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# REFLEXIVE SUBSPACES OF SOME ORLICZ SPACES

BY

EMMANUELLE LAVERGNE (Lens)

**Abstract.** We show that when the conjugate of an Orlicz function $\phi$ satisfies the growth condition $\Delta^0$, then the reflexive subspaces of $L^\phi$ are closed in the $L^1$-norm. For that purpose, we use (and give a new proof of) a result of J. Alexopoulos saying that weakly compact subsets of such $L^\phi$ have equi-absolutely continuous norm.

**Introduction.** Bretagnolle and Dacunha-Castelle showed in [3] that an Orlicz space $L^\phi$ embeds into $L^1$ (meaning that there exists an isomorphism of this space onto a subspace of $L^1$) if and only if $\phi$ is 2-concave (recall that a function $f$ is *$r$-concave* if $f(x^{1/r})$ is concave). If $\phi$ is an Orlicz function whose conjugate $\phi^*$ satisfies the condition $\Delta^0$ (see below for the definition), then $\phi$ is equivalent, for every $r > 1$, to an $r$-concave Orlicz function (Proposition 4) and hence $L^\phi$ embeds into $L^1$. In this paper, we show that for such Orlicz functions $\phi$, the reflexive subspaces of $L^\phi$ are actually closed in the $L^1$-norm (and so the $L^\phi$-topology is the same as the $L^1$-topology). In order to prove this, we shall use a result of J. Alexopoulos (Theorem 1), saying that, when $\phi^* \in \Delta^0$, the weakly compact subsets of $L^\phi$ have equi-absolutely continuous norm, and we shall begin by giving a new proof of this result, using a recent characterization, due to P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza (see [6, Theorem 4]), of the weakly compact operators defined on a subspace of the Morse–Transue space $M^\psi$, when $\psi \in \Delta^0$.

**1. Notation.** We shall consider Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$ (see [7], [13]). By an *Orlicz function*, we shall understand a non-decreasing convex function $\phi : [0, +\infty] \to [0, +\infty]$ such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. To avoid pathologies, we shall assume that $\phi$ has the following additional properties: $\phi$ is continuous at 0, strictly convex, and moreover,

$$\lim_{x \to +\infty} \frac{\phi(x)}{x} = +\infty.$$

This is essentially to exclude the case of $\phi(x) = ax$, and so of $L^1$.

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Let $\phi$ be an Orlicz function. If $\phi'$ is the left derivative of $\phi$, then, for every $x > 0$,

$$\phi(x) = \int_0^x \phi'(t)\, dt.$$

The *Orlicz space* $L^\phi(\Omega)$ is the space of all equivalence classes of measurable functions $f : \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_\Omega \phi\bigg(\frac{|f(t)|}{C}\bigg)\, d\mathbb{P}(t) < +\infty.$$

Then for all $f \in L^\phi(\Omega)$, we define the *Luxemburg norm* of $f$ as the infimum of all possible constants $C$ such that the above integral is $\leq 1$. With this norm, $L^\phi(\Omega)$ is a Banach space.

The *Morse–Transue space* $M^\phi(\Omega)$ is the subspace of $L^\phi(\Omega)$ generated by $L^\infty(\Omega)$, or equivalently, the subspace of all functions $f$ for which the above integral is finite for all $C > 0$.

To every Orlicz function $\phi$ is associated the *conjugate Orlicz function* $\phi^*$ defined by

$$\phi^* : [0, +\infty) \to [0, +\infty), \qquad x \mapsto \sup\{xy - \phi(y);\, y \geq 0\}.$$

(Observe that $\phi^*(x) < \infty$ since $\phi(x)/x$ tends to $\infty$.)

The function $\phi^*$ is itself strictly convex. It should also be noticed that for all Orlicz functions $\phi$, we have

$$(\phi^*)^* = \phi.$$

Moreover, if $\phi_1$ and $\phi_2$ are two Orlicz functions such that $\phi_1(x) \leq \phi_2(x)$ whenever $x \geq x_0$, then there exists $y_0$ such that $\phi_2^*(y) \leq \phi_1^*(y)$ for all $y \geq y_0$.

