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ULTRASMOOTHNESS IN DENDROIDS

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Abstract. The class of ultrasmooth dendroids is contained in the class of smooth dendroids and contains the class of locally connected dendroids. In this paper we study relationships between ultrasmoothness and smoothness in dendroids and we characterize ultrasmooth dendroids.

Introduction. In 1970, J. J. Charatonik and C. Eberhart [2] introduced the definition of smoothness in dendroids. That paper also contains a wide study of this concept. Later, in 1988, E. E. Grace and E. J. Vought [3] characterized smooth dendroids.

With the purpose of characterizing dendrites, L. Lum [5] defined 1978 the concept of ultrasmoothness and he raised the problem of characterizing ultrasmooth dendroids. The purpose of the present paper is to give some such characterizations.

A *continuum* is a compact, connected metric space, and a *subcontinuum* is a continuum contained in some topological space. The *hyperspace* 2^X (resp. $C(X)$) of a continuum X is the space of closed subsets (resp. subcontinua of X) with the topology induced by the Hausdorff metric. These two hyperspaces are again continua (see [4] for concepts and results relating to hyperspaces). The term "mapping" will be used for a continuous function, and $\mathrm{Cl}(A)$ denotes the closure of the set A.

Let $\{A_n\}_{n\in\mathbb{N}} \subseteq 2^X$. We recall some facts about $\liminf A_n$ and $\limsup A_n$ (see [6, Definition 4.8, p. 56] for the definitions):

1) $\liminf A_n \subseteq \limsup A_n$.
2) $\lim A_n$ with respect to the Hausdorff metric exists if and only if $\liminf A_n = \limsup A_n$, and in this case the three limits coincide.
3) $\liminf A_n = \{\lim x_n : x_n \in A_n\}$ and $\limsup A_n = \{\lim x_{n_j} : x_{n_j} \in A_{n_j}\}$ where $\lim x_n$ denotes the usual limit in the continuum X.

Let X be any continuum. A *Whitney map* $\mu : 2^X \to \mathbb{R}$ is a mapping such that $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(A) < \mu(B)$ whenever $A \subsetneq B$.

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Whitney maps exist for every continuum X (see [8]). In this paper, we consider Whitney maps restricted to $C(X)$.

A continuum is called a *dendroid* if it is arcwise connected and if the intersection of any two of its subcontinua is connected. It is not dificult to see that subcontinua of dendroids are dendroids and that for any two points p, q in a dendroid X there exists a unique arc $pq \subseteq X$ joining p and q. We define $[pq) = pq - \{q\}$, $(pq] = pq - \{p\}$ and $(pq) = pq - \{p, q\}$.

A point p in a dendroid X is called a *ramification point* if $X - \{p\}$ has at least three arc-components. A *fan* is a dendroid with exactly one ramification point, called the *vertex*. In particular, the *harmonic fan* is the cone over the set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$. The set of ramification points of X will be denoted by $R(X)$.

Let X be a dendroid and $p \in X$. For $a, b \in X$ we define $a \leq_p b$ if $a \in pb$. Clearly, the relation $\leq_p$ defines a partial order in X. X is called *ultrasmooth at p* if for every $x, y \in X$, there exists a retraction $r : X \to px \cup py$ which preserves the partial order $\leq_p$ ($a \leq_p b \Rightarrow r(a) \leq_p r(b)$). We call such a retraction a *$\leq_p$-retraction.*

X is called *smooth at p* if $\lim pa_n = pa$ whenever $\{a_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $a \in X$. If X is smooth at p, it follows easily that X is locally connected at p.

We define

$$S(X) = \{p \in X : X \text{ is smooth at } p\},$$
$$U(X) = \{p \in X : X \text{ is ultrasmooth at } p\}.$$

X is called *smooth* (resp. *ultrasmooth*) if $S(X) \neq \emptyset$ (resp. $U(X) \neq \emptyset$).

The following facts are easy consequences of the definition of ultrasmoothness:

(i) Each dendrite (locally connected dendroid) is ultrasmooth at every point (see [7, Theorem 3.24, p. 49]).

(ii) Subdendroids of ultrasmooth dendroids are ultrasmooth.

Let X be a dendroid, $p \in S(X)$ and $x \in X$. We define the mapping $\psi_{p,x} : X \to px$ by

$$\psi_{p,x}(y) = \begin{cases} \text{the point in } px \text{ with } \mu(p\psi_{p,x}(y)) = \mu(py) & \text{if } \mu(py) \leq \mu(px), \\ x & \text{if } \mu(py) \geq \mu(px). \end{cases}$$

It follows from the properties of μ and the smoothness at p that $\psi_{p,x}$ is well defined and is actually a $\leq_p$-retraction.

If p, x are points in a dendroid X, the arc-component of $X - \{p\}$ containing x will be denoted by $\mathcal{A}_p(x)$.

We will prove in Section 1 that $U(X) \subseteq S(X)$ and therefore the class of ultrasmooth dendroids is a subclass of the class of smooth dendroids. We also give examples to show that both inclusions are proper and we investigate the existence of $\leq_p$-retractions when $p \in S(X)$. As a corollary, we prove that every smooth fan is ultrasmooth. In Section 2 we study certain subsets $L(P)$ of X which are our main tool for characterizing ultrasmoothness. In Section 3 we prove that a dendroid X is ultrasmooth at p iff for every tree $T \subseteq X$ such that $p \in T$, there exists a $\leq_p$-retraction $r : X \to T$. Moreover, we provide an example that shows that this result is no longer true if the tree T is replaced by the more general case of a dendrite. Finally, in this section, we prove a theorem which relates ultrasmoothness to the sets $L(P)$ and we characterize ultrasmooth dendroids by means of a list of forbidden subdendroids. From now on X will denote a dendroid.

