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AN ELEMENTARY EXACT SEQUENCE OF MODULES WITH AN APPLICATION TO TILED ORDERS

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Abstract. Let $m \geq 2$ be an integer. By using m submodules of a given module, we construct a certain exact sequence, which is a well known short exact sequence when $m = 2$. As an application, we compute a minimal projective resolution of the Jacobson radical of a tiled order.

Let R be a ring with an identity, and let M be a right R-module. For R-submodules X, Y of M, there is an elementary short exact sequence

$$0 \to X \cap Y \xrightarrow{\eta} X \oplus Y \xrightarrow{\varphi} X + Y \to 0$$

where $\eta(t) = (t, -t)$ for $t \in X \cap Y$ and $\varphi(x, y) = x + y$ for $(x, y) \in X \oplus Y$. In this paper, we extend this elementary short exact sequence to the case of more than two R-submodules of a given right R-module, and as an application, we compute a minimal projective resolution of the Jacobson radical of a tiled order given by Fujita and Oshima [5], which provides a tiled order of finite global dimension without neat primitive idempotent (see [1], [4] for neat primitive idempotents, [3], [4], [6]–[8], [10], [13] for global dimension of tiled orders), and e.g. [11], [12], [14], [15] for further facts on tiled orders).

In [6], Jansen and Odenthal found a series of tiled orders having large global dimension. In order to compute the global dimension of their tiled orders, they used a short exact sequence constructed with three irreducible lattices. We begin by clarifying the short exact sequence used in [6].

Proposition 1. *Let X, Y, Z be R-submodules of a right R-module M. Let*

$$(X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y) \xrightarrow{\psi} X \oplus Y \oplus Z \xrightarrow{\varphi} X + Y + Z \to 0$$

be a sequence of R-modules and R-homomorphisms defined by

$$\varphi(x, y, z) = x + y + z, \qquad \psi(x_0, y_0, z_0) = (x_0 - y_0,\ y_0 - z_0,\ z_0 - x_0)$$

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for all $(x, y, z) \in X \oplus Y \oplus Z$ *and* $(x_0, y_0, z_0) \in (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)$. *Then*:

(1) $\operatorname{Ker} \psi \cong X \cap Y \cap Z$, $\operatorname{Im} \psi \subset \operatorname{Ker} \varphi$ *and* φ *is surjective.*

(2) *The following are equivalent*:

(a) $\operatorname{Im} \psi = \operatorname{Ker} \varphi$.

(b) $(X + Y) \cap Z \subset X + (Y \cap Z)$.

(c) *For any* $(x, y, z) \in \operatorname{Ker} \varphi$, *there exists* $x_0 \in X \cap Z$ *such that* $x_0 - x \in Y$.

(3) *If two of* X, Y, Z *are related by inclusion*, *then* $\operatorname{Im} \psi = \operatorname{Ker} \varphi$.

(4) *Suppose that* $X \cap Y \cap Z = Y \cap Z$. *Then* $\operatorname{Im} \psi \cong (X \cap Z) \oplus (Y \cap X)$. *If the equivalent conditions of* (2) *hold*, *then there is a short exact sequence*

$$0 \to (X \cap Z) \oplus (Y \cap X) \to X \oplus Y \oplus Z \xrightarrow{\varphi} X + Y + Z \to 0.$$

Proof. (1) Straightforward.

(2) (a)$\Rightarrow$(b) Take an arbitrary $x + y = z \in (X + Y) \cap Z$ where $x \in X$, $y \in Y$, $z \in Z$. Then $(x, y, -z) \in \operatorname{Ker} \varphi$. Hence $(x, y, -z) = (x_0 - y_0, y_0 - z_0, z_0 - x_0)$ for some $(x_0, y_0, z_0) \in (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)$. Then $z = x_0 - z_0 \in X + (Y \cap Z)$.

(b)$\Rightarrow$(c) Let $(x, y, z) \in \operatorname{Ker} \varphi$. Then $z = -x - y \in (X + Y) \cap Z \subset X + (Y \cap Z)$. Hence $z = -x_0 + z_0$ for some $x_0 \in X$ and $z_0 \in Y \cap Z$. Hence $x_0 - x = x_0 + y + z = y + z_0 \in Y$.

(c)$\Rightarrow$(a) Take an arbitrary $(x, y, z) \in \operatorname{Ker} \varphi$. Then there exists $x_0 \in X \cap Z$ such that $x_0 - x \in Y$. Put $y_0 = x_0 - x$ and $z_0 = y_0 - y$. Then $y_0 = x_0 - x \in Y \cap X$ and $z_0 = y_0 - y = y_0 + x + z = x_0 + z \in Z \cap Y$. Hence $(x, y, z) = \psi(x_0, y_0, z_0) \in \operatorname{Im} \psi$.