We shall also use some growth conditions for Orlicz functions. We shall say that $\phi$ satisfies the $\Delta_2$ *condition* (and write $\phi \in \Delta_2$) if there exists a constant $K > 1$ such that for all $x$ large enough,

$$\phi(2x) \leq K\phi(x).$$

We shall say (see [6] and [7]) that $\psi$ satisfies the $\Delta^0$ *condition* (and write $\psi \in \Delta^0$) if there exists a constant $\beta > 1$ such that

$$\lim_{x\to+\infty} \frac{\psi(\beta x)}{\psi(x)} = +\infty.$$

It should be noticed that if $\phi$ is an Orlicz function such that $\psi = \phi^* \in \Delta^0$, then $\phi \in \Delta_2$. Indeed, $\phi \in \Delta_2$ if and only if there exists $\beta > 1$ such that for all $x$ large enough (see [13, II.2.3]),

$$\frac{\psi(\beta x)}{\psi(x)} \geq 2\beta.$$

Let $\phi$ be an Orlicz function and let $\psi$ be its complementary Orlicz function. We shall assume that $\phi \in \Delta_2$. Then, isomorphically,

$$L^\phi = (M^\psi)^*, \quad L^\psi = (L^\phi)^*,$$

and so

$$(M^\psi)^{**} = L^\psi.$$

Moreover, $M^\psi = L^\psi$ if and only if $\psi \in \Delta_2$.

**2. Equi-absolutely continuous norms of relatively weakly compact subsets of an Orlicz space.** We first recall that if $\phi$ is an Orlicz function, then we say that $\mathcal{K} \subseteq L^\phi$ has *equi-absolutely continuous norm* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}(E) < \delta \ \Rightarrow\ \sup\{\|\chi_E f\|_{L^\phi};\ f \in \mathcal{K}\} < \varepsilon.$$

Every such $\mathcal{K}$ is relatively weakly compact, and, under the assumption $\phi^* \in \Delta^0$, J. Alexopoulos ([2]) proved the converse:

THEOREM 1. *Let $\phi$ be an Orlicz function such that $\psi = \phi^* \in \Delta^0$. Then every relatively weakly compact subset of $L^\phi$ has equi-absolutely continuous norm.*

We are going to give a new proof of this result, using a criterion of weak compactness proved by P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza (see [6, Theorem 4]).

THEOREM 2. *Let $\psi$ be an Orlicz function such that $\psi \in \Delta^0$, $X$ be a subspace of $M^\psi$, and $Y$ be a Banach space. Then for every bounded linear operator $T : X \to Y$, $T$ is weakly compact if and only if for some* (*and then all*) $p \in [1, +\infty[$,

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall f \in X, \quad \|T(f)\| \le C_\varepsilon \|f\|_p + \varepsilon \|f\|_\psi.$$

*Proof of Theorem 1.* We first prove that if $X$ is a reflexive subspace of $L^\phi$, then the closed unit ball $B_X$ of $X$ has equi-absolutely continuous norm. $B_X$ is also weakly compact, because $X$ is reflexive. Moreover, as $L^\phi = (M^\psi)^*$, $B_X$ is weak* compact, and so $X$ is weak* closed in $L^\phi$ (by Banach–Dieudonné's theorem). So there exists $Z \subseteq M^\psi$ such that $X = Z^\perp$. Then $X$ is isometrically isomorphic to $(M^\psi/Z)^*$. Let us denote by

$$\Pi : M^\psi \to M^\psi/Z$$

the canonical projection. As $(M^\psi/Z)^*$ is isometrically isomorphic to $X$, $M^\psi/Z$ is reflexive, and so $\Pi$ is weakly compact. We can now use Theorem 2.

Let $\alpha > 0$, $g \in B_X$ and $A$ be a measurable subset of $\Omega$. We have

$$\begin{aligned}\|g\chi_A\|_\phi &\le 2\sup\{|\langle g\chi_A, f\rangle|;\ f \in M^\psi,\ \|f\|_\psi \le 1\}\\ &= 2\sup\{|\langle g, f\chi_A\rangle|;\ f \in M^\psi,\ \|f\|_\psi \le 1\}\\ &= 2\sup\{|\langle g, \Pi(f\chi_A)\rangle|;\ f \in M^\psi,\ \|f\|_\psi \le 1\}\\ &\le 2\|g\|_\phi \sup\{\|\Pi(f\chi_A)\|;\ f \in M^\psi,\ \|f\|_\psi \le 1\}\\ &\le 2\sup\{C_\alpha\|f\chi_A\|_1 + \alpha\|f\chi_A\|_\psi;\ f \in M^\psi,\ \|f\|_\psi \le 1\}.\end{aligned}$$

