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TILTING SLICE MODULES OVER MINIMAL 2-FUNDAMENTAL ALGEBRAS

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Abstract. A class of finite-dimensional algebras whose Auslander–Reiten quivers have starting but not generalized standard components is investigated. For these components the slices whose slice modules are tilting are considered. Moreover, the endomorphism algebras of tilting slice modules are characterized.

Introduction. We shall denote by K a fixed algebraically closed field. Moreover, we shall only consider finite-dimensional associative K-algebras with a unit element that will be assumed to be basic and connected.

For a given algebra A we shall denote by $\mathrm{mod}(A)$ the category of finite-dimensional left A-modules, and by Γ_A the Auslander–Reiten quiver of A [2, 3]. We shall denote by $\underline{\mathrm{mod}}(A)$ (respectively, $\overline{\mathrm{mod}}(A)$) the stable category of $\mathrm{mod}(A)$ modulo projectives (resp., injectives). If $f \in \mathrm{Hom}_A(X,Y)$ then $\underline{f}$ (resp., $\overline{f}$) denotes its coset modulo projectives (resp., injectives).

We are interested in minimal 2-fundamental algebras, introduced in [8]. In many cases, their Auslander–Reiten quiver contains a component at the beginning that is not generalized standard in the sense of Skowroński [12] and contains projective vertices. Therefore it is reasonable to generalize the notion of slice introduced in [7] and study when a slice module is a tilting module. That was done in [9], where a postprojective (resp., preinjective) slice $\mathcal{S}$ was defined and a slice module $M_{\mathcal{S}}$ considered. It was shown that there are only finitely many postprojective (resp., preinjective) slices $\mathcal{S}$ whose slice modules $M_{\mathcal{S}}$ are tilting (resp., cotilting).

In the present paper our objective is to provide a detailed description of such slices $\mathcal{S}$. Our first main result is

THEOREM 1. *For a minimal* 2*-fundamental algebra A let $\mathcal{C}$ be the starting* (*resp., ending*) *component in Γ_A that is not generalized standard. Let $\mathcal{S} =$*

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$\{X_i\}_{i=1}^t$ be a postprojective (resp., preinjective) slice in $\mathcal{C}$. Then the slice module $X_{\mathcal{S}} = \bigoplus_{i=1}^t X_i$ is a tilting (resp., cotilting) A-module if and only if $\mathcal{S}$ is contained in the postprojective starting cone C_{sc} (resp., preinjective ending cone C_{ec}) in $\mathcal{C}$.

We also have the following characterization for $\mathrm{End}_A(X_{\mathcal{S}})^{\mathrm{op}}$, where $\mathcal{S}$ is a postprojective (resp., preinjective) slice contained in C_{sc} (resp., C_{ec}).

THEOREM 2. *For a minimal 2-fundamental algebra A let $\mathcal{C}$ be the starting (resp., ending) component in Γ_A that is not generalized standard. Let $\mathcal{S}$ be a postprojective (resp., preinjective) slice in $\mathcal{C}$, contained in C_{sc} (resp., C_{ec}). Then $\mathrm{End}_A(X_{\mathcal{S}})^{\mathrm{op}}$ is a (t,p,s)-algebra (resp., $(t,p,s)^{\mathrm{op}}$-algebra).*

The definitions of a (t,p,s)-algebra (resp., $(t,p,s)^{\mathrm{op}}$-algebra) are given in Section 3. The integers t,p,s are uniquely determined by the algebra A.

We shall use freely all information on Auslander–Reiten sequences and irreducible homomorphisms that can be found in [2, 3]. For background on the representation theory of algebras we refer to [1, 3, 10, 11].

1. Preliminaries. Following Gabriel [6] one can associate a bound quiver (Q_A, I_A) to a finite-dimensional basic K-algebra A in such a way that $A \cong KQ_A/I_A$, where KQ_A is the path algebra of Q_A and I_A is a two-sided ideal in KQ_A contained in the square of the two-sided ideal generated by the arrows. The algebra A is said to be *triangular* if Q_A has no oriented cycle. We shall use the standard notation. To every vertex x of Q_A we can attach a simple A-module S_x, its projective cover P_x and its injective envelope E_x.

An algebra A is said to be *special biserial* (see [13]) if there exists a bound quiver (Q_A, I_A) with $A \cong KQ_A/I_A$ such that:

(1) Every vertex of Q_A is the source of at most two arrows.
(2) Every vertex of Q_A is the sink of at most two arrows.
(3) For every arrow α in Q_A there exists at most one arrow β (respectively, γ) such that $\alpha\beta \notin I_A$ (resp., $\gamma\alpha \notin I_A$).

Throughout the paper we shall always consider special biserial algebras of the form KQ_A/I_A with (Q_A, I_A) satisfying the above conditions.

Let $A = KQ_A/I_A$ be special biserial. Then A is called a *string algebra* (see [4]) if I_A is generated only by paths. For a string algebra A there is a full classification of indecomposable finite-dimensional left A-modules [5, 14]. Every such module X is a string module or a band module. In the first case X is induced by a walk w in (Q_A, I_A), which is denoted by writing $X = X(w)$. The other case will not be used here. Moreover, an algorithm for computing Auslander–Reiten sequences for string modules is due to Skowroński and Waschbüsch [13]. Recall that if $X(w)$ is a noninjective string A-module then we have the Auslander–Reiten sequence

$$0 \to X(w) \to X(w_L) \oplus X(w_R) \to X(w_{LR}) = \tau^{-1}(X(w)) \to 0.$$

If $X(w)$ is nonprojective then we have the Auslander–Reiten sequence

$$0 \to \tau(X(w)) = X(w_{L^{-1}R^{-1}}) \to X(w_{L^{-1}} \oplus X(w_{R^{-1}}) \to X(w) \to 0.$$

A triangular string algebra $A = KQ_A/I_A$ is said to be $\tilde{\mathbb{A}}_m$-*separated* provided that for any two subquivers Q', Q'' in Q_A of type $\tilde{\mathbb{A}}_m$ such that $KQ' \cap I_A = 0 = KQ'' \cap I_A$ we have $Q'_0 \cap Q''_0 = \emptyset$, where Q'_0, Q''_0 denote the sets of vertices of Q', Q'', respectively.

A triangular string $\tilde{\mathbb{A}}_m$-separated algebra $A = KQ_A/I_A$ is said to be 2-*fundamental* [8] if it is connected and the following conditions are satisfied:

(i) There exist exactly two full subquivers Q', Q'' of type $\tilde{\mathbb{A}}_m$ in (Q_A, I_A) such that $KQ' \cap I_A = 0 = KQ'' \cap I_A$ and the quiver $\bar{Q}_A$ obtained from Q_A by removing the arrows from Q' and Q'' and identifying the vertices of Q' with a vertex $0'$ and the vertices of Q'' with a vertex $0''$ is a tree.

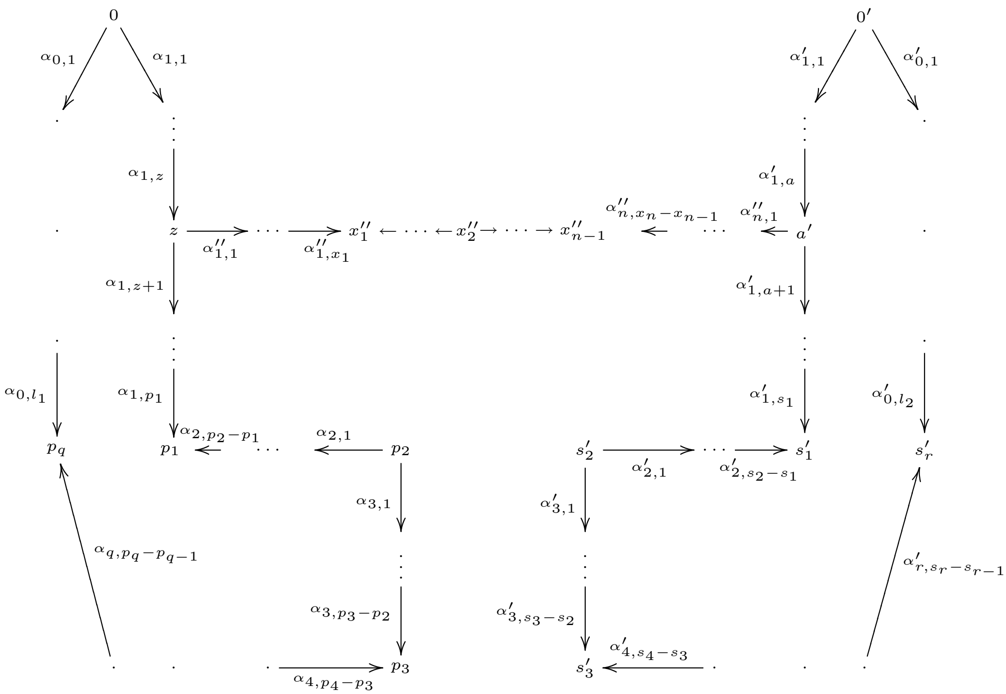
(ii) For $0^j = 0'$ or $0''$ there exists either a maximal path v in $\bar{Q}_A$ starting at 0^j such that $v \notin I_A$, or a maximal path u in $\bar{Q}_A$ ending at 0^j such that $u \notin I_A$. If v (treated as a path in Q_A) starts at some vertex x in Q^j that is a sink of two maximal paths v_1, v_2 in Q^j then $v_1v \notin I_A$ or $v_2v \notin I_A$. If u (treated as a path in Q_A) ends at some vertex y in Q^j that is a source of two maximal paths u_1, u_2 in Q^j then $uu_1 \notin I_A$ or $uu_2 \notin I_A$.