(3) If $X \subset Y$, then $(X + Y) \cap Z = Y \cap Z \subset X + (Y \cap Z)$. If $Y \subset X$, then $(X + Y) \cap Z = X \cap Z \subset X + (Y \cap Z)$. If $X \subset Z$, then $(X + Y) \cap Z = X + (Y \cap Z)$ by the modular law. If $Z \subset X$, then $(X + Y) \cap Z \subset Z \subset X + (Y \cap Z)$. If $Y \subset Z$, then $(X + Y) \cap Z \subset X + Y = X + (Y \cap Z)$. If $Z \subset Y$, then $(X + Y) \cap Z \subset Z \subset X + (Y \cap Z)$.

(4) Since $X \cap Y \cap Z = Z \cap Y$, we can define an R-isomorphism

$$\theta : (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y) \to (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)$$

by $\theta(x, y, z) = (x - z, y - z, z)$ for all $(x, y, z) \in (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)$. Then we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \cap Y \cap Z \arrow[r, "\eta"] \arrow[d, equal] & (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y) \arrow[d, "\theta"] \\
0 \arrow[r] & X \cap Y \cap Z \arrow[r, "i"] & (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)
\end{tikzcd}$$

where $\eta(t) = (t,t,t)$ and $i(t) = (0,0,t)$ for all $t \in X \cap Y \cap Z$. Hence

$$\operatorname{Im}\psi \cong \operatorname{Coker}\eta \cong \operatorname{Coker} i \cong (X\cap Z) \oplus (Y \cap X). \ \blacksquare$$

In what follows, D is a discrete valuation ring with a unique maximal ideal πD and a quotient field K.

Let $n \geq 2$ be an integer, and let $\{\lambda_{ij} \mid 1 \leq i,j \leq n\}$ be a set of n^2 integers satisfying $\lambda_{ii} = 0$, $\lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}$ for all $1 \leq i,j,k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a semiperfect Noetherian D-subalgebra of the full $n \times n$ matrix algebra $\mathbb{M}_n(K)$, and Λ is a D-order in $\mathbb{M}_n(K)$ (see [9]). Following [7] and [13], we call such a D-order Λ a *tiled D-order* in $\mathbb{M}_n(K)$ (see also Chapter 13 of [11]). We note that Λ is *basic* if and only if $\lambda_{ij} + \lambda_{ji} > 0$ for all $1 \leq i,j \leq n$ with $i \neq j$.

Let $V = K^n = (K,\dots,K)$ be a simple right $\mathbb{M}_n(K)$-module. Assume that $a_1,\dots,a_n$ are integers satisfying $a_i + \lambda_{ij} \geq a_j$ for all $1 \leq i,j \leq n$. Then $L = (\pi^{a_1}D,\dots,\pi^{a_n}D)$ is a right Λ-submodule of V. We call L an *irreducible right Λ-lattice* in V (see [10] and [15]).

The following fact is well known (see Lemmas 1.9, 1.10 of [6]).

Corollary 2. *Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D-order in $\mathbb{M}_n(K)$, and let X, Y, Z be irreducible right Λ-lattices in $V = K^n$. Then*:

(1) *There is an exact sequence*

$$0 \to X \cap Y \cap Z \xrightarrow{\eta} (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)$$
$$\xrightarrow{\psi} X \oplus Y \oplus Z \xrightarrow{\varphi} X + Y + Z \to 0$$

of right Λ-lattices.

(2) *Suppose that $X \cap Y \cap Z = Y \cap Z$. Then there is a short exact sequence*

$$0 \to (X\cap Z) \oplus (Y \cap X) \to X \oplus Y \oplus Z \to X + Y + Z \to 0$$

of right Λ-lattices.

Proof. (1) By Proposition 1(1), it is sufficient to show that $\operatorname{Im}\psi = \operatorname{Ker}\varphi$. Put $X = (X_1,\dots,X_n)$, $Y = (Y_1,\dots,Y_n)$, $Z = (Z_1,\dots,Z_n)$ where X_j, Y_j, Z_j $(1 \leq j \leq n)$ are nonzero ideals of D. Let $(x,y,z) \in \operatorname{Ker}\varphi$, and let $x = (x_j), y = (y_j), z = (z_j)$ where $x_j \in X_j$, $y_j \in Y_j$, $z_j \in Z_j$ for $j = 1,\dots,n$. Then $x_j + y_j + z_j = 0$ for each $1 \leq j \leq n$. Since D is a discrete valuation ring, X_j, Y_j, Z_j can be linearly ordered by inclusion, for each $1 \leq j \leq n$. Hence by (3) and (2) of Proposition 1, we can find $x_0 = (x_{0j}) \in X \cap Z$ such that $x_0 - x \in Y$. Hence Proposition 1(2) implies that $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.