1. Smoothness and ultrasmoothness. In this section, it will be proved that $U(X) \subseteq S(X)$ and therefore the class of ultrasmooth dendroids is a subclass of the class of smooth dendroids. We give examples that show that the two inclusions are proper and we prove some results which will be used to characterize ultrasmoothness.

THEOREM 1. *A point $p \in X$ is in $S(X)$ iff for every $x \in X$, there exists a $\leq_p$-retraction $r : X \to px$.*

Proof. Assume that $p \in S(X)$ and let μ be a Whitney map for $C(X)$. Given $x \in X$, the mapping $\psi_{p,x} : X \to px$ as defined in the introduction is a $\leq_p$-retraction.

To prove the converse, let $\{a_n\}_{n\in\mathbb{N}} \subseteq X$ converge to $a \in X$. Since $\{p, a\} \subseteq \liminf pa_n$ and by [7, Proposition 3.7, p. 35], $\liminf pa_n$ is a dendroid, we obtain

$$pa \subseteq \liminf pa_n \tag{1}$$

Suppose that there is $y \in (\limsup pa_n) - pa$ and consider a $\leq_p$-retraction $r : X \to py$. Then $y = \lim y_{n_j}$ for some $y_{n_j} \in pa_{nj}$, so that $y_{n_j} \leq_p a_{n_j}$ for each $n_j \in \mathbb{N}$ and therefore $r(y_{n_j}) \leq_p r(a_{n_j}) \leq_p y$. Since these points are all contained in the arc py, the order $\leq_p$ is preserved under the limit operation, so that $y = r(y) = \lim r(y_{n_j}) \leq_p \lim r(a_{n_j}) = r(a) \leq_p y$. This proves that $r(a) = y$ and thus $a \notin py$. Assume that $pa \cap py = pb$. Since $r(a) = y$ and $r(b) = b$, there is a point $z \in ba$ such that $r(z) \in (by)$. Since $z \in ya \subseteq \liminf y_n a_n$, we have $z = \lim z_n$, where $z_n \in y_n a_n$, and it follows that $r(y_{n_j}) \leq_p r(z_{n_j}) \leq_p y$, which implies, by considering the limit, that $r(z) = y$, a contradiction which proves that

$$\limsup pa_n \subseteq pa. \tag{2}$$

It follows from (1) and (2) that $\lim pa_n$ exists and equals pa, so that $p \in S(X)$. ∎

Corollary 1. *For every dendroid X, we have $U(X) \subseteq S(X)$, and so X ultrasmooth $\Rightarrow X$ smooth.*

The following examples, which are given in [5], exhibit dendroids $\mathbb{P}$ and $2\mathbb{P}$ such that $S(\mathbb{P}) \neq U(\mathbb{P}) \neq \emptyset$, $S(2\mathbb{P}) \neq \emptyset$ and $U(2\mathbb{P}) = \emptyset$.

Example 1. Given $a, b \in \mathbb{R}^2$ we denote by ab the rectilinear segment from a to b. A comb $\mathbb{P}$ is defined as follows: $\mathbb{P} = \mathbf{v}a \cup \mathbf{v}q_1 \cup \bigcup_{n\in\mathbb{N}} q_n a_n$ where $\mathbf{v} = (0,0)$, $a = (0,1)$, $q_n = (1/n, 0)$, $a_n = (1/n, 1)$. It is clear that $\mathbf{v} \in S(\mathbb{P})$ and we will prove that $\mathbf{v} \notin U(\mathbb{P})$. Suppose that $r : \mathbb{P} \to \mathbf{v}a \cup \mathbf{v}a_1$ is a retraction. Since $\lim a_n = a$, $r(a_n) \in \mathbf{v}a$ for n large enough. We fix N such that $r(a_N) \in \mathbf{v}a$. On the other hand, since $q_N \in \mathbf{v}a_1$, $r(q_N) = q_N$. Now we notice that $q_N \leq_{\mathbf{v}} a_N$ but $r(q_N)$ and $r(a_N)$ are not $\leq_{\mathbf{v}}$-comparable. Therefore $U(\mathbb{P}) = \mathbb{P} - \mathbf{v}a$ and $S(\mathbb{P}) = U(\mathbb{P}) \cup \{\mathbf{v}\}$.

Example 2. Let $2\mathbb{P} = \mathbb{P} \cup \mathbb{P}^*$ where $\mathbb{P}^*$ is the reflection of $\mathbb{P}$ in the y-axis. With the notation of Example 1, it is clear that $S(2\mathbb{P}) = 2\mathbb{P} - (va]$. We now prove that $U(2\mathbb{P})$ is empty. Let $\mathbf{u} = (u, v) \in 2\mathbb{P}$, $u > 0$. Choose $x = (x_1, x_2)$ and $y = (0, y_2) \in 2\mathbb{P}$ so that $x_1 < 0$ and $y_2 \neq 0$. With the same arguments used in Example 1, it is not difficult to see that there is no $\leq_{\mathbf{u}}$-retraction $r : 2\mathbb{P} \to \mathbf{u}x \cup \mathbf{u}y$. The case $u < 0$ is analogous. If $u = 0$ and $v \neq 0$, then $(u, v) \notin S(2\mathbb{P})$ implies $(u, v) \notin U(2\mathbb{P})$. Finally, the proof that $v \notin U(2\mathbb{P})$ is identical to the proof in Example 1.