A 2-fundamental algebra A is said to be *minimal* if the graph obtained from the quiver $\bar{Q}_A$ by forgetting orientations of the arrows is of the form $0' — \cdots — 0''$.

Following Auslander and Reiten [2, 3] we can attach to any K-algebra A its Auslander–Reiten quiver Γ_A. We shall not distinguish between indecomposable A-modules and vertices of Γ_A. A *component* in Γ_A will always mean a connected component. A component $\mathcal{C}$ of Γ_A is said to be *starting* (resp., *ending*) (see [8]) if there is no nonzero morphism $f : X \to Y$ between indecomposable modules X, Y such that $Y \in \mathcal{C}$ and $X \notin \mathcal{C}$ (resp., $X \in \mathcal{C}$ and $Y \notin \mathcal{C}$).

We say that a component $\mathcal{C}$ of Γ_A is *generalized standard* (see [12]) if $\mathrm{rad}^\infty(X, Y) = 0$ for any indecomposable $X, Y \in \mathcal{C}$. Recall that $\mathrm{rad}^\infty(\mathrm{mod}(A))$ is the intersection of all positive powers of the Jacobson radical $\mathrm{rad}(\mathrm{mod}(A))$.

Let $\underline{p} = (p_1, \ldots, p_q)$, $\underline{s} = (s_1, \ldots, s_r)$ and $\underline{x} = (x_1, \ldots, x_n)$ be strictly increasing sequences of nonnegative integers. Let $l_1, l_2 \geq 1$ be integers. Consider a quiver $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ of the following form:



Let $I^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ be the two-sided ideal of $KQ^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ generated by the paths $\alpha_{1,z}\alpha''_{1,1}$, $\alpha'_{1,a}\alpha''_{n,1}$. Set $A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)} = KQ^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}/I^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. We can also consider the quiver $Q^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ dual to $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Let $I^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ denote the two-sided ideal in $KQ^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ generated by $\alpha''_{1,1}\alpha_{1,z}$, $\alpha''_{n,1}\alpha'_{1,a}$. Set $A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)} = KQ^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}/I^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

Recall from [9] the following lemma.

Lemma 1.1. *Let A be a minimal 2-fundamental algebra.*

(1) *If Γ_A contains a starting component that is not generalized standard then there are strictly increasing sequences $\underline{p}$, $\underline{x}$, $\underline{s}$ of nonnegative integers and integers $l_1, l_2 \geq 1$ such that $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.*

(2) *If Γ_A contains an ending component that is not generalized standard then there are strictly increasing sequences $\underline{p}$, $\underline{x}$, $\underline{s}$ of nonnegative integers and integers $l_1, l_2 \geq 1$ such that $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.*

Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$, and consider the unique starting component $\mathcal{C}$ in Γ_A. Then $\mathcal{C}$ contains all indecomposable projective left A-modules and it is not generalized standard (see [8; Theorem 5.7]). A *postprojective slice* in $\mathcal{C}$ is a set $\mathcal{S} = \{N_1, \ldots, N_t\}$ of vertices of $\mathcal{C}$ such that:

(0) $\mathcal{S}$ consists only of postprojective modules.
(1) There is no oriented cycle of irreducible morphisms in $\mathcal{C}$ between modules from $\mathcal{S}$.
(2) If $M_0 \to M_1 \to \cdots \to M_m$ is a path in $\mathcal{C}$ such that $M_0, M_m \in \mathcal{S}$ then $M_1, \ldots, M_{m-1} \in \mathcal{S}$.
(3) $\mathcal{S}$ contains exactly one representative of every τ-orbit of projective A-modules.

For $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ one can dually define a preinjective slice in the unique ending component of Γ_A.

If $\mathcal{S}$ is a postprojective (resp., preinjective) slice in $\mathcal{C}$, then $X_{\mathcal{S}} = \bigoplus_{i=1}^{t} N_i$ is called the *postprojective* (resp., *preinjective*) *slice module* of $\mathcal{S}$.

2. Slices in cones. For $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) and the starting (resp., ending) component $\mathcal{C}$ in Γ_A we define some walks in $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $Q^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$). First consider the case of $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Then we put $u = \alpha_{1,z+1}\alpha_{1,z+2}\ldots\alpha_{1,p_1}$ in $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ if $z \neq p_1$, and $u = p_1$ if $z = p_1$. Further, $v = \alpha'_{1,a+1}\alpha'_{1,a+2}\ldots\alpha'_{1,s_1}$ if $a \neq s_1$, and $v = s'_1$ if $a = s_1$. Now consider the case $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Then we put $u = \alpha_{1,p_1}\ldots\alpha_{1,z+1}$ in $Q^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ if $z \neq p_1$, and $u = p_1$ if $z = p_1$. Further, $v = \alpha'_{1,s_1}\ldots\alpha'_{1,a+1}$ if $a \neq s_1$, and $v = s'_1$ if $a = s_1$.

For a starting (resp., ending) component $\mathcal{C}$ in Γ_A we define the postprojective (resp., preinjective) *starting cone* $C_{\rm sc}$ (resp., *ending cone* $C_{\rm ec}$) in $\mathcal{C}$ to consist of all vertices X of $\mathcal{C}$ such that:

(1) X is postprojective (resp., preinjective).
(2) There is an integer $i_X \geq 0$ and a path (maybe trivial) in $\mathcal{C}$ from X to $X(u^{-1}_{R^{i_X}})$ (resp., from $X(u_{L^{-i_X}})$ to X).
(3) There is an integer $j_X \geq 0$ and a path (maybe trivial) in $\mathcal{C}$ from X to $X(v_{L^{j_X}})$ (resp., from $X(v^{-1}_{R^{-j_X}})$ to X).

A module X in $C_{\rm sc}$ or $C_{\rm ec}$ is defined to be *inner* if $i_X \geq 1$ or $j_X \geq 1$ respectively.

For the quiver $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ we define

$$p = p_2 - p_1 + p_4 - p_3 + \cdots + p_{q-1} - p_{q-2} + l_1,$$
$$s = s_2 - s_1 + s_4 - s_3 + \cdots + s_{r-1} - s_{r-2} + l_2,$$

$$p' = p_q - p_{q-1} + p_{q-2} - p_{q-3} + \cdots + p_3 - p_2 + p_1,$$
$$s' = s_r - s_{r-1} + s_{r-2} - s_{r-3} + \cdots + s_3 - s_2 + s_1.$$

LEMMA 2.1. *Let* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

(1) *For any integers* $j, i \geq 0$ *we have*
$$X((u^{-1})_{L^j R^{p'+i}})/X((\alpha''_{1,2} \dots \alpha''_{1,x_1})_{R^i}) \cong X((u^{-1})_{L^{p+j}}).$$

(2) *For any integers* $j, i \geq 0$ *we have*
$$X(v_{L^{s'+i}R^j})/X((\alpha''^{-1}_{n,x_n-x_{n-1}} \dots \alpha''^{-1}_{n,2})_{L^i}) \cong X(v_{R^{s+j}}).$$

Proof. First we prove the condition in (1) for $i = 0$ by induction on $j \geq 0$. If $j = 0$ then
$$X((u^{-1})_{R^{p'}}) \cong X(u^{-1}\alpha^{-1}_{1,z} \dots \alpha^{-1}_{1,1}\alpha_{0,1} \dots \alpha_{0,l_1}\alpha^{-1}_{q,p_q-p_{q-1}} \dots \alpha^{-1}_{q,1}\alpha_{q-1,1} \dots$$
$$\dots \alpha_{q-1,p_{q-1}-p_{q-2}}\alpha^{-1}_{q-2,p_{q-2}-p_{q-3}} \dots \alpha_{2,p_2-p_1}\alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}\alpha''_{1,1} \dots \alpha''_{1,x_1}).$$
Moreover,
$$X((u^{-1})_{L^p}) \cong X(\alpha^{-1}_{1,p_1} \cdots \alpha^{-1}_{1,1}\alpha_{0,1} \dots \alpha_{0,l_1}\alpha^{-1}_{q,p_q-p_{q-1}} \dots \alpha^{-1}_{q,1}\alpha_{q-1,1} \dots$$
$$\dots \alpha_{q-1,p_{q-1}-p_{q-2}}\alpha^{-1}_{q-2,p_{q-2}-p_{q-3}} \dots \alpha_{2,p_2-p_1}\alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}).$$
Hence it is clear that $X((u^{-1})_{R^{p'}})/X(\alpha''_{1,2} \dots \alpha''_{1,x_1}) \cong X((u^{-1})_{L^p})$.

Now we assume that the condition in (1) holds for some $j_0 \geq 0$ and set $j = j_0 + 1$. Then $X((u^{-1})_{L^{j_0+1}R^{p'}}) \cong X(((u^{-1})_{L^{j_0}R^{p'}})_L)$. Similarly, $X((u^{-1})_{L^{p+j_0+1}}) \cong X(((u^{-1})_{L^{p+j_0}})_L)$. Furthermore, by the inductive assumption, $X((u^{-1})_{L^{j_0}R^{p'}})/X(\alpha''_{1,2} \cdots \alpha''_{1,x_1}) \cong X((u^{-1})_{L^{p+j_0}})$. Then the Skowroński–Waschbüsch algorithm gives $X(((u^{-1})_{L^{j_0}R^{p'}})_L)/X(\alpha''_{1,2} \dots \alpha''_{1,x_1}) \cong X((u^{-1})_{L^{p+j_0}})_L)$, which finishes the proof of (1) in the case $i = 0$.

Now, applying the Skowroński–Waschbüsch algorithm, we get (1) for all i, j.

Similarly one can prove (2).