(2) This follows from (1) and Proposition 1(4). $\blacksquare$

Lemma 3. *Let X, Y, Z be nonzero ideals of a principal ideal domain. Then $(X+Y) \cap Z \subset X + (Y \cap Z)$.*

Proof. Since each nonzero ideal of a principal ideal domain is generated by a product of prime elements, it suffices to show that $\max\{\min\{\alpha,\beta\},\gamma\}$

$\geq \min\{\alpha, \max\{\beta, \gamma\}\}$ for any integers $\alpha, \beta, \gamma \geq 0$. If $\alpha \leq \beta \leq \gamma$, then $\max\{\min\{\alpha, \beta\}, \gamma\} = \gamma \geq \alpha = \min\{\alpha, \max\{\beta, \gamma\}\}$. Similarly, we can check the remaining cases. ∎

REMARK. (1) The converse of Proposition 1(3) does not hold in general. By Lemma 3, we can find such examples among ideals of principal ideal domains. In fact, for example, let $R = \mathbb{Z}$ be the ring of integers, and let $X = 2\mathbb{Z}$, $Y = 3\mathbb{Z}$, $Z = 5\mathbb{Z}$. Then $(2\mathbb{Z}+3\mathbb{Z})\cap 5\mathbb{Z} = 5\mathbb{Z} \subset \mathbb{Z} = 2\mathbb{Z}+(3\mathbb{Z}\cap 5\mathbb{Z})$, but no two of $2\mathbb{Z}$, $3\mathbb{Z}$, $5\mathbb{Z}$ are related by inclusion.

(2) The sequence of Proposition 1 is not exact in general. In fact, let $R = \mathbb{Z}[t]$ be the polynomial ring over $\mathbb{Z}$ in the indeterminate t, and let $X = 2R$, $Y = tR$, $Z = (2+t)R$. Then $(X+Y)\cap Z \not\subset X + (Y\cap Z)$, because $2 + t \notin X + (Y \cap Z)$.

Next, we explore analogous elementary exact sequences constructed by using more than three submodules of a given module.

PROPOSITION 4. *Let R be an arbitrary ring and let $X_1, \ldots, X_m = X_0$ be R-submodules of a right R-module M, where $m \geq 3$. Let*

$$\bigoplus_{i=1}^{m}(X_i \cap X_{i-1}) \overset{\psi}{\to} \bigoplus_{i=1}^{m} X_i \overset{\varphi}{\to} \sum_{i=1}^{m} X_i \to 0$$

be a sequence of R-modules and R-homomorphisms defined by

$$\psi(y_1, \ldots, y_m) = (y_1-y_2,\ \ldots,\ y_{m-1}-y_m,\ y_m-y_1) \ \text{and}\ \varphi(x_1, \ldots, x_m)=\sum_{i=1}^{m} x_i$$

for $(y_1, \ldots, y_m) \in \bigoplus_{i=1}^{m}(X_i \cap X_{i-1})$ and $(x_1, \ldots, x_m) \in \bigoplus_{i=1}^{m} X_i$. Then:

(1) $\operatorname{Ker}\psi \cong \bigcap_{i=1}^{m} X_i$, $\operatorname{Im}\psi \subset \operatorname{Ker}\varphi$ *and φ is surjective.*
(2) *For any fixed $1 \leq a \leq m$, the following two statements are equivalent:*
   (a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.
   (b) *For any $(x_i) \in \operatorname{Ker}\varphi$, there exists $y \in X_a \cap X_m$ such that $y - (x_a + \cdots + x_t) \in X_{t+1}$ for all t ($a \leq t \leq a + m - 3$), where the indices are counted modulo m.*
(3) *Suppose that there exist $1 \leq a, b \leq m$ such that $X_a \subset X_i \subset X_b$ for all $1 \leq i \leq m$. Then the following two statements are equivalent:*
   (a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.
   (b) *$X_a \subset X_{a+1} \subset \cdots \subset X_{b-1} \subset X_b$ and $X_a \subset X_{a-1} \subset \cdots \subset X_{b+1} \subset X_b$, where the indices are counted modulo m.*

Proof. (1) Straightforward.
(2) We can assume that $a = 1$ by shifting the indices.