We recall that a *comb* is any dendroid homeomorphic to the $\mathbb{P}$ of Example 1. Let X be a dendroid and $\mathcal{P}$ the set of combs contained in X. Let $P \in \mathcal{P}$ and $h : \mathbb{P} \to P$ be a fixed homeomorphism. We say that $h((0,0)) = v(P)$ is the *vertex* of P. Moreover, we set $b(P) = h(\mathbb{P} \cap \{(0, y) : y \in \mathbb{R}\})$ and $P^* = P - b(P)$. Let $L(P)$ denote the arc-component of $X - \{v(P)\}$ containing P^*.

Lemma 1. *Let $p, q, x, y \in X$ (p and q are not necessarily different). Assume that $p \in S(X)$, $p \notin \mathcal{A}_q(x)$, $\mathcal{A}_q(x) \neq \mathcal{A}_q(y)$ and $\mathrm{Cl}(\mathcal{A}_q(x)) \cap \mathcal{A}_q(y) \neq \emptyset$. Then there exists $P \in \mathcal{P}$ with $q = v(P)$. Moreover, $\mathcal{A}_q(x) = L(P)$ and $y \notin L(P)$.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_q(x)$ with $\lim a_n = a \in \mathcal{A}_q(y)$. We may assume that $a_n \neq a_m$ if $n \neq m$. For each $n > 1$, let $q_n \in \mathcal{A}_q(x)$ satisfy $qa_1 \cap qa_n = qq_n$. Suppose that there exists a subsequence $\{q_{n_j}\}_{j\in\mathbb{N}}$ such that $\lim q_{n_j} = q$. We may assume $q_{n_j} \neq q_{n_k}$ if $j \neq k$. Let $P = \mathrm{Cl}(\bigcup_{j\in\mathbb{N}} qa_{n_j})$. Since $p \notin \mathcal{A}_q(x)$ and $p \in S(X)$, we see that $P = qa \cup \bigcup_{j\in\mathbb{N}} qa_{n_j}$ is a comb with vertex q. Moreover, $x \in L(P)$, so that $\mathcal{A}_q(x) = L(P)$ and $y \notin L(P)$ as desired. Therefore, we only need to prove that $\{q_{n_j}\}_{j\in\mathbb{N}}$ exists. Otherwise there exists $q_0 \in (qa_1]$ such that $q_0 \in qa_n$ for every $n \in \mathbb{N}$. Since $p \notin \mathcal{A}_q(x)$, we obtain $pa_n = pq \cup qq_0 \cup q_0 a_n$, which leads to $\lim pa_n = pq_0 \cup pa$. This contradicts the smoothness at p and proves the lemma. ∎

PROPOSITION 1. *Let $p \in U(X)$. Then $\mathcal{A}_p(x)$ is closed in $X - \{p\}$ for each $x \in X$.*

Proof. By Corollary 1, $p \in S(X)$. Assume that some $\mathcal{A}_p(x)$ is not closed in $X - \{p\}$. By Lemma 1, p is the vertex of some comb contained in X, and hence, by Example 1, $p \notin U(X)$. ∎

We summarize the previous results in the following theorem.

THEOREM 2. *Let X be a dendroid and $p \in U(X)$. Then $p \in S(X)$ and $\mathcal{A}_p(x)$ is closed in $X - \{p\}$ for each $x \in X$.*

The converse of Theorem 2 is not necessarily true, as the following example shows:

EXAMPLE 3. Let $\mathbb{P}$ be the comb in Example 1, $p = (-1, 0)$ and $Y = \mathbb{P} \cup p\mathbf{v}$, so that p is an endpoint of Y, and therefore the only arc-component of $Y - \{p\}$ is closed in $Y - \{p\}$. Moreover, $p \in S(Y)$. Let a and a_1 be as in Example 1. Suppose that there exists a $\le_p$-retraction $r : Y \to pa \cup pa_1$. Notice that $r^{-1}(p\mathbf{v}) = p\mathbf{v}$. Therefore the mapping $\gamma = r|\mathbb{P} : \mathbb{P} \to \mathbf{v}a \cup \mathbf{v}a_1$ is a $\le_{\mathbf{v}}$-retraction, contrary to what was proved in Example 1.

PROPOSITION 2. *Let $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n, p\} \subseteq X$, where $p \in S(X)$. Assume that $X - \{p\} = \bigcup_{i=1}^n H_i$ where each H_i is open in $X - \{p\}$, $\mathcal{A}_p(x_i) \subseteq H_i$ and $H_i \cap H_j = \emptyset$ for $i \neq j$. Then there exists a $\le_p$-retraction $r : X \to \bigcup_{i=1}^n px_i$.*

Proof. Define $r : X \to \bigcup_{i=1}^n px_i$ as $r(w) = \psi_{p,x_i}(w)$ if $w \in H_i \cup \{p\}$, where ψ_{p,x_i} is defined in the introduction. Then r is a $\le_p$-retraction. ∎

The following lemma is a consequence of [6, Lemma 3.2, p. 37] and [2, Theorem 9, p. 309].

LEMMA 2. *Let X be a smooth dendroid and M a subcontinuum of X such that $M \cap S(X) \neq \emptyset$. Then the quotient space $Y = X/M$ is a dendroid which is smooth at $M \in Y$.*

PROPOSITION 3. *Let $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n, p\} \subseteq X$, where $p \in S(X)$. Assume that $\mathrm{Cl}(\mathcal{A}_p(x_i)) \cap \mathcal{A}_p(x_j) = \emptyset$ for $i \neq j$. Then there exists a $\le_p$-retraction $r : X \to \bigcup_{i=1}^n px_i$.*

Proof. Since every metric space X is completely normal, there are open sets U_i such that $\mathcal{A}_p(x_i) \subseteq U_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$.