LEMMA 2.2. *Let* $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

(1) *For any integers* $j, i \geq 0$ *we have*
$$X(u_{L^{-j}R^{-p'-i}})/X(u_{L^{-j-p}}) \cong X((\alpha''^{-1}_{1,2} \cdots \alpha''^{-1}_{1,x_1})_{R^{-i}}).$$

(2) *For any integers* $j, i \geq 0$ *we have*
$$X((v^{-1})_{L^{-s'-i}R^{-j}})/X((v^{-1})_{R^{-s-j}}) \cong X((\alpha''_{n,x_n-x_{n-1}} \dots \alpha''_{n,2})_{L^{-i}}).$$

Proof. Use arguments dual to those in the proof of Lemma 2.1.

For a fixed postprojective (resp., preinjective) slice $\mathcal{S}$, we shall consider the slice module $X_{\mathcal{S}} = \bigoplus_{i=1}^t X(w_i)$. Moreover, we shall fix an indexing of the walks $w_1, \dots, w_t$ in such a way that $X(w_1)$ belongs to the τ^{-1}-orbit of

P_{z-1} (resp., τ-orbit of E_{z-1}) and $X(w_t)$ belongs to the τ^{-1}-orbit of $P_{(a-1)'}$ (resp., τ-orbit of $E_{(a-1)'}$). Then we can identify $\mathcal{S}$ with the set of vertices of a subquiver $X(w_1)$—$X(w_2)$—$\cdots$—$X(w_t)$ in $\mathcal{C}$, where — stands for a left or right arrow. For $X(w_i), X(w_j) \in \mathcal{S}$ with $i < j$ we define $l_{\mathcal{S}}(X(w_i), X(w_j))$ and $r_{\mathcal{S}}(X(w_i), X(w_j))$ to be the numbers of left and of right arrows in $X(w_i)$—$\cdots$—$X(w_j)$, respectively.

Let $\mathcal{S}_0$ denote the postprojective (resp., preinjective) slice formed by the projective (resp., injective) vertices of $\mathcal{C}$.

Lemma 2.3. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ be a postprojective slice in $\mathcal{C}$.*

(1) *If $X(w_1) \cong \tau^{-c}(P_{z-1})$ and $X(w_{i_0}) \cong \tau^{-d}(P_z)$ then*
$$l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p + c - d, \quad r_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p' + d - c.$$
(2) *If $X(w_t) \cong \tau^{-e}(P_{(a-1)'})$ and $X(w_{j_0}) \cong \tau^{-f}(P_{a'})$ then*
$$l_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s' + f - e, \quad r_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s + e - f.$$
(3) *If $X(w_1) \cong \tau^{-c}(P_{z-1})$ and $X(w_{i_0}) \cong \tau^{-d}(P_i)$ for some $i \in \{0, 1, \ldots, p_q + l_1 - 1\} \setminus \{z-1\}$, then*
$$l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = l_{\mathcal{S}_0}(P_{z-1}, P_i) + c - d,$$
$$r_{\mathcal{S}}(X(w_1), X(w_{i_0})) = r_{\mathcal{S}_0}(P_{z-1}, P_i) + d - c.$$
(4) *If $X(w_t) \cong \tau^{-c}(P_{(a-1)'})$ and $X(w_{j_0}) \cong \tau^{-d}(P_{i'})$ for some $i \in \{0, 1, \ldots, s_r + l_2 - 1\} \setminus \{a-1\}$ then*
$$l_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = l_{\mathcal{S}_0}(P_{i'}, P_{(a-1)'}) + d - c,$$
$$r_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = r_{\mathcal{S}_0}(P_{i'}, P_{(a-1)'}) + c - d.$$

Proof. We prove (1) by induction on $c + d$. If $c + d = 0$ then $c = 0 = d$. Thus $X(w_1) \cong P_{z-1}$ and $X(w_{i_0}) \cong P_z$. Furthermore, replacement of $X(w_i)$ with $\tau^{-1}(X(w_i))$ does not affect $l_{\mathcal{S}}(X(w_1), X(w_{i_0}))$ and $r_{\mathcal{S}}(X(w_1), X(w_{i_0}))$ for $1 < i < i_0$. Therefore $l_{\mathcal{S}}(P_{z-1}, P_z) = p$ and $r_{\mathcal{S}}(P_{z-1}, P_z) = p'$ as desired.

Now assume that (1) holds for $c + d \le n_0$, and let $c + d = n_0 + 1$. Then either $\tau(\mathcal{S}) = \{\tau(X(w_i))\}_{i=1}^t$ is a postprojective slice in $\mathcal{C}$, or there is $X(w_{j_0}) \in \mathcal{S}$ that is projective.

If $\tau(\mathcal{S})$ is a postprojective slice in $\mathcal{C}$ then by the inductive assumption
$$l_{\tau(\mathcal{S})}(\tau(X(w_1)), \tau(X(w_{i_0}))) = p + (c-1) - (d-1) = p + c - d.$$
But the left hand side equals $l_{\mathcal{S}}(X(w_1), X(w_{i_0}))$. Similarly we show that $r_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p' + d - c$.

If there is a projective vertex $X(w_{j_0}) \in \mathcal{S}$ then choose one with j_0 minimal. If $j_0 = 1$ then $X(w_1) \cong P_{z-1}$. Then it is easily seen that every shift of P_z along its τ^{-1}-orbit decreases $l_{\mathcal{S}}(X(w_1), X(w_{i_0}))$ by 1 and increases

$r_{\mathcal{S}}(X(w_1), X(w_{i_0}))$ by 1. Moreover, shifts of the other indecomposable projective A-modules along their τ^{-1}-orbits do not affect these numbers. Therefore (1) is satisfied. If $j_0 > 1$ then we proceed similarly.

The proofs of (2)–(4) are similar.

LEMMA 2.4. *Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. Let $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ be a preinjective slice in $\mathcal{C}$.*

(1) *If $X(w_1) \cong \tau^c(E_{z-1})$ and $X(w_{i_0}) \cong \tau^d(E_z)$ then*
$$l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p' + d - c, \quad r_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p + c - d.$$
(2) *If $X(w_t) \cong \tau^e(E_{(a-1)'})$ and $X(w_{j_0}) \cong \tau^f(E_{A'})$ then*
$$l_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s + e - f, \quad r_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s' + f - e.$$
(3) *If $X(w_1) \cong \tau^c(E_{z-1})$ and $X(w_{i_0}) \cong \tau^d(E_i)$ for some $i \in \{0, 1, \ldots, p_q + l_1 - 1\} \setminus \{z-1\}$ then*
$$l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = l_{\mathcal{S}_0}(E_{z-1}, E_i) + d - c,$$
$$r_{\mathcal{S}}(X(w_1), X(w_{i_0})) = r_{\mathcal{S}_0}(E_{z-1}, E_i) + c - d.$$
(4) *If $X(w_t) \cong \tau^e(E_{(a-1)'})$ and $X(w_{j_0}) \cong \tau^f(E_{i'})$ for some $i \in \{0, 1, \ldots, s_r + l_2 - 1\} \setminus \{a-1\}$ then*
$$l_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = l_{\mathcal{S}_0}(E_{i'}, E_{(a-1)'}) + e - f,$$
$$r_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = r_{\mathcal{S}_0}(E_{i'}, E_{(a-1)'}) + f - e.$$

Proof. Dual to the proof of Lemma 2.3.

LEMMA 2.5. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ be a postprojective slice in $\mathcal{C}$.*

(1) *If $X(w_1) \cong \tau^{-c}(P_{z-1})$ and $X(w_{i_1}) \cong \tau^{-e}(P_{j''})$ for some $j \in \{1, \ldots, x_n - 1\}$ then*
$$l_{\mathcal{S}}(X(w_1), X(w_{i_1})) = p + l_{\mathcal{S}_0}(P_z, P_{j''}) + c - e,$$
$$r_{\mathcal{S}}(X(w_1), X(w_{i_1})) = p' + r_{\mathcal{S}_0}(P_z, P_{j''}) + e - c.$$
(2) *If $X(w_1) \cong \tau^{-c}(P_{z-1})$ and $X(w_{i_2}) \cong \tau^{-f}(P_{j'})$ for some $j \in \{0, 1, \ldots, s_r + l_2 - 1\}$ then*
$$l_{\mathcal{S}}(X(w_1), X(w_{i_2})) = p + l_{\mathcal{S}_0}(P_z, P_{j'}) + c - f,$$
$$r_{\mathcal{S}}(X(w_1), X(w_{i_2})) = p' + r_{\mathcal{S}_0}(P_z, P_{j'}) + f - c.$$
(3) *If $X(w_t) \cong \tau^{-e}(P_{(a-1)'})$ and $X(w_{j_1}) \cong \tau^{-d}(P_{i''})$ for some $i \in \{1, \ldots, x_n - 1\}$ then*
$$l_{\mathcal{S}}(X(w_{j_1}), X(w_t)) = s' + l_{\mathcal{S}_0}(P_{i''}, P_{a'}) + d - e,$$
$$r_{\mathcal{S}}(X(w_{j_1}), X(w_t)) = s + r_{\mathcal{S}_0}(P_{i''}, P_{a'})) + e - d.$$

(4) *If* $X(w_t)\cong\tau^{-e}(P_{(a-1)'})$ *and* $X(w_{j_2})\cong\tau^{-c}(P_i)$ *for some* $i\in\{0,1,\dots,p_q+l_1-1\}$ *then*

$$l_{\mathcal{S}}(X(w_{j_2}),X(w_t)) = s'+l_{\mathcal{S}_0}(P_i,P_{a'})+c-e,$$
$$r_{\mathcal{S}}(X(w_{j_2}),X(w_t)) = s+r_{\mathcal{S}_0}(P_i,P_{a'})+e-c.$$