(a)⇒(b) Let $(x_i) \in \operatorname{Ker}\varphi$. Since $\operatorname{Ker}\varphi = \operatorname{Im}\psi$, we have $x_i = y_i - y_{i+1}$ $(1 \le i \le m)$ for some $(y_i) \in \bigoplus_{i=1}^m (X_i \cap X_{i-1})$, where $y_{m+1} := y_1$. Put $y := y_1 \in X_1 \cap X_m$. Then, for $t = 1, \ldots, m-2$,

$$y - (x_1 + \cdots + x_t) = y_1 - \sum_{i=1}^{t} (y_i - y_{i+1}) = y_{t+1} \in X_{t+1} \cap X_t \subset X_{t+1}.$$

(b)⇒(a) Let $(x_i) \in \operatorname{Ker}\varphi$. Then, by (a), there exists $y \in X_1 \cap X_m$ such that $y-(x_1+\cdots+x_t) \in X_{t+1}$ for $1 \le t \le m-2$. Put $y_1 := y$ and $y_i := y-(x_1+\cdots+x_{i-1})$ for $2 \le i \le m$. Then $y_1 = y = y-(x_1+\cdots+x_m) = y_m - x_m$, and for $2 \le i \le m$, $y_i = y-(x_1+\cdots+x_{i-1}) = y_{i-1}-x_{i-1}$. Hence $x_i = y_i - y_{i+1}$ for $1 \le i \le m$, where $y_{m+1} = y_1$. Since $y_1 \in X_1 \cap X_m$ and $y_{t+1} = y_t - x_t \in X_{t+1}$ for $1 \le t \le m-2$, it follows that $y_i \in X_i \cap X_{i-1}$ for $1 \le i \le m-1$, and $y_m = y_{m-1} - x_{m-1} \in X_{m-1} \cap X_m$, because $y_m = y_1 + x_m \in X_m$. Hence $(x_i) = \psi(y_i) \in \operatorname{Im}\psi$.

(3) Without loss of generality, we can assume that $a = 1$.

(⇒) Let $2 \le r < b$ and take an arbitrary $x \in X_r$. For $1 \le i \le m$, we set

$$x_i := \begin{cases} -x & \text{if } i = r, \\ x & \text{if } i = b, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x_i) \in \operatorname{Ker}\varphi$. It follows from (2) that there exists $y \in X_1 \cap X_m$ such that $y+x = y-(x_1+\cdots+x_r) \in X_{r+1}$. Hence $x \in X_{r+1}$, because $y \in X_1 \subset X_{r+1}$, and we get $X_r \subset X_{r+1}$. Therefore $X_1 \subset X_2 \subset \cdots \subset X_{b-1} \subset X_b$.

Let $b < s \le m$ and take an arbitrary $x \in X_s$. For $1 \le i \le m$, set

$$x_i := \begin{cases} x & \text{if } i = s, \\ -x & \text{if } i = b, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x_i) \in \operatorname{Ker}\varphi$. It follows from (2) that there exists $y \in X_1 \cap X_m$ such that $y + x = y + (x_m + \cdots + x_s) = y - (x_1 + \cdots + x_{s-1}) \in X_{s-1}$. Hence $X_s \subset X_{s-1}$ for all $b < s \le m$.

(⇐) Let $(x_i) \in \operatorname{Ker}\varphi$. Put $y := x_1 \in X_1 = X_1 \cap X_m$. If $1 \le t < b$, then $y - (x_1 + \cdots + x_t) = -(x_2 + \cdots + x_t) \in X_t \subset X_{t+1}$. If $b \le t \le m-2$, then $y - (x_1 + \cdots + x_t) = y + (x_m + \cdots + x_{t+1}) \in X_{t+1}$. Hence (2) yields $\operatorname{Ker}\varphi = \operatorname{Im}\psi$. ∎

Remark. We notice that the sequence of Proposition 4 is not always exact, even if $X_1, \ldots, X_m$ can be linearly ordered by inclusion. In fact, for example, consider the submodules $X_1 = 4\mathbb{Z} \subset X_3 = 2\mathbb{Z} \subset X_2 = X_4 = \mathbb{Z}$ of $M = \mathbb{Z}$. Then it follows from Proposition 4(3) that the sequence

$$\bigoplus_{i=1}^{4} (X_i \cap X_{i-1}) \xrightarrow{\psi} \bigoplus_{i=1}^{4} X_i \xrightarrow{\varphi} \sum_{i=1}^{4} X_i \to 0$$

is not exact. However, if we interchange the indices of X_2 and X_3, then $X_1 = 4\mathbb{Z} \subset X_2 = 2\mathbb{Z} \subset X_3 = X_4 = \mathbb{Z}$ and the sequence is exact.