Let $M = \{w \in X : w \in pu \text{ for some } u \notin \bigcup_{i=1}^n U_i\}$. Since $p \in M$, $M \neq \emptyset$. Clearly, M is a connected subset of X and by the smoothness at p, M is closed. By Lemma 2, $Y = X/M$ is a dendroid which is smooth at M. Let $\varphi : X \to Y$ be the quotient map and define $H_i = \varphi(U_i) - \{M\}$, $i \in \{1, \ldots, n\}$. Then H_i is an open subset of $Y - \{M\}$, $\mathcal{A}_M(x_i) \subseteq H_i$, $Y - \{M\} = \bigcup_{i=1}^n H_i$ and $H_i \cap H_j = \emptyset$ for $i \neq j$. By Proposition 2, there exists a $\leq_M$-retraction $\varrho : Y \to \bigcup_{i=1}^n M\varphi(x_i)$. Let $h : \bigcup_{i=1}^n M\varphi(x_i) \to \bigcup_{i=1}^n px_i$ be the natural homeomorphism. Since φ is a monotone mapping, $a \leq_p b$ implies $\varphi(a) \leq_M \varphi(b)$, and the mapping $r = h \circ \varrho \circ \varphi : X \to \bigcup_{i=1}^n px_i$ is the desired $\leq_p$-retraction. ■

Corollary 2. *Let $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n, p\} \subseteq X$, where $p \in S(X)$ and $\mathcal{A}_p(x_i) \neq \mathcal{A}_p(x_j)$ if $i \neq j$. Assume that each $\mathcal{A}_p(x_i)$ is closed in $X - \{p\}$. Then there exists a $\leq_p$-retraction $r : X \to \bigcup_{i=1}^n px_i$.*

Since a smooth fan X is smooth at the vertex p and the arc-components of $X - \{p\}$ are closed in $X - \{p\}$, we obtain the following corollary:

Corollary 3. *Every smooth fan is ultrasmooth.*

Proposition 4. *Let $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n, p, q\} \subseteq X$. Assume that*

(i) $p \in S(X)$,
(ii) $px_i \cap px_j = pq$ *for all* $i, j \in \{1, \ldots, n\}$, $i \neq j$,
(iii) $\mathrm{Cl}(\mathcal{A}_q(x_i)) \cap \mathcal{A}_q(x_j) = \emptyset$ *for* $i \neq j$.

Then there exists a $\leq_p$-retraction $r : X \to \bigcup_{i=1}^n px_i$.

Proof. Let $E = pq$ and $Y = X/E$. By Lemma 2, Y is a dendroid smooth at E. Let $\varphi : X \to Y$ be the quotient map. Since $\varphi|\mathcal{A}_q(x_i) : \mathcal{A}_q(x_i) \to \mathcal{A}_E(\varphi(x_i))$ is a homeomorphism, it follows from (iii) that $\mathrm{Cl}(\mathcal{A}_E(\varphi(x_i))) \cap \mathcal{A}_E(\varphi(x_j)) = \emptyset$ for $i \neq j$. Therefore, by Proposition 3, there exists a $\leq_E$-retraction $\varrho : Y \to \bigcup_{i=1}^n E\varphi(x_i)$. Let $f = \varphi^{-1} : \bigcup_{i=1}^n E\varphi(x_i) \to \bigcup_{i=1}^n qx_i$ be the natural homeomorphism and define $r : X \to \bigcup_{i=1}^n px_i$ as follows:

$$r(a) = \begin{cases} \psi_{p,q}(a) & \text{if either } \mu(pa) \leq \mu(pq) \text{ or } \varrho(\varphi(a)) = E, \\ \min\{\psi_{p,x_i}(a), f(\varrho(\varphi(a)))\} & \text{if } \mu(pa) \geq \mu(pq) \text{ and } \varrho(\varphi(a)) \in E\varphi(x_i), \end{cases}$$

where the minimum is taken in the arc px_i with respect to the order $\leq_p$.

Since r is well defined and the functions defining r are continuous, so is r, and it is not difficult to verify that it is a $\leq_p$-retraction. ■

Corollary 4. *Let $p \in S(X)$. Assume that for every $q \in R(X)$, every arc-component of $X - \{q\}$ which does not contain p is closed in $X - \{q\}$. Then $p \in U(X)$.*

We recall some terminology: A point p of a dendroid X is called *terminal* if it is an endpoint of any arc containing it.

A *tree* is a dendroid which has only finitely many terminal points (and therefore a finite number of ramification points).

Let T be a tree contained in X, denote by $R(T)$ the set of ramification points of T, and let $p \in T \cap S(X)$. For the purposes of the following proposition, we set

$$R^*(T) = \begin{cases} R(T) & \text{if } p \text{ is a terminal point of } T, \\ R(T) \cup \{p\} & \text{otherwise.} \end{cases}$$

PROPOSITION 5. *Let T be a tree contained in X and $p \in T \cap S(X)$. Assume that for each $q \in R^*(T)$ and $x \in T$, $\mathcal{A}_q(x)$ is closed in $X - \{q\}$. Then there exists a $\leq_p$-retraction $r : X \to T$.*

Proof. We proceed by induction on $|R^*(T)| = k$. Proposition 4 proves the case $k = 1$. Suppose the assertion is true for every tree T with $|R^*(T)| < k$ and let T be a tree for which $|R^*(T)| = k$. We consider two cases:

