REFLEXIVE SUBSPACES OF SOME ORLICZ SPACES

BY

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Abstract. We show that when the conjugate of an Orlicz function $\phi$ satisfies the growth condition $\Delta^0$, then the reflexive subspaces of $L^\phi$ are closed in the $L^1$-norm. For that purpose, we use (and give a new proof of) a result of J. Alexopoulos saying that weakly compact subsets of such $L^\phi$ have equi-absolutely continuous norm.

Introduction. Bretagnolle and Dacunha-Castelle showed in [3] that an Orlicz space $L^\phi$ embeds into $L^1$ (meaning that there exists an isomorphism of this space onto a subspace of $L^1$) if and only if $\phi$ is 2-concave (recall that a function $f$ is $r$-concave if $f(x^{1/r})$ is concave). If $\phi$ is an Orlicz function whose conjugate $\phi^*$ satisfies the condition $\Delta^0$ (see below for the definition), then $\phi$ is equivalent, for every $r > 1$, to an $r$-concave Orlicz function (Proposition 4) and hence $L^\phi$ embeds into $L^1$. In this paper, we show that for such Orlicz functions $\phi$, the reflexive subspaces of $L^\phi$ are actually closed in the $L^1$-norm (and so the $L^\phi$-topology is the same as the $L^1$-topology). In order to prove this, we shall use a result of J. Alexopoulos (Theorem 1), saying that, when $\phi^* \in \Delta^0$, the weakly compact subsets of $L^\phi$ have equi- absolutely continuous norm, and we shall begin by giving a new proof of this result, using a recent characterization, due to P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza (see [6, Theorem 4]), of the weakly compact operators defined on a subspace of the Morse–Transue space $M^\psi$, when $\psi \in \Delta^0$.

1. Notation. We shall consider Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$ (see [7], [13]). By an Orlicz function, we shall understand a non-decreasing convex function $\phi : [0, +\infty] \to [0, +\infty]$ such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. To avoid pathologies, we shall assume that $\phi$ has the following additional properties: $\phi$ is continuous at 0, strictly convex, and moreover,

$$\lim_{x \to +\infty} \frac{\phi(x)}{x} = +\infty.$$ 

This is essentially to exclude the case of $\phi(x) = ax$, and so of $L^1$. 

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Let $\phi$ be an Orlicz function. If $\phi'$ is the left derivative of $\phi$, then, for every $x > 0$,

$$
\phi(x) = \int_0^x \phi'(t) \, dt.
$$

The Orlicz space $L^\phi(\Omega)$ is the space of all equivalence classes of measurable functions $f : \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$
\int_\Omega \phi\left( \frac{|f(t)|}{C} \right) \, d\mathbb{P}(t) < +\infty.
$$

Then for all $f \in L^\phi(\Omega)$, we define the Luxemburg norm of $f$ as the infimum of all possible constants $C$ such that the above integral is $\leq 1$. With this norm, $L^\phi(\Omega)$ is a Banach space.

The Morse–Transue space $M^\phi(\Omega)$ is the subspace of $L^\phi(\Omega)$ generated by $L^\infty(\Omega)$, or equivalently, the subspace of all functions $f$ for which the above integral is finite for all $C > 0$.

To every Orlicz function $\phi$ is associated the conjugate Orlicz function $\phi^*$ defined by

$$
\phi^* : [0, +\infty) \to [0, +\infty), \quad x \mapsto \sup\{xy - \phi(y) ; y \geq 0\}.
$$

(Observe that $\phi^*(x) < \infty$ since $\phi(x)/x$ tends to $\infty$.)

The function $\phi^*$ is itself strictly convex. It should also be noticed that for all Orlicz functions $\phi$, we have

$$(\phi^*)^* = \phi.$$

Moreover, if $\phi_1$ and $\phi_2$ are two Orlicz functions such that $\phi_1(x) \leq \phi_2(x)$ whenever $x \geq x_0$, then there exists $y_0$ such that $\phi_2^*(y) \leq \phi_1^*(y)$ for all $y \geq y_0$.