The following is a generalization of Corollary 2.

COROLLARY 5. *Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D-order in $\mathbb{M}_n(K)$, and let $L_1 = (L_{11}, \ldots, L_{1n}), \ldots, L_m = (L_{m1}, \ldots, L_{mn}) = L_0$ be irreducible right Λ-lattices in $V = K^n$, where $m \geq 3$. For each $1 \leq j \leq n$, let a_j, b_j be integers in $\{1, \ldots, m\}$ such that $L_{a_j,j} \subset L_{ij} \subset L_{b_j,j}$ for all $1 \leq i \leq m$. Let*

$$\bigoplus_{i=1}^{m}(L_i \cap L_{i-1}) \xrightarrow{\psi} \bigoplus_{i=1}^{m} L_i \xrightarrow{\varphi} \sum_{i=1}^{m} L_i \to 0$$

be a sequence of Λ-lattices and Λ-homomorphisms defined by

$$\psi(y_1, \ldots, y_m) = (y_1 - y_2, \ldots, y_{m-1} - y_m, y_m - y_1), \ \varphi(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i$$

for $(y_1, \ldots, y_m) \in \bigoplus_{i=1}^{m}(L_i \cap L_{i-1})$ and $(x_1, \ldots, x_m) \in \bigoplus_{i=1}^{m} L_i$.

(1) *The following statements are equivalent*:
   (a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.
   (b) *For each $1 \leq j \leq m$, $L_{i,j} \subset L_{i+1,j}$ for all $i \in \{1, \ldots, m\}$ with $a_j \leq i < b_j \pmod{m}$ and $L_{i,j} \subset L_{i-1,j}$ for all $i \in \{1, \ldots, m\}$ with $a_j \geq i > b_j \pmod{m}$.*

(2) *Suppose that the equivalent conditions of* (1) *hold. Then there is an exact sequence*

$$0 \to \bigcap_{i=1}^{m} L_i \to \bigoplus_{i=1}^{m}(L_i \cap L_{i-1}) \xrightarrow{\psi} \bigoplus_{i=1}^{m} L_i \xrightarrow{\varphi} \sum_{i=1}^{m} L_i \to 0$$

of right Λ-lattices. In particular, if $\bigcap_{i=1}^{m} L_i = L_{m-1} \cap L_m$, then there is a short exact sequence

$$0 \to \bigoplus_{i=1}^{m-1}(L_i \cap L_{i-1}) \to \bigoplus_{i=1}^{m} L_i \to \sum_{i=1}^{m} L_i \to 0$$

of right Λ-lattices.

Proof. Apply Proposition 4 and the arguments used in the proof of Corollary 2. ∎

REMARK. Condition (b) always holds if $m = 3$.

As an application of our elementary exact sequence, we compute a minimal projective resolution of the Jacobson radical of a tiled D-order given by Fujita and Oshima [5].

We use the following notations. Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D-order in $\mathbb{M}_n(K)$, and let $J(\Lambda)$ be the Jacobson radical of Λ. For each $1 \le i \le n$, let $e_i \in \mathbb{M}_n(K)$ be the matrix whose (i,i)-entry is 1 and the other entries are 0. For each $1 \le i \le n$, let P_i be the irreducible right Λ-lattice

$$P_i = (\pi^{\lambda_{i1}} D, \ldots, \pi^{\lambda_{in}} D)$$

in $V = K^n$, and let

$$J_i = P_i J(\Lambda) \simeq e_i J(\Lambda)$$

be the radical of $P_i \simeq e_i \Lambda$ for $1 \le i \le n$. Moreover, we put

$$S_i := P_i / J_i$$

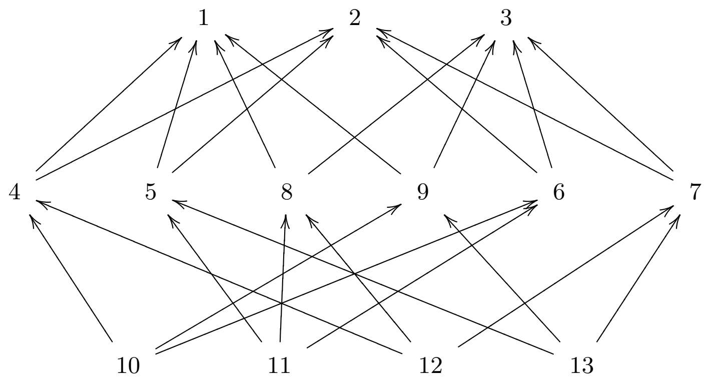
for each $1 \le i \le n$. Then P_i $(1 \le i \le n)$ are the indecomposable projective right Λ-modules, and S_i $(1 \le i \le n)$ are the simple right Λ-modules.