Proof. To prove (1), observe that for $X(w_{i_0})\cong\tau^{-d}(P_z)$ the arguments from the proof of Lemma 2.3 yield

$$l_{\mathcal{S}}(X(w_{i_0}),X(w_{i_1})) = l_{\mathcal{S}_0}(P_z,P_{j''})+d-e,$$
$$r_{\mathcal{S}}(X(w_{i_0}),X(w_{i_1})) = r_{\mathcal{S}_0}(P_z,P_{j''})+e-d.$$

Then the equalities $l_{\mathcal{S}}(X(w_1),X(w_{i_0}))=p+c-d$ and $r_{\mathcal{S}}(X(w_1),X(w_{i_0}))=p'+d-c$ from Lemma 2.3 imply

$$\begin{aligned} l_{\mathcal{S}}(X(w_1),X(w_{i_1}) &= l_{\mathcal{S}}(X(w_1),X(w_{i_0}))+l_{\mathcal{S}}(X(w_{i_0}),X(w_{i_1}))\\ &= p+c-d+l_{\mathcal{S}_0}(P_z,P_{j''})+d-e\\ &= p+l_{\mathcal{S}_0}(P_z,P_{j''})+c-e \end{aligned}$$

and

$$\begin{aligned} r_{\mathcal{S}}(X(w_1),X(w_{i_1}) &= r_{\mathcal{S}}(X(w_1),X(w_{i_0}))+r_{\mathcal{S}}(X(w_{i_0}),X(w_{i_1}))\\ &= p'+d-c+r_{\mathcal{S}_0}(P_z,P_{j''})+e-d\\ &= p'+r_{\mathcal{S}_0}(P_z,P_{j''})+e-c. \end{aligned}$$

Similar arguments yield (2)–(4).

LEMMA 2.6. *Let* $A\cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *and* $\mathcal{C}$ *be the ending component in* Γ_A. *Let* $\mathcal{S}=\{X(w_i)\}_{i=1}^{t}$ *be a preinjective slice in* $\mathcal{C}$.

(1) *If* $X(w_1)\cong\tau^{c}(E_{z-1})$ *and* $X(w_{i_1})\cong\tau^{e}(E_{j''})$ *for some* $j\in\{1,\dots,x_n-1\}$ *then*

$$l_{\mathcal{S}}(X(w_1),X(w_{i_1})) = p'+l_{\mathcal{S}_0}(E_z,E_{j''})+e-c,$$
$$r_{\mathcal{S}}(X(w_1),X(w_{i_1})) = p+r_{\mathcal{S}_0}(E_z,E_{j''})+c-e.$$

(2) *If* $X(w_1)\cong\tau^{c}(E_{z-1})$ *and* $X(w_{i_2})\cong\tau^{f}(E_{j'})$ *for some* $j\in\{0,1,\dots,s_r+l_2-1\}$ *then*

$$l_{\mathcal{S}}(X(w_1),X(w_{i_2})) = p'+l_{\mathcal{S}_0}(E_z,E_{j'})+f-c,$$
$$r_{\mathcal{S}}(X(w_1),X(w_{i_2})) = p+l_{\mathcal{S}_0}(E_z,E_{j'})+c-f.$$

(3) *If* $X(w_t)\cong\tau^{e}(E_{(a-1)'})$ *and* $X(w_{j_1})\cong\tau^{d}(E_{i''})$ *for some* $i\in\{1,\dots,x_n-1\}$ *then*

$$l_{\mathcal{S}}(X(w_{j_1}),X(w_t)) = s+l_{\mathcal{S}_0}(E_{i''},E_{a'})+e-d,$$
$$r_{\mathcal{S}}(X(w_{j_1}),X(w_t)) = s'+r_{\mathcal{S}_0}(E_{i''},E_{a'})+d-e.$$

(4) *If* $X(w_t) \cong \tau^e(E_{(a-1)'})$ *and* $X(w_{j_2}) \cong \tau^c(E_i)$ *for some* $i \in \{0, 1, \ldots, p_q + l_1 - 1\}$ *then*
$$l_{\mathcal{S}}(X(w_{j_2}), X(w_t)) = s + l_{\mathcal{S}_0}(E_i, E_{a'}) + e - c,$$
$$r_{\mathcal{S}}(X(w_{j_2}), X(w_t)) = s' + r_{\mathcal{S}_0}(E_i, E_{a'}) + c - e.$$

Proof. Dual to the proof of Lemma 2.5.

For a given postprojective (resp., preinjective) slice $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ in the starting (resp., ending) component $\mathcal{C}$ we can define $\mathcal{S}' = \{X(v_i)\}_{i=1}^t$, where for each $i = 1, \ldots, t$ the walk v_i satisfies:
$$X(v_i) = \begin{cases} \tau(X(w_i)) & \text{if } X(w_i) \text{ is not projective,} \\ X(w_i) & \text{otherwise} \end{cases}$$
(resp.,
$$X(v_i) = \begin{cases} \tau^{-1}(X(w_i)) & \text{if } X(w_i) \text{ is not injective,} \\ X(w_i) & \text{otherwise).} \end{cases}$$

LEMMA 2.7. *Let* $A \cong A^{(1)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ (*resp.*, $A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$) *and* $\mathcal{C}$ *be the starting* (*resp.*, *ending*) *component in* Γ_A. *Let* $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ *be a postprojective* (*resp.*, *preinjective*) *slice in* $\mathcal{C}$. *Then* $\mathcal{S}' = \{X(v_i)\}_{i=1}^t$ *is a postprojective* (*resp.*, *preinjective*) *slice in* $\mathcal{C}$.

Proof. It is easily seen that if $\mathcal{S}$ consists only of postprojective A-modules then so does $\mathcal{S}'$. Furthermore by [9; Lemma 3.2] there is no finite oriented cycle in $\mathcal{C}$ consisting of modules from $\mathcal{S}'$.

Consider a path $X_0 \to X_1 \to \cdots \to X_m$ in $\mathcal{C}$ such that $X_0, X_m \in \mathcal{S}'$. If $X_0 = \tau(X_0')$ and $X_m = \tau(X_m')$ for some $X_0', X_m' \in \mathcal{S}$ then we have the path $X_0' \to X_1' \to \cdots \to X_{m-1}' \to X_m'$ in $\mathcal{C}$, where $\tau(X_i') = X_i$ for each $i = 1, \ldots, m-1$. Since $\mathcal{S}$ is a postprojective slice, we have $X_1', \ldots, X_{m-1}' \in \mathcal{S}$ and they are not projective, because $\tau(X_i') = X_i$. Thus $X_1, \ldots, X_{m-1} \in \mathcal{S}'$.

If $X_0 = \tau(X_0')$ and $X_m = X_m'$ is projective, and $X_0', X_m' \in \mathcal{S}$, then each X_i is projective, because the above path in $\mathcal{C}$ connects postprojective modules. Now we show inductively on m that $X_1, \ldots, X_{m-1} \in \mathcal{S}'$. If $m = 1$ this is obvious. Assume that this holds for all paths with the above properties whose length is m_0. Let $X_0 \to \cdots \to X_{m_0+1}$ be a path with $X_0, \ldots, X_{m_0+1}$ postprojective such that $X_0, X_{m_0+1} \in \mathcal{S}'$ and $X_{m_0+1} \in \mathcal{S}$ is projective. Then so is X_{m_0}. If $X_{m_0} \notin \mathcal{S}$ then we use the fact that $\mathcal{S}$ contains exactly one representative of any τ^{-1}-orbit of projective A-modules and that a path in $\mathcal{C}$ whose source and target belong to $\mathcal{S}$ passes only through vertices from $\mathcal{S}$. These two facts imply that $\tau^{-1}(X_{m_0}) \in \mathcal{S}$. Thus $X_{m_0} \in \mathcal{S}'$ and so $X_1, \ldots, X_{m_0-1} \in \mathcal{S}'$ by the inductive assumption. Similarly if $X_{m_0} \in \mathcal{S}$ then $X_{m_0} \in \mathcal{S}'$, and we obtain the same conclusion.

If $X_0 \in \mathcal{S}$ is projective and $X_m = \tau(X_m')$ for some $X_m' \in \mathcal{S}$ that is not projective, then we shall prove the required condition inductively on the length m of paths with these properties. The case $m = 1$ is obvious. Assume that, for every path of length m_0 in $\mathcal{C}$ which starts at a projective $X_0 \in \mathcal{S}$ and ends at $X_{m_0} \in \mathcal{S}'$ such that $X_{m_0} = \tau(X'_{m_0})$ with $X'_{m_0} \in \mathcal{S}$, we have all $X_1, \ldots, X_{m_0-1}$ in $\mathcal{S}'$. Consider now a path $X_0 \to X_1 \to \cdots \to X_{m_0+1}$ with the above properties. If X_1 is projective and $X_1 \in \mathcal{S}$ then it is clear that $X_1 \in \mathcal{S}'$, and $X_2, \ldots, X_{m_0} \in \mathcal{S}'$ by the inductive assumption. If X_1 is projective and $X_1 \notin \mathcal{S}$ then as above we find that X_1 must belong to $\mathcal{S}'$, because $\mathcal{S}$ is a postprojective slice, and so contains a representative of the τ^{-1}-orbit of X_1. Since any path starting at X_0 and ending at that representative can only pass through vertices from $\mathcal{S}$, the representative must be X_1.

If X_1 is not projective then $X_1 = \tau(X'_1)$ for some nonprojective $X'_1 \in \mathcal{S}$. But we have the path $X_0 \to X_1 \to X \to X'_1$, where X is from the τ^{-1}-orbit of X_0. Thus $X_1, X \in \mathcal{S}$, which is impossible, because $X_0 \in \mathcal{S}$. This contradiction finishes the proof in the case in question.