Using Hölder's inequality for Orlicz spaces, we get

$$\|f\chi_A\|_1 = \int_\Omega |f|\chi_A\, d\mathbb{P} \le \|f\|_\psi\|\chi_A\|_\phi \le \|\chi_A\|_\phi.$$

On the other hand, for every positive constant $C$,

$$\int_\Omega \phi\Big(\frac{\chi_A}{C}\Big)\, d\mathbb{P} = \int_A \phi\Big(\frac{1}{C}\Big)\, d\mathbb{P} = m(A)\phi\Big(\frac{1}{C}\Big),$$

and so

$$\|\chi_A\|_\phi = \frac{1}{\phi^{-1}(1/m(A))}.$$

We also have

$$\|f\chi_A\|_\psi \le \|f\|_\psi \le 1.$$

Let $\varepsilon > 0$. Let us choose $\alpha$ such that $4\alpha < \varepsilon$, and $\delta > 0$ such that

$$m(A) < \delta \ \Rightarrow\ \frac{1}{\phi^{-1}(1/m(A))} \le \frac{\alpha}{C_\alpha}\,.$$

Thus we get

$$\|g\chi_A\|_\phi \le 4\alpha < \varepsilon$$

whenever $m(A) < \delta$; so $B_X$ has equi-absolutely continuous norm. ∎

We now assume that $\mathcal{K}$ is a relatively weakly compact subset of $L^\phi$. We use the following theorem (see [4, Theorem 11.17]):

THEOREM 3 (Davis, Figiel, Johnson, Pełczyński). *Let $K$ be a weakly compact subset of a Banach space $X$. Then there exist a reflexive space $Y$ and a bounded linear one-to-one operator $U$ from $Y$ into $X$ such that $K \subseteq U(B_Y)$.*

Let $\alpha > 0$, $g \in B_X$ and $A$ be a measurable subset of $\Omega$. By the theorem above, there exists $h \in B_Y$ such that $g = U(h)$. Denote by $U^* : L^\psi \to Y^*$ the dual operator, and $T$ its restriction to $M^\psi$. As $Y^*$ is reflexive, we can use Theorem 2 to obtain

$$\begin{aligned}\|g\chi_A\|_\phi &\le 2\sup\{|\langle g\chi_A, f\rangle|;\ f\in M^\psi,\ \|f\|_\psi\le 1\}\\ &= 2\sup\{|\langle g, f\chi_A\rangle|;\ f\in M^\psi,\ \|f\|_\psi\le 1\}\\ &= 2\sup\{|\langle U(h), f\chi_A\rangle|;\ f\in M^\psi,\ \|f\|_\psi\le 1\}\\ &= 2\sup\{|\langle h, U^*(f\chi_A)\rangle|;\ f\in M^\psi,\ \|f\|_\psi\le 1\}\\ &\le 2\sup\{\|T(f\chi_A)\|;\ f\in M^\psi,\ \|f\|_\psi\le 1\}\\ &\le 2\sup\{C_\alpha\|f\chi_A\|_1+\alpha\|f\chi_A\|_\psi;\ f\in M^\psi,\ \|f\|_\psi\le 1\}\\ &\le 4\alpha\end{aligned}$$

as above. ∎

**3. Reflexive subspaces of $L^\phi$ when $\phi^*\in\Delta^0$.** We begin by the following consequence of the embedding theorem of Bretagnolle and Dacunha-Castelle quoted in the introduction.

PROPOSITION 4. *Let $\phi$ be an Orlicz function $\phi^*\in\Delta^0$. Then $L^\phi$ embeds into $L^1$.*

*Proof.* Let us observe that condition $\Delta^0$ for $\psi=\phi^*$ implies that the lower Matuszewska–Orlicz index at infinity of $\psi$ is $\alpha_\psi^\infty=+\infty$ (see [11]). In fact, if $\beta>1$ and $x_0>1$ are such that

$$\psi(\beta x)\ge C\psi(x)\quad \text{for every } x\ge x_0,$$

we can deduce that setting $q=\ln(C)/\ln(\beta)$ we have

$$\psi(tx)\ge C^{-1}t^q\psi(x)\quad \text{for every } x\ge x_0 \text{ and } t\ge 1,$$

and consequently $\alpha_\psi^\infty\ge q$. Since $C$ is arbitrary, $\alpha_\psi^\infty=+\infty$.