(i) $p \in R^*(T)$. Since $X - \{p\}$ is completely normal there exist open sets U_i, $i = 1, 2$, contained in $X - \{p\}$ such that for each $x \in T$, $\mathcal{A}_p(x)$ is contained either in U_1 or in U_2. Moreover, we may choose U_i so that $|U_i \cap R^*(T)| < k$ $(i = 1, 2)$. Let $M = \{w \in X : w \in pu$ for some $u \notin U_1 \cup U_2\}$. Let $Y = X/M$ and let $\varphi : X \to Y$ be the quotient map. Then, by Lemma 2, Y is a dendroid and $M = \varphi(p) \in S(Y)$. On the other hand, the sets $Y_i = \varphi(U_i) \cup \{M\}$ are closed in Y. Moreover, since $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2$ consists of exactly one point, the sets Y_1 and Y_2 are arcwise connected, so they are subdendroids of Y. Since $\varphi(T)$ is homeomorphic to T, for $i = 1, 2$, $T_i = (\varphi(U_i) \cup \{M\}) \cap \varphi(T)$ is a tree contained in Y_i, and $|T_i| < k$. It follows from the hypothesis that for each $x \in T$, $\mathcal{A}_M(\varphi(x)) \cup \{M\} = \varphi(\mathcal{A}_p(x) \cup \{p\})$ is closed in Y. Since $\mathcal{A}_M(\varphi(x)) = (\mathcal{A}_M(\varphi(x)) \cup \{M\}) \cap (Y - \{M\})$, $\mathcal{A}_M(\varphi(x))$ is closed in $Y - \{M\}$. It follows, by induction hypothesis, that there exists a $\leq_M$-retraction $\varrho_i : \varphi(U_i) \cup \{M\} \to T_i$, $i = 1, 2$. Let $h : \varphi(T) \to T$ be the natural homeomorphism ($h = \varphi^{-1}|\varphi(T)$) and define $r : X \to T$ as $r(a) = h(\varrho_i(\varphi(a)))$ if $\varphi(a) \in \varphi(U_i) \cup \{M\}$, $i = 1, 2$. Then r is the desired retraction.

(ii) $p \notin R^*(T)$. Then p is a terminal point of T, so there is a point $q \in R(T)$ for which $pq \cap R(T) = \{q\}$. Let $E = pq$ and $Y = X/E$. Then, by Lemma 2, Y is a dendroid smooth at E. Let $\varphi : X \to Y$ be the quotient map. Then $\varphi(T)$ is a tree and $E \in R(T)$, so that we may apply (i): Let $\varrho : Y \to \varphi(T)$ be a $\leq_E$-retraction and let $\{x_1, \ldots, x_n\}$ be the set of terminal points of T which are different from p.

We define $r : X \to T$ as follows:

$$r(a) = \begin{cases} \psi_{p,q}(a) & \text{if either } \mu(pa) \leq \mu(pq) \text{ or } \varrho(\varphi(a)) = E, \\ \min\{\psi_{p,x_i}(a), (\varphi^{-1}(\varrho(\varphi)))(a)\} & \text{if } \mu(pa) \geq \mu(pq) \text{ and } \varrho(\varphi(a)) \in Ex_i. \end{cases}$$

As before, the minimum is taken in the arc px_i with respect to the order $\leq_p$.

Since r is well defined and the functions defining r are continuous, so is r, and it is not difficult to verify that it is a $\leq_p$-retraction. ∎

2. The sets $L(P)$. Let $P, Q \in C(X)$, $P \cap Q = \emptyset$. We say that uv is the *minimal arc* from P to Q if $P \cap uv = \{u\}$ and $Q \cap uv = \{v\}$. If $p \in P$, we say that $\{p\}$ is the minimal arc from $\{p\}$ to P.

In what follows, we assume that $\mathcal{P} \neq \emptyset$. We will find a relationship between the sets $L(P)$ and $U(X)$.

PROPOSITION 6. *$U(X) = \bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X)$.*

Proof. The inclusion $U(X) \subseteq S(X)$ has been established in Corollary 1. Assume that $p \notin L(P)$ for some $P \in \mathcal{P}$, and let pq be the minimal arc from p to P. Then $q \in b(P)$. If $q \neq v(P)$, then $p \notin S(X)$, so that $p \notin U(X)$. If $q = v(P)$, we proceed as in Example 3 to prove that $p \notin U(X)$.

Now, let $p \in \bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X)$ and $q \in X$. Then, by Lemma 1, any arc-component of $X - \{q\}$ which does not contain p is closed in $X - \{q\}$, since otherwise there would exist $P \in \mathcal{P}$ such that $p \notin L(P)$. It follows from Corollary 4 that $p \in U(X)$. ∎

With the aid of the following definitions we will establish necessary and sufficient conditions for $U(X)$ to be nonempty.

Let $P, Q \in \mathcal{P}$ and assume:

(i) $P \cap Q = \emptyset$. If uv is the minimal arc from P to Q, we write:

$$\begin{aligned} &P ** Q && \text{if } u \in P^* \text{ and } v \in Q^*,\\ &P *b\, Q && \text{if } u \in P^* \text{ and } v \notin Q^*,\\ &P\, bb\, Q && \text{if } u \notin P^* \text{ and } v \notin Q^*. \end{aligned}$$

(ii) $P \cap Q \neq \emptyset$. We write

$$\begin{aligned} &P ** Q && \text{if } P^* \cap Q^* \neq \emptyset,\\ &P *b\, Q && \text{if } P^* \cap Q^* = \emptyset,\ b(P) \cap b(Q) = \emptyset \text{ and } P^* \cap b(Q) \neq \emptyset,\\ &P\, bb\, Q && \text{if } P^* \cap Q^* = \emptyset \text{ and } b(P) \cap b(Q) \neq \emptyset. \end{aligned}$$

Then

$$\begin{aligned} P ** Q &\Rightarrow L(P) \cap L(Q) \supseteq P^* \cup Q^*,\\ P *b\, Q &\Rightarrow L(P) \cap L(Q) \supseteq Q^*,\\ P\, bb\, Q &\Rightarrow L(P) \cap L(Q) = \emptyset. \end{aligned}$$

It then follows that $P\, bb\, Q \Leftrightarrow L(P) \cap L(Q) = \emptyset$.