We shall also use some growth conditions for Orlicz functions. We shall say that $\phi$ satisfies the $\Delta_2$ condition (and write $\phi \in \Delta_2$) if there exists a constant $K > 1$ such that for all $x$ large enough,

$$
\phi(2x) \leq K \phi(x).
$$

We shall say (see [6] and [7]) that $\psi$ satisfies the $\Delta^0$ condition (and write $\psi \in \Delta^0$) if there exists a constant $\beta > 1$ such that

$$
\lim_{x \to +\infty} \frac{\psi(\beta x)}{\psi(x)} = +\infty.
$$

It should be noticed that if $\phi$ is an Orlicz function such that $\psi = \phi^* \in \Delta^0$, then $\phi \in \Delta_2$. Indeed, $\phi \in \Delta_2$ if and only if there exists $\beta > 1$ such that for all $x$ large enough (see [13, II.2.3]),

$$
\frac{\psi(\beta x)}{\psi(x)} \geq 2\beta.
$$
Let \( \phi \) be an Orlicz function and let \( \psi \) be its complementary Orlicz function. We shall assume that \( \phi \in \Delta_2 \). Then, isomorphically,
\[
L^\phi = (M^\psi)^*, \quad L^\psi = (L^\phi)^*,
\]
and so
\[
(M^\psi)^{**} = L^\psi.
\]
Moreover, \( M^\psi = L^\psi \) if and only if \( \psi \in \Delta_2 \).

2. Equi-absolutely continuous norms of relatively weakly compact subsets of an Orlicz space. We first recall that if \( \phi \) is an Orlicz function, then we say that \( K \subseteq L^\phi \) has equi-absolutely continuous norm if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
P(E) < \delta \Rightarrow \sup\{\|\chi_E f\|_{L^\phi}; f \in K\} < \varepsilon.
\]
Every such \( K \) is relatively weakly compact, and, under the assumption \( \phi^* \in \Delta^0 \), J. Alexopoulos ([2]) proved the converse:

**Theorem 1.** Let \( \phi \) be an Orlicz function such that \( \psi = \phi^* \in \Delta^0 \). Then every relatively weakly compact subset of \( L^\phi \) has equi-absolutely continuous norm.

We are going to give a new proof of this result, using a criterion of weak compactness proved by P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza (see [6, Theorem 4]).

**Theorem 2.** Let \( \psi \) be an Orlicz function such that \( \psi \in \Delta^0 \), \( X \) be a subspace of \( M^\psi \), and \( Y \) be a Banach space. Then for every bounded linear operator \( T: X \to Y \), \( T \) is weakly compact if and only if for some (and then all) \( p \in [1, +\infty[ \),
\[
\forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall f \in X, \quad \|T(f)\| \leq C_\varepsilon \|f\|_{\psi} + \varepsilon \|f\|_p.
\]

**Proof of Theorem 1.** We first prove that if \( X \) is a reflexive subspace of \( L^\phi \), then the closed unit ball \( B_X \) of \( X \) has equi-absolutely continuous norm. \( B_X \) is also weakly compact, because \( X \) is reflexive. Moreover, as \( L^\phi = (M^\psi)^* \), \( B_X \) is weak* compact, and so \( X \) is weak* closed in \( L^\phi \) (by Banach–Dieudonné’s theorem). So there exists \( Z \subseteq M^\psi \) such that \( X = Z^\perp \). Then \( X \) is isometrically isomorphic to \((M^\psi/Z)^*\). Let us denote by
\[
\Pi : M^\psi \to M^\psi/Z
\]
the canonical projection. As \((M^\psi/Z)^*\) is isometrically isomorphic to \( X \), \( M^\psi/Z \) is reflexive, and so \( \Pi \) is weakly compact. We can now use Theorem 2.
Let \( \alpha > 0 \), \( g \in B_X \) and \( A \) be a measurable subset of \( \Omega \). We have
\[
\|g\chi_A\|_\phi \leq 2 \sup \{ |\langle g\chi_A, f \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \}
= 2 \sup \{ |\langle g, f\chi_A \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \}
= 2 \sup \{ |\langle g, \Pi(f\chi_A) \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \}
\leq 2 \|g\|_\phi \sup \{ \|\Pi(f\chi_A)\|; f \in M^\psi, \|f\|_\psi \leq 1 \}
\leq 2 \sup \{ C_\alpha \|f\chi_A\|_1 + \alpha \|f\chi_A\|_\psi; f \in M^\psi, \|f\|_\psi \leq 1 \}.
\]
Using Hölder’s inequality for Orlicz spaces, we get
\[
\|f\chi_A\|_1 = \int_{\Omega} |f|\chi_A \ d\mathbb{P} \leq \|f\|_\psi \|\chi_A\|_\phi \leq \|\chi_A\|_\phi.
\]
On the other hand, for every positive constant \( C \),
\[
\int_{\Omega} \phi \left( \frac{\chi_A}{C} \right) \ d\mathbb{P} = \int_{A} \phi \left( \frac{1}{C} \right) \ d\mathbb{P} = m(A) \phi \left( \frac{1}{C} \right),
\]
and so
\[
\|\chi_A\|_\phi = \frac{1}{\phi^{-1}(1/m(A))}.
\]
We also have
\[
\|f\chi_A\|_\psi \leq \|f\|_\psi \leq 1.
\]
Let \( \varepsilon > 0 \). Let us choose \( \alpha \) such that \( 4\alpha < \varepsilon \), and \( \delta > 0 \) such that
\[
m(A) < \delta \Rightarrow \frac{1}{\phi^{-1}(1/m(A))} \leq \frac{\alpha}{C_\alpha}.
\]
Thus we get
\[
\|g\chi_A\|_\phi \leq 4\alpha < \varepsilon
\]
whenever \( m(A) < \delta \); so \( B_X \) has equi-absolutely continuous norm. \( \blacksquare \)