Example 6. We compute minimal projective resolutions of J_i $(1 \le i \le 13)$ for the following basic $(0,1)$-tiled D-order Λ in $\mathbb{M}_{13}(K)$ where $\boldsymbol{\pi} = \pi D$ (see [15] and Chapter 13 of [11]):

$$\Lambda := \begin{pmatrix}
D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & D & D & D & D \\
\boldsymbol{\pi} & D & \boldsymbol{\pi} & D & D & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & D & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & D & D & D & D & D & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & D & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D
\end{pmatrix}.$$

Let $F := D/\boldsymbol{\pi}$ be the residue field, and let A be the F-algebra $\Lambda/\mathbb{M}_{13}(\boldsymbol{\pi})$. It follows from [2] that the link graph of Λ is obtained from the Gabriel quiver $\mathcal{Q}(A)$ of A by adding the arrows from non-domains in $\mathcal{Q}(A)$ to non-ranges in $\mathcal{Q}(A)$ to the set of arrows of $\mathcal{Q}(A)$. Note that $\mathcal{Q}(A)$ is the quiver

$\mathcal{Q}(\Lambda)$:



Step 1. Since $J_1/J_1J(\Lambda) \cong S_4 \oplus S_5 \oplus S_8 \oplus S_9$, J_1 has the projective cover

$$\varphi : P_4 \oplus P_5 \oplus P_8 \oplus P_9 \twoheadrightarrow J_1, \qquad (x_4, x_5, x_8, x_9) \mapsto x_4 + x_5 + x_8 + x_9.$$

Note also that the modules P_4, P_8, P_5, P_9 satisfy condition (b) of Corollary 5 in that order. Moreover, $P_4 \cap P_9 = P_{10}$, $P_8 \cap P_4 = P_{12}$, $P_5 \cap P_8 = P_{11}$, $P_9 \cap P_5 = P_{13}$ and $P_4 \cap P_8 \cap P_5 \cap P_9 = J_{10}$. Hence, by Corollary 5, we have the exact sequence

$$0 \to J_{10} \xrightarrow{\eta} P_{10} \oplus P_{12} \oplus P_{11} \oplus P_{13} \xrightarrow{\psi} P_4 \oplus P_8 \oplus P_5 \oplus P_9 \xrightarrow{\varphi} J_1 \to 0.$$

Note that $P_{10} \oplus P_{12} \oplus P_{11} \oplus P_{13} \overset{\psi}{\twoheadrightarrow} \operatorname{Im}\psi$ is a projective cover, because $\operatorname{Im}\eta \subset (P_{10} \oplus P_{11} \oplus P_{12} \oplus P_{13})J(\Lambda)$. Similarly, we get the exact sequences

$$0 \to J_{10} \to P_{12} \oplus P_{10} \oplus P_{11} \oplus P_{13} \to P_4 \oplus P_6 \oplus P_5 \oplus P_7 \to J_2 \to 0,$$

$$0 \to J_{10} \to P_{10} \oplus P_{11} \oplus P_{12} \oplus P_{13} \to P_6 \oplus P_8 \oplus P_7 \oplus P_9 \to J_3 \to 0.$$

Step 2. Note that J_i $(4 \le i \le 9)$ have the following projective covers:

$$\begin{aligned} 0 &\to J_{10} \to P_{10} \oplus P_{12} \to J_4 \to 0,\\ 0 &\to J_{10} \to P_{11} \oplus P_{13} \to J_5 \to 0,\\ 0 &\to J_{10} \to P_{10} \oplus P_{11} \to J_6 \to 0,\\ 0 &\to J_{10} \to P_{12} \oplus P_{13} \to J_7 \to 0,\\ 0 &\to J_{10} \to P_{11} \oplus P_{12} \to J_8 \to 0,\\ 0 &\to J_{10} \to P_{10} \oplus P_{13} \to J_9 \to 0. \end{aligned}$$

Step 3. Note that $X := (D, \ldots, D) \cong J_{10} = J_{11} = J_{12} = J_{13}$ and $X/XJ(\Lambda) \cong S_1 \oplus S_2 \oplus S_3$. Hence by Corollary 2, we get the exact sequence

$$0 \to P_1 \cap P_2 \cap P_3 \xrightarrow{h} (P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2) \xrightarrow{g} P_1 \oplus P_2 \oplus P_3 \xrightarrow{f} X \to 0.$$