Since $\mathcal{S}$ contains exactly one representative from every τ^{-1}-orbit of indecomposable projective A-modules, the same holds for $\mathcal{S}'$ by construction. Consequently, $\mathcal{S}'$ is a postprojective slice in $\mathcal{C}$.

If $\mathcal{S}$ is a preinjective slice in $\mathcal{C}$, one proceeds dually.

Proposition 2.8. *Suppose that the assumptions of Lemma* 2.7 *hold.*

(1) *If* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (*resp.*, $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) *and* $X(w_1) \cong \tau^{-c}(P_{z-1})$, $X(w_{i_0}) \cong \tau^{-e}(P)$ (*resp.*, $X(w_1) \cong \tau^{c}(E_{z-1})$, $X(w_{i_0}) \cong \tau^{e}(E)$) *for some indecomposable projective* (*resp.*, *injective*) *A-module P* (*resp.*, *E*) *and* $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p+1$ (*resp.*, $r_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p+1$) *then there is a nonzero homomorphism* $f : \tau^{-1}(X(w_{i_0})) \to X(w_1)$ (*resp.*, $g : X(w_1) \to \tau(X(w_{i_0}))$) *such that* $\underline{f} \neq 0$ (*resp.*, $\overline{g} \neq 0$) *for* $c \neq 0$.

(2) *If* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (*resp.*, $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) *and* $X(w_t) \cong \tau^{-e}(P_{(a-1)'})$, $X(w_{j_0}) \cong \tau^{-d}(P)$ (*resp.*, $X(w_t) \cong \tau^{e}(E_{(a-1)'})$, $X(w_{j_0}) \cong \tau^{d}(E)$) *for some indecomposable projective* (*resp.*, *injective*) *A-module P* (*resp.*, *E*) *and* $r_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s+1$ (*resp.*, $l_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s+1$) *then there is a nonzero homomorphism* $f : \tau^{-1}(X(w_{j_0})) \to X(w_t)$ (*resp.*, $g : X(w_t) \to \tau(X(w_{j_0}))$) *such that* $\underline{f} \neq 0$ (*resp.*, $\overline{g} \neq 0$) *for* $e \neq 0$.

Proof. Assume that $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $X(w_1) \cong \tau^{-c}(P_{z-1})$, $X(w_{i_0}) \cong \tau^{-e}(P)$ for some indecomposable projective A-module P. Furthermore, assume that $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p+1$. Thus $P \ncong P_{z-1}$.

The first case we consider is $P \cong P_i$ for some $i \in \{0, 1, \ldots, p_q + l_1 - 1\} \setminus \{z-1\}$. Then $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = l_{\mathcal{S}_0}(P_{z-1}, P_i) + c - e = p+1$ by Lemma 2.3, hence $c = e - l_{\mathcal{S}_0}(P_{z-1}, P_i) + p + 1$. Now we prove (1) inductively on c. If

$c = 1$ then the equation $l_{\mathcal{S}_0}(P_{z-1}, P_i) = e + p$ has exactly one solution in the case $e = 0$ and $i = z$. Define $f : \tau^{-1}(P_z) \to \tau^{-1}(P_{z-1})$ to be the composition of an epimorphism $f_1 : \tau^{-1}(P_z) \to X((u^{-1})_L)$ and a monomorphism $f_2 : X((u^{-1})_L) \to \tau^{-1}(P_{z-1})$. It is clear that $\underline{f} = \underline{f_2 f_1} \neq 0$ as desired.

Now assume that for any $c \leq c_0$ the desired f can be found and let $c = c_0+1$. Then we deduce from $c_0+1 = e - l_{\mathcal{S}_0}(P_{z-1}, P_i) + p + 1$ that $c_0 = e - l_{\mathcal{S}_0}(P_{z-1}, P_i) + p$. If $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p+1$ then either $X(w_{i_0})$ is not projective and $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), X((w_{i_0})_{L^{-1}R^{-1}})) = p+1$, or $X(w_{i_0})$ is projective by Lemma 2.7. If $X(w_{i_0})$ is projective but not simple then there exists an irreducible homomorphism $P_{i+1} \to P_i$ and $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), P_{i+1}) = p+1$. If $X(w_{i_0})$ is a simple projective A-module then $X(w_{i_0}) \cong S_{p_j}$ for an odd j and we replace either S_{p_j} with $P_{p_{j-1}-1}$, or S_{p_1} with P_z. In the former case, for $j > 0$, we obviously have $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), P_{p_{j-1}-1}) = p + 1$. In the latter case $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), P_z) = p$, which means that $X((w_1)_{L^{-1}R^{-1}}) \cong P_{z-1}$. Then, as above, we obtain a homomorphism $f : \tau^{-1}(P_z) \to X(w_1) \cong X((u^{-1})_{LR^2})$ that factorizes through $X((u^{-1})_L)$ but not through $X(u^{-1})$, which shows that $\underline{f} \neq 0$.

Now we can consider the other subcases. If $X(w_{i_0})$ is not projective, then $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), X((w_{i_0})_{L^{-1}R^{-1}})) = p + 1$ and by the inductive assumption there is a nonzero homomorphism $f' : \tau^{-1}(X((w_{i_0})_{L^{-1}R^{-1}})) \to X((w_1)_{L^{-1}R^{-1}})$ such that $\underline{f'} \neq 0$. Then $\underline{f} = \tau^{-1}(\underline{f'}) \neq 0$ as desired (in fact, it factorizes through $X((u^{-1})_{L^{c_0+1}})$).

If $X(w_{i_0})$ is projective but not simple then $X(w_{i_0}) \cong P_i$ and we have $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), P_{i+1}) = p + 1$. Then the inductive assumption yields a nonzero homomorphism $f' : \tau^{-1}(P_{i+1}) \to X((w_1)_{L^{-1}R^{-1}})$ that factorizes through $X((u^{-1})L^{c_0})$ and satisfies $\underline{f'} \neq 0$. But then for $P_{i+1} \cong X(w)$ we have $w_{i_0} = w_L$. Therefore there is a nonzero homomorphism $f'' : \tau^{-1}(P_i) \to X(w_1)$ that factorizes through $X((u^{-1})_{L^{c_0+1}})$ and satisfies $\underline{f''} \neq 0$.

If $X(w_{i_0})$ is a simple projective A-module and $X(w_{i_0}) \cong S_{p_j}$ for an odd $j > 1$ and we have replaced S_{p_j} by $P_{p_{j-1}-1}$, then $l_{\mathcal{S}'}(X((w_1)_{L^{-1}R^{-1}}), P_{p_{j-1}-1}) = p + 1$. The inductive assumption yields a nonzero homomorphism $h : \tau^{-1}(P_{p_{j-1}-1}) \to X((w_1)_{L^{-1}R^{-1}})$ that factorizes through $X((u^{-1})_{L^{c_0}})$ and satisfies $\underline{h} \neq 0$. Further, we have an inclusion $h_1 : S_{p_j} \to \tau^{-1}(P_{p_{j-1}-1})$ that induces an embedding $h_2 : \tau^{-1}(S_{p_j}) \to \tau^{-2}(P_{p_{j-1}-1})$. Moreover, h induces a homomorphism $h' : \tau^{-2}(P_{p_{j-1}-1}) \to X(w_1)$ that factorizes through $X((u^{-1})_{L^{c_0+1}})$ and satisfies $\tau^{-1}(\underline{h}) = \underline{h'}$. Then $h'h_2 : \tau^{-1}(S_{p_j}) \to X(w_1)$ is nonzero and does not factorize through $X((u^{-1})_{L^{c_0}})$. Thus $\underline{h'h_2} \neq 0$. This finishes the inductive proof of the first case.

Now we consider the second case: $P \cong P_{j''}$ for some $j \in \{1, \ldots, x_n - 1\}$. Then $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p + l_{\mathcal{S}_0}(P_z, P_{j''}) + c - e = p + 1$ by Lemma 2.5, and hence $l_{\mathcal{S}_0}(P_z, P_{j''}) + c - e = 1$. In this case we shall prove inductively on c that there is a nonzero homomorphism $f : \tau^{-1}(X(w_{i_0})) \to X(w_1)$ that factorizes through $X((u^{-1})_{L^c})$ and satisfies $\underline{f} \neq 0$ for $c \geq 1$.

If $c = 0$ then $l_{\mathcal{S}_0}(P_z, P_{j''}) - e = 1$, hence $l_{\mathcal{S}_0}(P_z, P_{j''}) - 1 = e$. Thus it is easy to verify that, for $\tau^{-e}(P_{j''}) \cong X(w_{i_0})$, there is a walk w' such that $w_{i_0} = \alpha''_{1,2}w'$ provided $x_1 > 1$, and $w_{i_0} = e_{x''_1}w'$ provided $x_1 = 1$, where $e_{x''_1}$ is the trivial walk attached to the vertex x''_1. Thus $\tau^{-1}(X(w_{i_0})) \cong X((w_{i_0})_{LR})$, where $(w_{i_0})_{LR} = \alpha^{-1}_{1,p_1} \ldots \alpha^{-1}_{1,z+1}\alpha''_{1,1}(w_{i_0})_R$. Hence there is a nonzero homomorphism $f : \tau^{-1}(X(w_{i_0})) \to P_{z-1}$ that factorizes through $X((u^{-1})_{L^0}) = X(u^{-1})$. Thus $\underline{f} = 0$, but $c = 0$ and the required condition is satisfied.