We now assume that \( K \) is a relatively weakly compact subset of \( L^\phi \). We use the following theorem (see [4, Theorem 11.17]):

**Theorem 3** (Davis, Figiel, Johnson, Pełczyński). Let \( K \) be a weakly compact subset of a Banach space \( X \). Then there exist a reflexive space \( Y \) and a bounded linear one-to-one operator \( U \) from \( Y \) into \( X \) such that \( K \subseteq U(B_Y) \).

Let \( \alpha > 0 \), \( g \in B_X \) and \( A \) be a measurable subset of \( \Omega \). By the theorem above, there exists \( h \in B_Y \) such that \( g = U(h) \). Denote by \( U^*: L^\psi \rightarrow Y^* \) the dual operator, and \( T \) its restriction to \( M^\psi \). As \( Y^* \) is reflexive, we can
use Theorem 2 to obtain
\[ \|g\chi_A\|_\phi \leq 2 \sup \{ |\langle g\chi_A, f \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \} \]
\[ = 2 \sup \{ |\langle g, f\chi_A \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \} \]
\[ = 2 \sup \{ |\langle U(h), f\chi_A \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \} \]
\[ = 2 \sup \{ |\langle h, U^*(f\chi_A) \rangle|; f \in M^\psi, \|f\|_\psi \leq 1 \} \]
\[ \leq 2 \sup \{ \|T(f\chi_A)\|; f \in M^\psi, \|f\|_\psi \leq 1 \} \]
\[ \leq 2 \sup \{ C_\alpha \|f\chi_A\|_1 + \alpha \|f\chi_A\|_\psi; f \in M^\psi, \|f\|_\psi \leq 1 \} \]
\[ \leq 4\alpha \]
as above. 

3. Reflexive subspaces of $L^\phi$ when $\phi^* \in \Delta^0$. We begin by the following consequence of the embedding theorem of Bretagnolle and Dacunha-Castelle quoted in the introduction.

**Proposition 4.** Let $\phi$ be an Orlicz function $\phi^* \in \Delta^0$. Then $L^\phi$ embeds into $L^1$.

**Proof.** Let us observe that condition $\Delta^0$ for $\psi = \phi^*$ implies that the lower Matuszewska–Orlicz index at infinity of $\psi$ is $\alpha^\infty_\psi = +\infty$ (see [11]). In fact, if $\beta > 1$ and $x_0 > 1$ are such that
\[ \psi(\beta x) \geq C\psi(x) \quad \text{for every } x \geq x_0, \]
we can deduce that setting $q = \ln(C)/\ln(\beta)$ we have
\[ \psi(tx) \geq C^{-1}\beta^q \psi(x) \quad \text{for every } x \geq x_0 \text{ and } t \geq 1, \]
and consequently $\alpha^\infty_\psi \geq q$. Since $C$ is arbitrary, $\alpha^\infty_\psi = +\infty$.