By the duality of Matuszewska–Orlicz indices, the upper Matuszewska–Orlicz index of $\phi$ is $\beta_\phi^\infty=1$. As a consequence, $\phi$ is equivalent to an $r$-concave Orlicz function, for every $r>1$. But a result of Bretagnolle and Dacunha-Castelle tells us that any 2-concave Orlicz function space is isomorphic to a subspace of $L^1$. ∎

Our main result is:

THEOREM 5. *Let $\phi$ be an Orlicz function with $\phi^*\in\Delta^0$. Then the reflexive subspaces of $L^\phi$ are closed in the $L^1$-norm. In particular, the $L^1$- and $L^\phi$-norms are equivalent on reflexive subspaces of $L^\phi$.*

Together with Rosenthal's theorem (see [14, p. 268] or [8, p. 446]) this yields

COROLLARY 6. *Let $\phi$ be an Orlicz function such that $\phi^*\in\Delta^0$ and let $X$ be a reflexive subspace of $L^\phi$. Then there exist some $p>1$ and a probability* *density $u > 0$ such that the map*

$$j : X \to j(X) \subseteq L^p(u.\mathbb{P}), \qquad f \mapsto f/u,$$

*is an isomorphism.*

*Proof of Theorem 5.* First notice that $L^\phi(\Omega, \mathbb{P}) \subseteq L^1(\Omega, \mathbb{P})$. Indeed, $\phi$ is convex and $\phi'$ is non-decreasing, so

$$\phi(x) = \int_0^x \phi'(t)\,dt \geq \int_1^x \phi'(t)\,dt \geq (x-1)\phi'(1) \geq x\phi'(1).$$

Hence for every constant $C > 0$ and all $f \in L^\phi(\Omega, \mathbb{P})$, we have

$$\phi\left(\frac{|f(x)|}{C}\right) \geq \frac{\phi'(1)}{C}\,|f(x)| > 0,$$

and so

$$\int_\Omega \phi\left(\frac{|f|}{C}\right) d\mathbb{P} \geq \frac{\phi'(1)}{C}\,\|f\|_{L^1}.$$

Choosing $C = \|f\|_\phi$, we get

$$\|f\|_{L^\phi} \geq \phi'(1)\|f\|_{L^1}\,.$$

In particular, convergence in $L^\phi$-norm implies convergence in $L^1$-norm.

Let now $X$ be a reflexive subspace of $L^\phi(\Omega)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $X$ which converges in measure to a function $f$. We are going to prove that $(f_n)_{n\in\mathbb{N}}$ converges to $f$ for the Luxemburg norm of $L^\phi(\Omega)$. The unit closed ball $B_X$ of $X$ is weakly compact because $X$ is reflexive. Hence $B_X$ has an equi-absolutely continuous norm: for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\mathbb{P}(A) \leq \delta \ \Rightarrow\ \|g\chi_A\|_\phi \leq \varepsilon,\ \forall g \in B_X.$$

By homogeneity,

$$\mathbb{P}(A) \leq \delta \ \Rightarrow\ \|g\chi_A\|_\phi \leq \varepsilon\|g\|_\phi,\ \forall g \in X.$$

Fix $\varepsilon > 0$ and let $\delta > 0$ be associated to $\varepsilon$ as above. Since $(f_n)_{n\in\mathbb{N}}$ converges to $f$ in measure, there is an $n_0 \geq 0$ such that $\mathbb{P}(|f_n - f| \geq \varepsilon) \leq \delta$ for every $n \geq n_0$. Then for $n \geq n_0$,

$$\begin{aligned}\|f_n - f\|_\phi &\leq \|(f_n - f)\chi_{\{|f_n - f| \geq \varepsilon\}}\|_\phi + \|(f_n - f)\chi_{\{|f_n - f| \leq \varepsilon\}}\|_\phi \\ &\leq \varepsilon\|f_n - f\|_\phi + \varepsilon/\phi^{-1}(1)\,\cdot\end{aligned}$$