Assume that for every postprojective slice $\mathcal{S}$ of $\mathcal{C}$ and any $c \leq c_0$ the required condition holds. Consider now any postprojective slice $\mathcal{S}$ of $\mathcal{C}$ with the above $X(w_{i_0})$, $X(w_1)$ such that $c = c_0+1$. Then, since $p+l_{\mathcal{S}_0}(P_z, P_{j''})+c_0+1-e = p+1$, we have $l_{\mathcal{S}_0}(P_z, P_{j''})+c_0 = e$. As in the case $c = 0$, it is easily seen that $w_{i_0} = (u^{-1})_{L^{c_0}}\alpha''_{1,1}w''$. Thus $\tau^{-1}(X(w_{i_0})) \cong X((w_{i_0})_{LR})$, where $(w_{i_0})_{LR} = (u^{-1})_{L^{c_0+1}}\alpha''_{1,1}w'_R$, which shows that there is a nonzero homomorphism $f : \tau^{-1}(X(w_{i_0})) \to X(w_1)$ that factorizes through $X((u^{-1})_{L^{c_0+1}})$ but not through $X((u^{-1})_{L^{c_0}})$, because $w_1 = (u^{-1})_{L^{c_0}R^{c_0+1}}$. Thus $\underline{f} \neq 0$ and the second case is proved.

Finally, consider the third case: $P \cong P_{j'}$ for some $j \in \{0, 1, \ldots, s_r+l_2-1\}$. Then $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p+l_{\mathcal{S}_0}(P_z, P_{j'})+c-e = p+1$ by Lemma 2.5. Hence $e = l_{\mathcal{S}_0}((P_z, P_{j'})+c-1$. Further, it is easy to verify that $\tau^{-l_{\mathcal{S}_0}(P_z,P_{j'})+1}(P_{j'}) \cong X(w)$, where $w = \alpha''_{1,2}w'$ provided $x_2 - x_1 > 1$, and $w = \alpha''^{-1}_{2,x_3-x_2}w'$ provided $x_2 - x_1 = 1$. Thus $\tau^{-e}(P_{j'}) \cong \tau^{-l_{\mathcal{S}_0}(P_z,P_{j'})+1-c}(P_{j'}) \cong X(w_{L^cR^c})$ and $w_{L^cR^c} = u_{L^{c-1}}\alpha''_{1,1}w_{R^c}$. Moreover, $X(w_1) \cong \tau^{-c}(P_{z-1}) \cong X((u^{-1})_{L^cR^{c+1}})$, yielding an obvious nonzero homomorphism $f : \tau^{-1}(X(w_{i_0})) \to X(w_1)$ that factorizes through $X((u^{-1})_{L^c})$ but not through $X((u^{-1})_{L^{c-1}})$, because $w_{i_0} = w_{L^cR^c} = u_{L^{c-1}}\alpha''_{1,1}w_{R^c}$ and $\tau^{-1}(X(w_{i_0})) \cong X((w_{i_0})_{LR})$ with $(w_{i_0})_{LR} = u_{L^c}\alpha''_{1,1}w_{R^{c+1}}$. Consequently, $\underline{f} \neq 0$ as desired.

Dual arguments show (1) if $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Similarly one can prove (2). We omit the details.

3. Proofs of the main results. For a finite-dimensional K-algebra C, we recall the notions of tilting and cotilting C-module, introduced by Happel and Ringel [7] (see also [1]). A finite-dimensional C-module ${}_CT$ is said to be *tilting* (respectively, *cotilting*) if:

(1) $\operatorname{proj\,dim}({}_CT) \leq 1$ (resp., $\operatorname{inj\,dim}({}_CT) \leq 1$),
(2) $\operatorname{Ext}^1_C(T, T) = 0$,
(3) The number of nonisomorphic indecomposable summands of ${}_CT$ equals the rank of the Grothendieck group $K_0(C)$.

LEMMA 3.1. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) and $\mathcal{C}$ be the starting (resp., ending) component in Γ_A. Let $X, Y \in \mathcal{C}$ be indecomposable postprojective (resp., preinjective) A-modules with $Y \in C_{\rm sc}$ (resp., $X \in C_{\rm ec}$). If there is a nonzero nonisomorphism $f : X \to Y$ such that $\underline{f} \neq 0$ (resp., $\overline{f} \neq 0$) then there is a finite path $X \to \cdots \to Y$ in $\mathcal{C}$.*

Proof. Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Suppose that $X, Y \in \mathcal{C}$ are postprojective with $Y \in C_{\rm sc}$. We shall argue by induction on $n = \dim_K(Y)$.

If $n = 1$ then Y is a simple A-module that is postprojective. Thus Y is projective, and the assertion is obvious.

Now assume that the assertion holds for all Y as above with $\dim_K(Y) \leq n_0$. Consider $Y \in C_{\rm sc}$ with $\dim_K(Y) = n_0 + 1$. Suppose that there is a nonzero nonisomorphism $f : X \to Y$ such that $\underline{f} \neq 0$. Since $\underline{f} \neq 0$, Y is not projective. Consider the Auslander–Reiten sequence

$$0 \to Y(w_{L^{-1}R^{-1}}) \to Y(w_{L^{-1}}) \oplus Y(w_{R^{-1}}) \xrightarrow{(i_1,i_2)} Y(w) \to 0,$$

where w is a walk in $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ such that $Y \cong Y(w)$. Then there are homomorphisms $f_1 : X \to Y(w_{L^{-1}})$ and $f_2 : X \to Y(w_{R^{-1}})$ such that

$$f = (i_1, i_2) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Furthermore, $\underline{f}_1$, $\underline{f}_2$ are not both zero, because $\underline{f} \neq 0$. Suppose that $\underline{f}_1 \neq 0$. By the Skowroński–Waschbüsch algorithm, for Y postprojective, i_1, i_2 are monomorphisms. Therefore $\dim_K(Y(w_{L^{-1}})) \leq n_0$. If f_1 is not an isomorphism then the inductive assumption yields a finite path $X \to \cdots \to Y(w_{L^{-1}})$, because $\underline{f}_1 \neq 0$ and it is clear that $Y(w_{L^{-1}}) \in C_{\rm sc}$. Thus we have a finite path $X \to \cdots \to Y(w_{L^{-1}}) \to Y$ in $\mathcal{C}$. If f_1 is an isomorphism then $X \cong Y(w_{L^{-1}})$ and we have the path $X \to Y$ in $\mathcal{C}$. This finishes the proof for $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

If $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ one proceeds dually; we omit the details.

LEMMA 3.2. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) and $\mathcal{C}$ be the starting (resp., ending) component in Γ_A that is not generalized standard. Let $\mathcal{S} = \{X_i\}_{i=1}^t$ be a postprojective (resp., preinjective) slice in $\mathcal{C}$, contained in the postprojective starting cone $C_{\rm sc}$ (resp., preinjective ending cone $C_{\rm ec}$) in $\mathcal{C}$. Then the slice module $X_{\mathcal{S}} = \bigoplus_{i=1}^t X_i$ is tilting (resp., cotilting).*

Proof. We give the proof for $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ the starting component in Γ_A. If $\mathcal{S} = \{X_i\}_{i=1}^t$ is a postprojective slice in $C_{\rm sc}$, then $\operatorname{proj\,dim}(X_{\mathcal{S}})$

≤ 1 from [9; Corollary 2.5]. Since $\mathcal{S} \subset C_{\mathrm{sc}}$ and there is a nonzero homomorphism $f : \tau^{-1}(X_{i_0}) \to X_{i_1}$, suppose that $\underline{f} \neq 0$. Then Lemma 3.1 yields a finite path $\tau^{-1}(X_{i_0}) \to \cdots \to X_{i_1}$ in $\mathcal{C}$. Thus we have a path $X_{i_0} \to Y \to \tau^{-1}(X_{i_0}) \to \cdots \to X_{i_1}$ in $\mathcal{C}$ with $X_{i_0}, X_{i_1} \in \mathcal{S}$. Hence $\tau^{-1}(X_{i_0}) \in \mathcal{S}$, which contradicts the fact that $\mathcal{S}$ is a postprojective slice, because $X_{i_0}, \tau^{-1}(X_{i_0}) \in \mathcal{S}$. Consequently, $\underline{f} = 0$. Applying the Auslander–Reiten formula $D\underline{\mathrm{Hom}}_A(\tau^{-1}(X_{\mathcal{S}}), X_{\mathcal{S}}) = \mathrm{Ext}^1_A(X_{\mathcal{S}}, X_{\mathcal{S}})$ we see that $\mathrm{Ext}^1_A(X_{\mathcal{S}}, X_{\mathcal{S}}) = 0$, so $X_{\mathcal{S}}$ is tilting.

The case $A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ is dual; we omit the details.

Proof of Theorem 1. Let $\mathcal{S}$ be a postprojective slice in $\mathcal{C}$. If it is contained in C_{sc} then $X_{\mathcal{S}}$ is tilting by Lemma 3.2.

Now suppose that a postprojective slice $\mathcal{S} = \{X_i\}_{i=1}^t$ in $\mathcal{C}$ is such that $X_{\mathcal{S}} = \bigoplus_{i=1}^t X_i$ is tilting. Suppose to the contrary that $\mathcal{S}$ is not contained in C_{sc}. Let $X_1 \cong X(w_1) \cong \tau^{-c}(P_{z-1})$ and $X_t \cong X(w_t) \cong \tau^{-e}(P_{(a-1)'})$. Then Lemma 2.5 shows that $l_{\mathcal{S}}(X(w_1), X(w_t)) > p$ or $r_{\mathcal{S}}(X(w_1), X(w_t)) > s$.