LEMMA 3. *Let $P_0 \in \mathcal{P}$ and define $\mathcal{P}_0 = \{P \in \mathcal{P} : P_0 *b\, P\} \cup \{P_0\}$. Assume that X does not contain $P, Q \in \mathcal{P}$ such that $P\, bb\, Q$. Then $\bigcap\{L(P) : P \in \mathcal{P}_0\} \neq \emptyset$.*

Proof. Let $P \in \mathcal{P}_0$ and set $a_0 = v(P_0)$. Define $a_P \in b(P)$ so that a_0a_P is the minimal arc from $\{a_0\}$ to P. Then for any $P, Q \in \mathcal{P}_0$, either $a_P \in a_0a_Q$ or $a_Q \in a_0a_P$, since otherwise we would obtain $P\, bb\, Q$. It follows that for any $P, Q \in \mathcal{P}_0$, either $a_0a_P \subseteq a_0a_Q$ or $a_0a_Q \subseteq a_0a_P$. On the other hand, if $P, Q \in \mathcal{P}_0 - \{P_0\}$ and $a_0a_P \subseteq a_0a_Q$, then $P *b\, Q$, so that the minimal arc from P to Q is $b_P a_q$ where $b_P \in P^*$. Let $L = \bigcup\{a_0 a_P : P \in \mathcal{P}_0\}$. We consider two cases:

CASE 1: L is a closed subset of X. Then $L = a_0 a_Q$ for some $Q \in \mathcal{P}_0$ and $\bigcap\{L(P) : P \in \mathcal{P}_0\} \supseteq Q^* \neq \emptyset$.

CASE 2: L is not closed. Then L is homeomorphic to the interval $[0, 1)$ and it follows from [1, Lemma 3, p. 18] that $\mathrm{Cl}(L) = a_0 a$ for some $a \in X$. Let $P \in \mathcal{P}_0$. Then the minimal arc from a to P is ab_P, so that $a \in L(P)$ and therefore $a \in \bigcap\{L(P) : P \in \mathcal{P}_0\}$. ∎

THEOREM 3. *In a dendroid* X, $\bigcap\{L(P) : P \in \mathcal{P}\} = \emptyset$ *iff there exist* $P_0, Q_0 \in \mathcal{P}$ *such that* $L(P_0) \cap L(Q_0) = \emptyset$.

Proof. Clearly $L(P_0) \cap L(Q_0) = \emptyset$ implies $\bigcap\{L(P) : P \in \mathcal{P}\} = \emptyset$.

We now assume that for any $P, Q \in \mathcal{P}$, $L(P) \cap L(Q) \neq \emptyset$, so that X does not contain two elements $P, Q \in \mathcal{P}$ such that $P\ bb\ Q$. Fix $P_0 \in \mathcal{P}$. Then, by Lemma 4, $\emptyset \neq \bigcap\{L(P) : P \in \mathcal{P}_0\}$. Let $a \in \bigcap\{L(P) : P \in \mathcal{P}_0\}$. Then $a \in L(P_0)$. If $P \in \mathcal{P}$ and either $P ** P_0$ or $P *b\ P_0$ then $a \in L(P)$, therefore $a \in \bigcap\{L(P) : P \in \mathcal{P}\}$. ∎

The following lemma is easy to prove.

LEMMA 4. *For every arc* $tu \subseteq X$, $tu \cap S(X)$ *is a closed subset of* X.

THEOREM 4. *Assume that* $\bigcap\{L(P) : P \in \mathcal{P}\} \neq \emptyset$. *Then* $\bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X) \neq \emptyset$ *iff* $L(P) \cap S(X) \neq \emptyset$ *for each* $P \in \mathcal{P}$.

Proof. The implication "⇒" is clear.

To prove the converse, using the notation of Lemma 3, we consider three cases:

1) $\mathcal{P}_0 = \{P_0\}$. Let $p \in S(X) \cap L(P_0)$. Then clearly $p \in L(P)$ for every P such that $P ** P_0$ or $P *b\ P_0$, so that $\bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X) \neq \emptyset$.

2) $\mathcal{P}_0 \neq \{P_0\}$ and the set L in the proof of Lemma 3 (Case 1) is closed. Then $\bigcap\{L(P) : P \in \mathcal{P}_0\} = L(Q)$ where Q is as in that proof. If $p \in L(Q) \cap S(X)$, then $p \in \bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X)$.

3) $\mathcal{P}_0 \neq \{P_0\}$ and L is not closed (Case 2 in the proof of Lemma 3). We will prove that $a \in S(X)$. Let $w \in L(P_0) \cap S(X)$ and assume that $a \notin S(X)$. Then, by Lemma 4, $wa \cap S(X) = wb$, $b \neq a$. We notice that there must be a $Q \in \mathcal{P}_0$ such that $a_Q \in ba$, and it is not difficult to see that for such Q, $L(Q) \cap S(X) = \emptyset$, a contradiction which proves the theorem. ∎

3. Characterization of ultrasmooth dendroids. We start this section with an affirmative answer to a natural question. Let $p \in U(X)$. Is it posssible to replace "two points" by "a finite number of points" in the definition of ultrasmoothness? In this section we also characterize $U(X)$ using the sets $L(P)$ and we give a list of dendroids such that a smooth dendroid is ultrasmooth iff it does not contain any dendroid in the list.