Indeed, if $g_n = (f_n - f)\chi_{\{|f_n - f| \leq \varepsilon\}}$, then for every $C > 0$,

$$\int_\Omega \phi(|g_n|/C)\,d\mathbb{P} \leq \phi(\varepsilon/C),$$

and so if $C \geq \varepsilon/\phi^{-1}(1)$, then

$$\int_{\Omega} \phi(|g_n|/C)\, d\mathbb{P} \leq 1,$$

and hence $\|g_n\|_\phi \leq \varepsilon/\phi^{-1}(1)$·

For $0 < \varepsilon < 1$, we have obtained, for $n \geq n_0$,

$$\|f_n - f\|_\phi \leq \frac{1}{\phi^{-1}(1)} \frac{\varepsilon}{1-\varepsilon}.$$

So

$$\lim_{n\to+\infty} \|f_n - f\|_\phi = 0.$$

Hence, on $X$, the convergences in $L^\phi$-norm, in $L^1$-norm and in measure are equivalent. ■

Remark. Without the additional assumption on the Orlicz function $\phi$, Proposition 3 is no longer true, and $\phi \in \Delta_2$ does not suffice; indeed, one has the following example.

Example. There exists an Orlicz function $\phi$ such that $L^\phi(0,1)$ is reflexive (so $\phi \in \Delta_2$ and $\psi = \phi^* \in \Delta_2$), but not isomorphic to any subspace of any $L^p$ space, $1 \leq p < \infty$.

This space was contructed by F. Hernández and V. Peirats in [5]. It is based on the construction by J. Lindenstrauss and L. Tzafriri ([9, Theorem 3]) of a reflexive Orlicz sequence space which contains no complemented subspace isomorphic to any $\ell_p$, $1 \leq p \leq \infty$ ([10, Theorem 3]). More precisely, for every $2 \leq \alpha \leq \beta < +\infty$, they constructed an Orlicz function on $[0,1]$ such that $\ell_\phi$ contains a subspace isomorphic to $\ell_q$ for any $q$ such that $\alpha \leq q \leq \beta$ ([11, Theorem 1], or [12, Theorem 4.a.9]), but no complemented subspace isomorphic to any $\ell_p$. It is proved in [5] that the minimal (see [9, Definition 2]) Orlicz function $\phi$ constructed by Lindenstrauss and Tzafriri on $[0,1]$ has an extension $\phi$ to a minimal Orlicz function defined on $[0,+\infty[$, and that the Orlicz function space $L^\phi(0,1)$ contains a (complemented) subspace isomorphic to $\ell_\phi$, but no complemented subspace isomorphic to $\ell_p$ for $p \neq 2$.

This Orlicz space $L^\phi(0,1)$ is reflexive (because $1 < \alpha_\phi^\infty = \alpha$ and $\beta_\phi^\infty = \beta < +\infty$: see [5]) and cannot be isomorphic to a subspace of any $L^p$ space. Indeed, if $\beta > \alpha$, then $\ell_\phi$, and hence $L^\phi(0,1)$, contains a subspace isomorphic to $\ell_q$ for any $q \in [\alpha,\beta]$, and in particular with $q > 2$; hence $L^\phi(0,1)$ cannot be isomorphic to a subspace of $L^p$ for $1 \leq p \leq 2$, since these latter spaces have cotype 2, whereas the cotype of $L^p$ is $p$. On the other hand, $L^\phi(0,1)$ cannot be isomorphic to a subspace of any $L^p$ space for $p > 2$ since, by the Kadec–Pełczyński theorem (see [1, Theorem 6.4.8]), every non-Hilbertian reflexive subspace (which is the case of $L^\phi(0,1)$) of such an $L^p$ space must contain a complemented subspace isomorphic to $\ell_p$, and $L^{\phi}(0,1)$ contains no such subspace. ■

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Laboratoire de Mathématiques Lens (LML)
Équipe d'Accueil EA 2462
Fédération CNRS Nord-Pas-de-Calais FR 2956
Faculté des Sciences Jean Perrin
Université d'Artois
rue Jean Souvraz S.P. 18
62307 Lens Cedex, France
E-mail: lavergne@euler.univ-artois.fr

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