Consider the case $l_{\mathcal{S}}(X(w_1), X(w_t)) > p$. Take the minimal $i_0 \in \{1, \ldots, t\}$ such that $l_{\mathcal{S}}(X(w_1), X(w_{i_0})) = p + 1$. If $c > 1$ then Proposition 2.8 yields an $f : \tau^{-1}(X(w_{i_0})) \to X(w_1)$ with $\underline{f} \neq 0$. Thus $\mathrm{Ext}^1_A(X_{\mathcal{S}}, X_{\mathcal{S}}) \neq 0$ by the Auslander–Reiten formula, which contradicts the assumption that $X_{\mathcal{S}}$ is tilting. Hence $c = 0$. Then applying again Lemma 2.5 we find that $r_{\mathcal{S}}(X(w_1), X(w_t)) > s$. Take the maximal $j_0 \in \{1, \ldots, t\}$ such that $r_{\mathcal{S}}(X(w_{j_0}), X(w_t)) = s + 1$. If $e > 0$ Proposition 2.8 yields an $f : \tau^{-1}(X(w_{j_0})) \to X(w_t)$ with $\underline{f} \neq 0$, so again $\mathrm{Ext}^1_A(X_{\mathcal{S}}, X_{\mathcal{S}}) \neq 0$, a contradiction. Hence $e = 0$. Consequently, $P_{z-1}, P_{(a-1)'} \in \mathcal{S}$. Thus it is clear that $\mathcal{S} \subset C_{\mathrm{sc}}$ as desired.

In the case $r_{\mathcal{S}}(X(w_1), X(w_t)) > s$ we proceed symmetrically.

We omit the details of the case when $\mathcal{S}$ is a preinjective slice in $\mathcal{C}$.

LEMMA 3.3. *Let $A \cong A^{(1)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$* (*resp.*, *$A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$*) *and $\mathcal{C}$ be the starting* (*resp.*, *ending*) *component in Γ_A that is not generalized standard. Then*, *for an indecomposable projective* (*resp.*, *injective*) *A-module $X(w)$*:

(1) *$l_{\mathcal{S}_0}(P_z, X(w)) = d$* (*resp.*, *$r_{\mathcal{S}_0}(E_z, X(w)) = d$*), *where d is the minimal nonnegative integer such that w_{L^d}* (*resp.*, *$w_{R^{-d}}$*) *is of the form $w_{L^d} = \alpha^{-1}_{1,p_1} \ldots \alpha^{-1}_{1,z+1} \alpha''_{1,1} w'$* (*resp.*, *$w_{R^{-d}} = w' \alpha''_{1,1} \alpha^{-1}_{1,z+1} \ldots \alpha^{-1}_{1,p_1}$*).

(2) *$r_{\mathcal{S}_0}(X(w), P_{a'}) = b$* (*resp.*, *$l_{\mathcal{S}_0}(X(w), E_{a'}) = b$*), *where b is the minimal nonnegative integer such that w_{R^b}* (*resp.*, *$w_{L^{-b}}$*) *is of the form $w_{R^b} = w' \alpha''^{-1}_{n,1} \alpha'_{1,a+1} \ldots \alpha'_{1,s_1}$* (*resp.*, *$w_{L^{-b}} = \alpha'_{1,s_1} \ldots, \alpha'_{1,a+1} \alpha''^{-1}_{n,1} w'$*).

Proof. Consider the case of $A \cong A^{(1)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ and $\mathcal{C}$ the starting component in Γ_A. It is clear that $l_{\mathcal{S}_0}(P_z, P)$ is defined if $P \in \{P_z, P_{j''}, P_{i'}\}$ for all possible j and i. Now we proceed inductively on $d = l_{\mathcal{S}_0}(P_z, X(w))$. If

$d = 0$ then $X(w) \cong P_z$ and 0 is the minimal nonnegative integer d such that $w_{L^d} = \alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}\alpha''_{1,1} \dots \alpha''_{1,x_2-x_1}$.

Assume that (1) holds for all indecomposable projective A-modules P with $l_{\mathcal{S}_0}(P_z, P) \le d_0$. Suppose that $X(w)$ is an indecomposable projective A-module such that $l_{\mathcal{S}_0}(P_z, X(w)) = d_0 + 1$. Then consider $X(w_L)$. By the Skowroński–Waschbüsch algorithm, $w_L = \beta_1^{-1} \dots \beta_c^{-1}\alpha w$, where $\beta_c \dots \beta_1$ is a maximal nonzero path or is trivial, and α is an arrow. But $X(w)$ is projective, hence either $w = \delta_b \dots \delta_1$, $w = \gamma_1^{-1} \dots \gamma_e^{-1}$ or $w = \gamma_1^{-1} \dots \gamma_e^{-1}\delta_b \dots \delta_1$ for some integers b, e. If $w = \delta_b \dots \delta_1$ then $X(w_L)$ is projective with $l_{\mathcal{S}_0}(P_z, X(w_L)) = d_0$. Then by the inductive assumption and the Skowroński–Waschbüsch algorithm, $w_{L^{d_0+1}} = \alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}\alpha''_{1,1}w'$ and $d_0 + 1$ is minimal with this property.

If $w = \gamma_1^{-1} \dots \gamma_e^{-1}$ then the indecomposable A-module $X(\beta_1^{-1} \dots \beta_c^{-1}\alpha)$ is projective and $l_{\mathcal{S}_0}(P_z, X(\beta_1^{-1} \dots \beta_c^{-1}\alpha)) = d_0$, so by the inductive assumption $(\beta_1^{-1} \dots \beta_c^{-1}\alpha)_{L^{d_0}} = \alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}\alpha''_{1,1}w'$. Thus $(w_L)_{L^{d_0}} = w_{L^{d_0+1}} = \alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}\alpha''_{1,1}w'\gamma_1^{-1} \dots \gamma_e^{-1}$ and $d_0 + 1$ is minimal with this property.

If $w = \gamma_1^{-1} \dots \gamma_e^{-1}\delta_b \dots \delta_1$ then again $X(\beta_1^{-1} \dots \beta_c^{-1}\alpha)$ is projective and we get (1) just as for $w = \gamma_1^{-1} \dots \gamma_e^{-1}$.

Similarly one proves (2). If $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ then we use dual arguments.

LEMMA 3.4. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) and $\mathcal{C}$ be the starting (resp., ending) component in Γ_A that is not generalized standard. Let $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ be a postprojective (resp., preinjective) slice in $\mathcal{C}$, contained in $C_{\rm sc}$ (resp., $C_{\rm ec}$). Then*:

(1) *$P_{z-1}, P_{(a-1)'} \in \mathcal{S}$ (resp., $E_{z-1}, E_{(a-1)'} \in \mathcal{S}$).*

(2) *If $l_{\mathcal{S}}(P_{z-1}, X(w_{i_0})) = p$ (resp., $r_{\mathcal{S}}(E_{z-1}, X(w_{i_0})) = p$) then*
$$w_{i_0} = \alpha^{-1}_{1,p_1} \dots \alpha^{-1}_{1,z+1}\alpha''_{1,1}w' \ (\textit{resp.}, \ w_{i_0} = w'\alpha''_{1,1}\alpha^{-1}_{1,z+1} \dots \alpha^{-1}_{1,p_1}).$$

(3) *If $r_{\mathcal{S}}(X(w_{j_0}), P_{(a-1)'}) = s$ (resp., $l_{\mathcal{S}}(X(w_{j_0}), E_{(a-1)'}) = s$) then*
$$w_{j_0} = w''\alpha''^{-1}_{n,1}\alpha'_{1,a+1} \dots \alpha'_{1,s_1} \ (\textit{resp.}, \ w_{j_0} = \alpha'_{1,s_1} \dots \alpha'_{1,a+1}\alpha''_{n,1}w'').$$

Proof. Consider the case of $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ the starting component in Γ_A. Let $\mathcal{S} \subset C_{\rm sc}$ be a slice. Then by the definition of a postprojective starting cone, $P_{z-1}, P_{(a-1)'} \in \mathcal{S}$ and (1) holds.

To prove (2) consider an indecomposable projective A-module P and an integer $d \ge 0$ such that $X(w_{i_0}) \cong \tau^{-d}(P)$. Then Lemmas 2.3 and 2.5 show that $l_{\mathcal{S}}(P_{z-1}, X(w_{i_0}))$ is independent of the choice of the slice $\mathcal{S}$.

If $l_{\mathcal{S}}(P_{z-1}, X(w_{i_0})) = p$ then we deduce from Lemmas 2.3 and 2.5 that $l_{\mathcal{S}_0}(P_z, P) = d$ and $X(w_{i_0}) \cong \tau^{-d}(P)$. Hence, Lemma 3.3(1) and the Skowroński–Waschbüsch algorithm imply that $w_{i_0} = w_{L^dR^d}$ for some w such that $P \cong X(w)$ and the required shape of w_{i_0} is clear.

Similar arguments prove (3). For $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ the ending component in Γ_A, $\mathcal{S} \subset C_{\mathrm{ec}}$, we proceed dually.

LEMMA 3.5. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) and $\mathcal{C}$ be the starting (resp., ending) component in Γ_A that is not generalized standard. If X, Y are inner modules in C_{sc} (resp., C_{ec}) then any nonzero homomorphism $f : X \to Y$ is a linear combination of paths in C_{sc} (resp., C_{ec}) from X to Y.*

Proof. Suppose that $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ is the starting component in Γ_A. Consider two indecomposable inner A-modules X, Y in C_{sc}. We use induction on $n = \dim_K(Y)$. If $n = 1$ then Y is a simple A-module that is inner in C_{sc}. Thus Y is simple projective and the assertion is obvious.

Assume that it holds for all inner A-modules X, Y in C_{sc} such that $\dim_K(Y) \le n_0$.