THEOREM 5. *A point $p \in X$ is an element of $U(X)$ iff for every tree T such that $p \in T \subseteq X$, there exists a $\leq_p$-retraction $r : X \to T$.*

Proof. Let $p \in U(X)$. We consider two cases: (i) $\mathcal{P} \neq \emptyset$, so that Proposition 6 implies $p \in \bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X)$, and (ii) $\mathcal{P} = \emptyset$. In both cases, by Lemma 1, the hypotheses of Proposition 5 are satisfied. Thus there exists a $\leq_p$-retraction $r : X \to T$. The converse is immediate. ■

The following example shows a dendroid X, $p \in U(X)$ and a dendrite Y such that $p \in Y \subseteq X$ and there is no $\leq_p$-retraction of X onto Y.

EXAMPLE 4. Let X be the harmonic fan $X = \bigcup_{n\in\mathbb{N}}(0,0)(1,1/n) \cup (0,0)(1,0)$ where $(a,b)(c,d)$ denotes the rectilinear segment from (a,b) to (c,d).

Clearly $p = (0,0) \in U(X)$. Let $Y = \bigcup_{n\in\mathbb{N}}(0,0)(1/n,1/n^2) \cup (0,0)(1,0)$. Then Y is a dendrite. Suppose that there exists a $\leq_p$-retraction $r : X \to Y$. Then since r preserves $\leq_p$, $r(1,1/n) = (1,1/n^2)$, and since r is a retraction, $r(1,0) = (1,0)$, contradicting the continuity of r at $(1,0)$.

As a corollary, we obtain the theorem below.

THEOREM 6. *The following are equivalent for a dendroid X:*

(a) *X is ultrasmooth.*
(b) $\bigcap\{L(P) : P \in \mathcal{P}\} \cap S(X) \neq \emptyset$.
(c) *For all $P, Q \in \mathcal{P}$, $L(P) \cap L(Q) \neq \emptyset$ and $L(P) \cap S(X) \neq \emptyset$.*

THEOREM 7. *If $\mathcal{P} = \emptyset$, then $S(X) = U(X)$.*

Proof. We only need to prove that $S(X) \subseteq U(X)$. Let $p \in S(X)$ and $q \in X$. Suppose that $\mathcal{A}$ is a nonclosed arc-component of $X - \{q\}$ such that $p \notin \mathcal{A}$. Then, by Lemma 1, there exists $P \in \mathcal{P}$ with vertex q. But this is impossible since $\mathcal{P} = \emptyset$. Hence, by Corollary 4, $p \in U(X)$. ■

The converse of this theorem is not true, as shown by the following example.

EXAMPLE 5. Let $\mathbb{P}$ be the comb of Example 1 and $q_1, q_2 \in \mathbb{P}$. Let F be a harmonic fan with vertex at a point V and let B be the limit bar of F. Define a homeomorphism $f : B \to q_1q_2$ where $f(V) = q_1$. Then $X = \mathbb{P} \cup_f F$, the space obtained by attaching F to $\mathbb{P}$ by means of f, is a dendroid such that $S(X) = U(X)$ and $\mathcal{P} \neq \emptyset$.

The proof of the following lemma is left to the reader.

LEMMA 5. *Let X be a dendroid and $A, B \in X$. Then $A \in S(X)$ and $B \notin S(X)$ iff there exists either a harmonic fan or a comb such that the* *limit arc vw of either one Y is contained in AB and $A \leq v < w \leq B$ where v denotes the vertex of Y.*

The dendroids P_ω which we now construct by induction will be used in Theorem 8.

Let $M = \{0,1,2,3\}$ and $M^* = \{\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in M^{\mathbb{N}} : \omega_i \neq 0$ for infinitely many indices $i\}$. With the notation of Example 1 and in Figure 2, we consider $q_{i+1}q_i \subseteq \mathbb{P}$, $i \in \mathbb{N}$. Given $\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in M^*$, we fix a homeomorphism $f_{\omega_i} : q_{i+1}q_i \to A_{\omega_i}B_{\omega_i}$ where $f_{\omega_i}(q_i) = B_{\omega_i}$. We define $P_{\omega_1} = P \cup_{f_{\omega_1}} X_{\omega_1}$. If P_{ω_n} has been constructed, then we set $P_{\omega_{n+1}} = P_{\omega_n} \cup_{f_{\omega_{n+1}}} X_{\omega_n}$. Finally, $P_\omega = \bigcup_{n \in \mathbb{N}} P_{\omega_n}$ (see Figure 2). Therefore $\mathbb{P} \subseteq P_{\omega_1} \subseteq P_{\omega_2} \subseteq \cdots \subseteq P_\omega$, P_ω is a dendroid, and it is easy to see, using Lemma 5, that $L(P_\omega) \cap S(X) = \emptyset$.

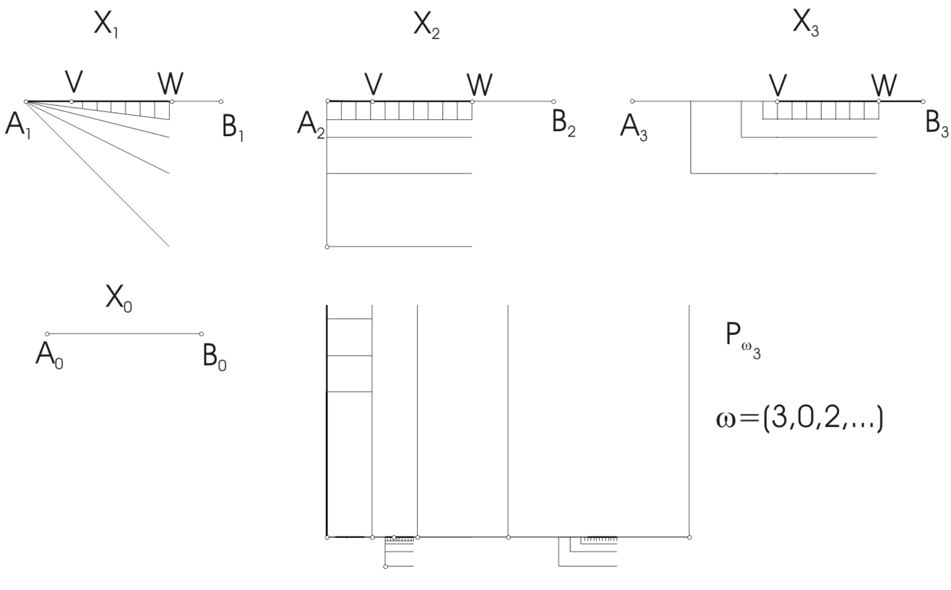

Fig. 1. Construction of the dendroids P_ω

THEOREM 8. *A smooth dendroid X is not ultrasmooth iff it contains one of the dendroids in Figure* 1 *below or one of the dendroids P_ω described above.*

