1\otimes i)(k' \otimes_k A^d) = A'^{d'}$.* ∎

The way of reducing an arbitrary linear derivation to its Jordan form is given in the following corollary of the above proposition and Proposition 2.4.

Corollary 2.5. *If d is a k-derivation of $k[X]$ and d' is a k'-derivation of $k'[X]$ such that $d'(x_i) = d(x_i)$ for $i=1,\dots,m$, then the following conditions are equivalent*:

(i) *$k[X]^d = k[y_1,\dots,y_r,y_{r+1}^p,\dots,y_m^p]$ for some k-linear basis $y_1,\dots,y_m$ of $k[X]_1$;*
(ii) *$k'[X]^{d'} = k'[z_1,\dots,z_r,z_{r+1}^p,\dots,z_m^p]$ for some k'-linear basis $z_1,\dots,z_m$ of $k'[X]_1$.* ∎

3. The general case. Now let d be a linear derivation of $k[X]$. Using Corollary 2.5 for the algebraic closure $\overline{k}$ of k, Proposition 1.1 and Theorems 1.2 and 1.3 for the Jordan matrix of the endomorphism $d|_{k[X]_1}$ over $\overline{k}$, we obtain the following theorems.

Theorem 3.1. *Let d be a linear derivation of $k[X]$, where k is a field of characteristic* 0. *Then*

$$k[X]^d = k[y_1,\dots,y_r]$$

for some linearly independent homogeneous polynomials $y_1,\dots,y_m$ of degree 1 *if and only if the Jordan matrix of $d|_{k[X]_1}$ satisfies the following conditions.*

(1) *There are exactly r Jordan blocks with zero eigenvalues.*

(2) *Nonzero eigenvalues of different Jordan blocks are pairwise different and linearly independent over* $\mathbb{N}$.

(3) *At most one Jordan block with zero eigenvalue has dimension greater than* 1, *and if such a block exists*, *it is of dimension* 2. ■

THEOREM 3.2. *Let d be a linear derivation of $k[X]$, where k is a field of characteristic $p > 0$. Then*

$$k[X]^d = k[y_1, \dots, y_r, y_{r+1}^p, \dots, y_m^p]$$

for some k-linear basis $y_1, \dots, y_m$ of $k[X]_1$ if and only if the Jordan matrix of $d|_{k[X]_1}$ satisfies the following conditions.

(1) *There are exactly r Jordan blocks with zero eigenvalues.*

(2) *Nonzero eigenvalues of different Jordan blocks are pairwise different and linearly independent over* $\mathbb{F}_p$.

(3) *At most one Jordan block has dimension greater than* 1, *and if such a block exists*, *then its dimension is* 2 *in the case of* $p > 2$, *and* 2 *or* 3 *for* $p = 2$. ■

Note that all the rings of constants mentioned in Theorems 3.1 and 3.2 are polynomial k-algebras. It is well known that in the case of $\operatorname{char} k = 0$ there exist linear derivations of $k[X]$ with rings of constants being polynomial k-algebras not generated by linear forms. Let us end with the following question.

QUESTION. Does there exist a linear derivation of $k[X]$, where $\operatorname{char} k = p > 0$, such that $k[X]^d$ is a polynomial k-algebra not of the form mentioned in Theorem 3.2?

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