Suppose that X, Y are inner A-modules in C_{sc} such that $\dim_K(Y) = n_0 + 1$. Observe that Y cannot be simple. Thus we have to consider two cases. In the first case, there is only one indecomposable inner A-module Z in C_{sc} such that we have an arrow $Z \to Y$ in $\mathcal{C}$. The Skowroński–Waschbüsch algorithm then yields an irreducible homomorphism $h : Z \to Y$ that is a monomorphism. Hence $\dim_K(Z) < \dim_K(Y)$ and every homomorphism $f : X \to Z$ is a linear combination of paths in C_{sc} from X to Z by the inductive assumption. Every homomorphism $g : X \to Y$ is then clearly of the form $g = hg_1$ and the assertion is obvious.

In the second case, there are exactly two indecomposable inner A-modules Z_1, Z_2 in C_{sc} such that we have irreducible homomorphisms $h_1 : Z_1 \to Y$, $h_2 : Z_2 \to Y$. Then the argument is similar.

If $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ then we argue dually.

We shall say a homomorphism $f : X \to Y$ between indecomposable A-modules X, Y is *combinatorial* provided it is either an isomorphism or a linear combination of finite compositions of finitely many irreducible homomorphisms. Then the above lemma says that any homomorphism between inner modules of C_{sc} or C_{ec} is combinatorial.

Let $X_1, \ldots, X_n$ be indecomposable A-modules. We shall denote by $\mathrm{End}^{\mathrm{comb}}_A(\bigoplus_{i=1}^n X_i)$ the subalgebra of $\mathrm{End}_A(\bigoplus_{i=1}^n X_i)$ consisting of all homomorphisms $(f_{lj}) : \bigoplus_{i=1}^n X_i \to \bigoplus_{i=1}^n X_i$ such that $f_{lj} : X_l \to X_j$ is combinatorial for all $1 \le l, j \le n$.

PROPOSITION 3.6. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ (resp., $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$) and $\mathcal{C}$ be the starting (resp., ending) component in Γ_A, that is not generalized standard. Let $\mathcal{S} = \{X(w_i)\}_{i=1}^t$ be a postprojective (resp., preinjective) slice in $\mathcal{C}$, contained in C_{sc} (resp., C_{ec}), and set $X_{\mathcal{S}} = \bigoplus_{i=1}^t X(w_i)$. Then:*

(1) *There are homomorphisms* $0 \neq f_1 : X(w_{i_0}) \to P_{z-1}$, $0 \neq f_2 : X(w_{j_0}) \to P_{(a-1)'}$ (*resp.*, $0 \neq f_1 : E_{z-1} \to X(w_{i_0})$, $0 \neq f_2 : E_{(a-1)'} \to X(w_{j_0})$) *such that*:

(1a) f_1, f_2 *are not combinatorial*,

(1b) $l_{\mathcal{S}}(P_{z-1}, X(w_{i_0})) = p$ (*resp.*, $r_{\mathcal{S}}(E_{z-1}, X(w_{i_0})) = p$) *and* i_0 *is maximal with this property*,

(1c) $r_{\mathcal{S}}(X(w_{j_0}), P_{(a-1)'}) = s$ (*resp.*, $l_{\mathcal{S}}(X(w_{j_0}), E_{(a-1)'}) = s$) *and* j_0 *is minimal with this property*,

(1d) *if* $0 \neq f : X(w_{i_1}) \to X(w_{j_1})$ *is not combinatorial then* f *factorizes through either* f_1 *or* f_2.

(2) $\mathrm{End}_A^{\mathrm{comb}}(X_{\mathcal{S}})$ *is a hereditary algebra of type* $\mathbb{A}_t$.

Proof. Consider the case of $A \cong A^{(1)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ and $\mathcal{C}$ the starting component of $\varGamma_A$. Let $\mathcal{S} \subset \mathcal{C}_{\mathrm{sc}}$ be a slice. Then $P_{z-1}, P_{(a-1)'} \in \mathcal{S}$ by Lemma 3.4. From Lemma 3.4 we know that if $l_{\mathcal{S}}(P_{z-1}, X(w_{i_0})) = p$ then $w_{i_0} = \alpha^{-1}_{1,p_1} \ldots \alpha^{-1}_{1,z+1} \alpha''_{1,1} w'$. Thus there is a nonzero homomorphism $f_1 : X(w_{i_0}) \to P_{z-1}$ whose image is $X(u^{-1})$; choose one with i_0 maximal. Then it is obvious that any other homomorphism $g : X(w_{i_1}) \to P_{z-1}$ factorizes through f_1. Furthermore, f_1 is clearly not combinatorial. Similarly there is a nonzero homomorphism $f_2 : X(w_{j_0}) \to P_{(a-1)'}$ whose image is $X(v)$; choose one with j_0 minimal. Then again any other homomorphism $h : X(w_{i_2}) \to P_{(a-1)'}$ factorizes through f_2, and f_2 is not combinatorial.

Now consider $0 \neq f : X(w_{i_1}) \to X(w_{j_1})$ that is not combinatorial. Then by Lemma 3.5 at least one of $X(w_{i_1})$, $X(w_{j_1})$ is not inner. If $X(w_{i_1})$ is not inner then neither is $X(w_{j_1})$ by the shape of $\mathcal{C}$. Since we have a nonzero homomorphism $f : X(w_{i_1}) \to X(w_{j_1})$ and both $X(w_{i_1}), X(w_{j_1}) \in \mathcal{C}_{\mathrm{sc}}$, we have a chain of irreducible monomorphisms $X(w_{i_1}) \xrightarrow{g_1} X(w_{i_2}) \xrightarrow{g_2} \cdots \to X(w_{i_d}) \xrightarrow{g_d} X(w_{j_1})$ by the Skowroński–Waschbüsch algorithm and the definition of $\mathcal{C}_{\mathrm{sc}}$. Furthermore, $f = k \cdot g_d \ldots g_2 g_1$ for some $k \in K^*$, which contradicts the fact that f is not combinatorial. Therefore $X(w_{i_1})$ is inner and $X(w_{j_1})$ is not. Hence there is a combinatorial homomorphism $g_1 : P_{z-1} \to X(w_{j_1})$ or $g_2 : P_{(a-1)'} \to X(w_{j_1})$ such that any homomorphism from an inner module factorizes through g_1 or g_2. Thus f factorizes through either f_1 or f_2 as desired.

To prove (2) we observe that the irreducible homomorphisms between $X(w_i)$, $X(w_j)$ and identical isomorphisms form a basis of $\mathrm{End}_A^{\mathrm{comb}}(X_{\mathcal{S}})$. Thus it is clear from the properties of a slice contained in $\mathcal{C}_{\mathrm{sc}}$ that the algebra in question is hereditary of type $\mathbb{A}_t$.

For $A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ we proceed dually.

Let B be a hereditary algebra of type $\mathbb{A}_m$. For positive integers c, b with $c + b < m$ we shall construct a new algebra $B(c, b)$ under some additional assumptions on B. Suppose that the Gabriel quiver Q_B of B is of the form $1 — \cdots — m$ and let c, b be positive integers such that $c+b<m$ and there exists the maximal vertex i_c with $r(1,i_c)=c$ and there exists the minimal vertex j_b with $l(j_b,m)=b$. Here $r(1,i_c)$ is the number of right arrows in the subquiver $1 — \cdots — i_c$ of Q_B, and $l(j_b,m)$ the number of left arrows in the subquiver $j_b — \cdots — m$ of Q_B. Then we consider a new quiver $Q_{B(c,b)}$ obtained from Q_B by adjoining two arrows $1 \xrightarrow{\gamma} i_c$, $j_b \xleftarrow{\delta} m$. By the maximality of i_c there is an arrow $i_c \xrightarrow{\alpha} (i_c+1)$ or $i_c = m$. By the minimality of j_b there is an arrow $(j_b-1) \xleftarrow{\beta} j_b$ or $j_b = 1$. Consider the two-sided ideal $I_{B(c,b)}$ in $KQ_{B(c,b)}$ generated by the paths $\gamma\alpha$ if $i_c \neq m$ and $\delta\beta$ if $j_b \neq 1$ and $\gamma\delta$ if $i_c = m$ and $\delta\gamma$ if $j_b = 1$. Then we call the algebra $B(c,b) = KQ_{B(c,b)}/I_{B(c,b)}$ an *(m,c,b)-algebra*. One can also consider the dual construction that leads to an *$(m,c,b)^{\mathrm{op}}$-algebra*.

Proof of Theorem 2. Let A be a minimal 2-fundamental algebra and let $\mathcal{C}$ be the starting component in Γ_A that is not generalized standard. By Lemma 1.1(1), $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ for some $\underline{p}$, $\underline{x}$, $\underline{s}$, l_1, l_2. Let $\mathcal{S} = \{X(w_i)\}_{i=1}^{t}$ be a postprojective slice in $\mathcal{C}$, contained in C_{sc}. By Proposition 3.6(2), $\mathrm{End}_A^{\mathrm{comb}}(X_{\mathcal{S}})$ is hereditary of type $\mathbb{A}_t$. Moreover, Proposition 3.6(1) shows that $\mathrm{End}_A(X_{\mathcal{S}})^{\mathrm{op}}$ is obtained from $\mathrm{End}_A^{\mathrm{comb}}(X_{\mathcal{S}})^{\mathrm{op}}$ by the above construction for $c=p$ and $b=s$. Therefore $\mathrm{End}_A(X_{\mathcal{S}})^{\mathrm{op}}$ is a (t,p,s)-algebra.

If A is a minimal 2-fundamental algebra and $\mathcal{C}$ is the ending component in Γ_A that is not generalized standard, then we proceed dually to conclude that $\mathrm{End}_A(X_{\mathcal{S}})^{\mathrm{op}}$ is a $(t,p,s)^{\mathrm{op}}$-algebra.

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