Proof. We notice that dendroids 1 to 6 of Figure 1 contain combs P and Q such that $L(P) \cap L(Q) = \emptyset$; dendroids 7 and 8, as well as the dendroids P_ω, contain a comb P such that $L(P) \cap S(X) = \emptyset$. Therefore by Theorem 7 none of them is ultrasmooth. Since the property of being ultrasmooth is hereditary, if X contains any of those dendroids, then X is not ultrasmooth.

Assume now that X is not ultrasmooth. Then, by Theorem 7, there are two possibilities:

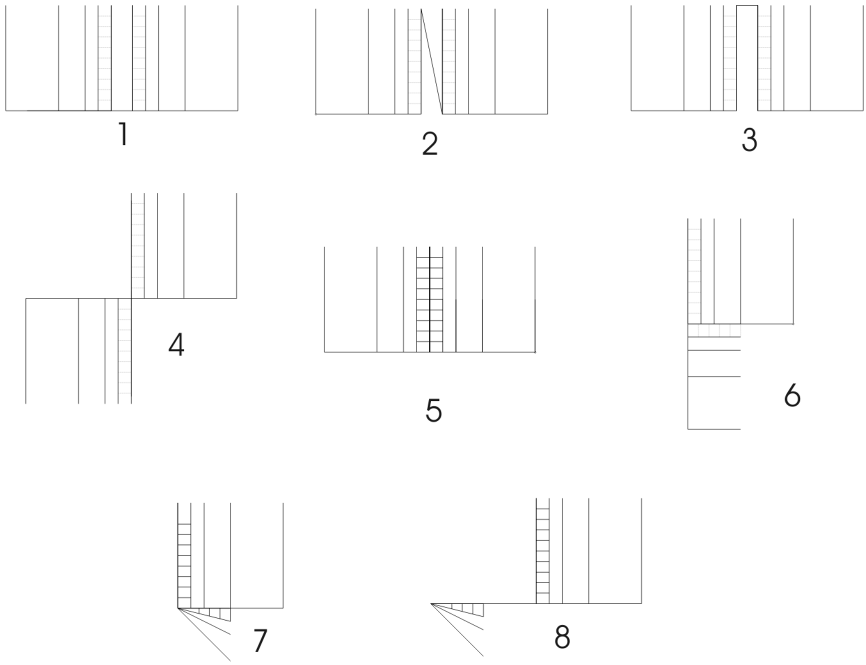

Fig. 2. These dendroids are not allowed

(I) X contains two combs P and Q such that $L(P) \cap L(Q) = \emptyset$, or
(II) X contains a comb P such that $L(P) \cap S(X) = \emptyset$.

In case (I), we consider the following subcases:

1) $P \cap Q = \emptyset$. Let ab be the minimal arc from P to Q and recall that $L(P) \cap L(Q) = \emptyset$ iff P bb Q and that $v(P)$ is the vertex of P. Three cases are possible:
   - $a = v(P)$ and $b = v(Q)$. Then $P \cup Q \cup ab$ contains dendroid 1.
   - $a = v(P)$ and $b \neq v(Q)$. Then $P \cup Q \cup ab$ contains dendroid 2.
   - $a \neq v(P)$ and $b \neq v(Q)$. Then $P \cup Q \cup ab$ contains dendroid 3 which is not smooth. Therefore X is not smooth (see [3]).

2) $P \cap Q \neq \emptyset$. Then we have four possibilities:
   - $v(P) = v(Q)$ and $P \cap Q = \{v(P)\}$. Then $P \cup Q$ contains dendroid 4.
   - $v(P) = v(Q)$ and $\{v(P)\} \subsetneq P \cap Q$. Then $P \cup Q$ contains dendroid 5 or 6.
   - $v(P) \neq v(Q)$ and $v(P) \in P \cap Q$. Then $P \cup Q$ contains dendroid 2.
   - $v(P) \neq v(Q)$, $v(P) \notin P \cap Q$ and $v(Q) \notin P \cap Q$. Then $P \cup Q$ contains dendroid 3. As noticed before, X is not smooth.

To deal with case (II), let $\lambda(X) = \{x \in X : X \text{ is locally connected at } x\}$ and consider the following cases:

1) $v(P) \in S(X)$. Then, since $S(X)$ is the component of $\lambda(X)$ containing $v(P)$ [2, Theorem 2, p. 299], we may have two cases:
   - There exists $u \in v(P)q_1$ (see Example 1 for notation) such that $v(P)u \cap \lambda(X) = \emptyset$. In this case, by Lemma 5, X contains dendroid 6 or 7.
   - There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq v(P)q_1$ such that $x_n \notin \lambda(X)$ for each $n \in \mathbb{N}$. Using again Lemma 5, we see that $X \supseteq P_\omega$ for some $\omega \in M^*$.
2) $v(P) \notin S(X)$. Then X contains dendroid 8 or 2. ∎

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