If we put $Y := \operatorname{Ker} f$, then we get two short exact sequences

$$0 \to Y \to P_1 \oplus P_2 \oplus P_3 \xrightarrow{f} X \to 0,$$
$$0 \to P_1 \cap P_2 \cap P_3 \xrightarrow{h} (P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2) \xrightarrow{g} Y \to 0.$$

Note that the projective covers of $P_1 \cap P_3$, $P_2 \cap P_1$, and $P_3 \cap P_2$ are given by

$$0 \to J_{10} \to P_8 \oplus P_9 \to P_1 \cap P_3 \to 0,$$
$$0 \to J_{10} \to P_4 \oplus P_5 \to P_2 \cap P_1 \to 0,$$
$$0 \to J_{10} \to P_6 \oplus P_7 \to P_3 \cap P_2 \to 0.$$

Hence $(P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2)$ has the projective cover

$$0 \to J_{10} \oplus J_{10} \oplus J_{10} \to P \xrightarrow{\theta} (P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2) \to 0,$$

where $P := P_4 \oplus P_5 \oplus P_6 \oplus P_7 \oplus P_8 \oplus P_9$. Note that

$$\operatorname{Im} h \subset [(P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2)]J(\Lambda).$$

Hence, the projective cover of Y has the form

$$0 \to Z \to P \xrightarrow{\theta \circ g} Y \to 0,$$

where

$$\begin{aligned} Z &:= \operatorname{Ker} \theta \circ g \\ &= \{(x_4, x_5, x_6, x_7, x_8, x_9) \in P \mid x_4 + x_5 = x_6 + x_7 = x_8 + x_9\}. \end{aligned}$$

STEP 4. Note that

$$\begin{aligned} P_{10} + P_{12} &= J_4 \subset P_4, & P_{11} + P_{13} &= J_5 \subset P_5, \\ P_{10} + P_{11} &= J_6 \subset P_6, & P_{12} + P_{13} &= J_7 \subset P_7, \\ P_{11} + P_{12} &= J_8 \subset P_8, & P_{10} + P_{13} &= J_9 \subset P_9. \end{aligned}$$

Hence, we obtain a Λ-homomorphism $\alpha : P_{10} \oplus P_{11} \oplus P_{12} \oplus P_{13} \to Z$ defined by

$$\alpha : \begin{pmatrix} x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}.$$

CLAIM 1. *If* $\operatorname{char} F \neq 2$ *then* α *is an isomorphism.*

Proof. First note that $2 \in D \setminus \pi D$ and 2 is invertible in D, because $\operatorname{char} F \neq 2$.

Let $(x_{10}, x_{11}, x_{12}, x_{13}) \in \operatorname{Ker} \alpha$. Then $2x_{10} = x_{10} + x_{12} + x_{10} + x_{11} = 0$, and similarly $2x_{11} = 0$, $2x_{12} = 0$, $2x_{13} = 0$. Hence $(x_{10}, x_{11}, x_{12}, x_{13}) = (0, 0, 0, 0)$, so that α is a monomorphism.

Let $(x_4, x_5, x_6, x_7, x_8, x_9) \in Z$. Since $x_4 + x_5 = x_6 + x_7 = x_8 + x_9$, we put

$$\begin{aligned}
2x_{10} &:= x_4 + x_6 - x_8 = x_9 - x_5 + x_6 = x_4 + x_9 - x_7,\\
2x_{11} &:= x_5 + x_6 - x_9 = x_8 - x_4 + x_6 = x_5 + x_8 - x_7,\\
2x_{12} &:= x_4 + x_7 - x_9 = x_8 - x_5 + x_7 = x_4 + x_8 - x_6,\\
2x_{13} &:= x_5 + x_7 - x_8 = x_9 - x_4 + x_7 = x_5 + x_9 - x_6.
\end{aligned}$$

Further, we put $x_i = (x_{i1}, \ldots, x_{i13}) \in P_i$ $(4 \le i \le 9)$. Then, for each $1 \le j \le 13$ with $j \ne 10$, we have $x_{4j} + x_{6j} - x_{8j} = x_{9j} - x_{5j} + x_{6j} = x_{4j} + x_{9j} - x_{7j} \in \pi D$. Hence $x_{10} \in P_{10}$. Similarly, we check that $x_{11} \in P_{11}$, $x_{12} \in P_{12}$, $x_{13} \in P_{13}$. Hence α is an epimorphism, because $\alpha(x_{10}, x_{11}, x_{12}, x_{13}) = (x_4, x_5, x_6, x_7, x_8, x_9)$. ■

CLAIM 2. *If* char $F \ne 2$, *then* gl.dim $\Lambda = 5$ *and* Λ *has no neat primitive idempotent.*

Proof. It follows from Steps 3, 4 and Claim 1 that J_k $(10 \le k \le 13)$ has the minimal projective resolution

$$0 \to \bigoplus_{i=10}^{13} P_i \to \bigoplus_{i=4}^{9} P_i \to \bigoplus_{i=1}^{3} P_i \to J_k \to 0.$$

Note that every P_i $(1 \le i \le 13)$ appears in the above resolution, and that minimal projective resolutions of J_k $(1 \le k \le 9)$ are given by connecting the above resolution to the sequences of Steps 1 and 2. Hence gl.dim $\Lambda = \sup\{\mathrm{pd}\, J_i \mid 1 \le i \le 13\} + 1 = 5$, and it follows from Proposition 1 of [4] that no e_i $(1 \le i \le 13)$ is neat. ■

STEP 5. Note that for any $x_{10} = (x_{10,1}, \ldots, x_{10,13}) \in J_{10} \subset P_{10}$, $x_{10j} \in \pi D$ for each $1 \le j \le 13$. Hence we get a Λ-homomorphism $\beta : J_{10} \to Z$ defined by

$$\beta(x_{10}) = (x_{10}, 0, x_{10}, 0, x_{10}, 0).$$

CLAIM 3. *If* char $F = 2$ *then* β *is a split monomorphism.*

Proof. For any $(x_4, x_5, x_6, x_7, x_8, x_9) \in Z$, put $y := x_4 + x_5 = x_6 + x_7 = x_8 + x_9$ and $z := x_4 - x_6 + x_8 = -x_5 + x_7 + x_8 = x_4 + x_7 - x_9$. Put $y = (y_1, \ldots, y_{13})$ and $z = (z_1, \ldots, z_{13})$. Then for each $1 \le j \le 13$ with $j \ne 12$, $z_j = x_{4j} - x_{6j} + x_{8j} = -x_{5j} + x_{7j} + x_{8j} = x_{4j} + x_{7j} - x_{9j} \in \pi D$. If $j = 12$, then $z_{12} = x_{4,12} - x_{6,12} + x_{8,12} = 2y_{12} - x_{5,12} - x_{6,12} - x_{9,12} \in \pi D$ because $2 \in \pi D$. Hence we get a Λ-homomorphism $\beta' : Z \to J_{10}$ defined by

$$\beta'(x_4, x_5, x_6, x_7, x_8, x_9) = x_4 - x_6 + x_8.$$

Since we can check that $\beta' \circ \beta = \mathrm{id}_{J_{10}}$, β is a split monomorphism. ■

CLAIM 4. *If* char $F = 2$, *then* gl.dim $\Lambda = \infty$.

Proof. It follows from Step 3 that there exists a long exact sequence

$$0 \to Z \to \bigoplus_{i=4}^{9} P_i \to \bigoplus_{i=1}^{3} P_i \to J_{10} \to 0.$$

Claim 3 shows that $Z \simeq J_{10} \oplus W$ for some right Λ-lattice W. Therefore $\mathrm{pd}\, J_{10} = \infty$ and $\mathrm{gl.dim}\, \Lambda = \infty$. ■

REMARK. Extending the $(0,1)$-tiled D-order Λ in $\mathbb{M}_{13}(K)$ of Example 6, for each $n \geq 14$, one can construct a basic $(0,1)$-tiled D-order Λ_n in $\mathbb{M}_n(K)$ such $\mathrm{gl.dim}\, \Lambda = 5$ if $\mathrm{char}\, F \neq 2$, and $\mathrm{gl.dim}\, \Lambda = \infty$ if $\mathrm{char}\, F = 2$. In fact, for example, let A_n be the F-algebra whose quiver $\mathcal{Q}(A_n)$ is obtained by adding arrows $n \to n-1 \to \cdots \to 13$ to the quiver $\mathcal{Q}(A)$ of the F-algebra $A = \Lambda/\mathbb{M}_{13}(\boldsymbol{\pi})$, and let Λ_n be the $(0,1)$-tiled D-order in $\mathbb{M}_n(K)$ such that $A_n = \Lambda_n/\mathbb{M}_n(\boldsymbol{\pi})$. Then $\mathrm{gl.dim}\, \Lambda_n = 5$ if $\mathrm{char}\, F \neq 2$ and $\mathrm{gl.dim}\, \Lambda_n = \infty$ if $\mathrm{char}\, F = 2$ as in Example 6.

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