By the duality of Matuszewska–Orlicz indices, the upper Matuszewska–Orlicz index of $\phi$ is $\beta^\infty_\phi = 1$. As a consequence, $\phi$ is equivalent to an $r$-concave Orlicz function, for every $r > 1$. But a result of Bretagnolle and Dacunha-Castelle tells us that any 2-concave Orlicz function space is isomorphic to a subspace of $L^1$. 

Our main result is:

**Theorem 5.** Let $\phi$ be an Orlicz function with $\phi^* \in \Delta^0$. Then the reflexive subspaces of $L^\phi$ are closed in the $L^1$-norm. In particular, the $L^1$- and $L^\phi$-norms are equivalent on reflexive subspaces of $L^\phi$.

Together with Rosenthal’s theorem (see [14, p. 268] or [8, p. 446]) this yields

**Corollary 6.** Let $\phi$ be an Orlicz function such that $\phi^* \in \Delta^0$ and let $X$ be a reflexive subspace of $L^\phi$. Then there exist some $p > 1$ and a probability
density $u > 0$ such that the map

$$j : X \to j(X) \subseteq L^p(u, \mathbb{P}), \quad f \mapsto f/u,$$

is an isomorphism.

Proof of Theorem 5. First notice that $L^\phi(\Omega, \mathbb{P}) \subseteq L^1(\Omega, \mathbb{P})$. Indeed, $\phi$ is convex and $\phi'$ is non-decreasing, so

$$\phi(x) = \int_0^x \phi'(t) \, dt \geq \int_1^x \phi'(t) \, dt \geq (x - 1)\phi'(1) \geq x\phi'(1).$$

Hence for every constant $C > 0$ and all $f \in L^\phi(\Omega, \mathbb{P})$, we have

$$\phi\left(\frac{|f(x)|}{C}\right) \geq \frac{\phi'(1)}{C} |f(x)| > 0,$$

and so

$$\int_\Omega \phi\left(\frac{|f|}{C}\right) \, d\mathbb{P} \geq \frac{\phi'(1)}{C} \|f\|_{L^1}.$$

Choosing $C = \|f\|_\phi$, we get

$$\|f\|_{L^\phi} \geq \phi'(1)\|f\|_{L^1}.$$

In particular, convergence in $L^\phi$-norm implies convergence in $L^1$-norm.

Let now $X$ be a reflexive subspace of $L^\phi(\Omega)$ and $(f_n)_{n \in \mathbb{N}}$ be a sequence in $X$ which converges in measure to a function $f$. We are going to prove that $(f_n)_{n \in \mathbb{N}}$ converges to $f$ for the Luxemburg norm of $L^\phi(\Omega)$. The unit closed ball $B_X$ of $X$ is weakly compact because $X$ is reflexive. Hence $B_X$ has an equi-absolutely continuous norm: for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\mathbb{P}(A) \leq \delta \Rightarrow \|g\chi_A\|_\phi \leq \varepsilon, \forall g \in B_X.$$

By homogeneity,

$$\mathbb{P}(A) \leq \delta \Rightarrow \|g\chi_A\|_\phi \leq \varepsilon\|g\|_\phi, \forall g \in X.$$

Fix $\varepsilon > 0$ and let $\delta > 0$ be associated to $\varepsilon$ as above. Since $(f_n)_{n \in \mathbb{N}}$ converges to $f$ in measure, there is an $n_0 \geq 0$ such that $\mathbb{P}(|f_n - f| \geq \varepsilon) \leq \delta$ for every $n \geq n_0$. Then for $n \geq n_0$,

$$\|f_n - f\|_\phi \leq \|(f_n - f)\chi_{\{|f_n - f| \geq \varepsilon\}}\|_\phi + \|(f_n - f)\chi_{\{|f_n - f| \leq \varepsilon\}}\|_\phi$$

$$\leq \varepsilon\|f_n - f\|_\phi + \varepsilon/\phi^{-1}(1).$$

Indeed, if $g_n = (f_n - f)\chi_{\{|f_n - f| \leq \varepsilon\}}$, then for every $C > 0$,

$$\int_\Omega \phi(|g_n|/C) \, d\mathbb{P} \leq \phi(\varepsilon/C),$$

$$\mathbb{P}(A) \leq \delta \Rightarrow \|g\chi_A\|_\phi \leq \varepsilon, \forall g \in B_X.$$
and so if $C \geq \varepsilon/\phi^{-1}(1)$, then
\[ \int \Omega \phi(|g_n|/C) d\mathbb{P} \leq 1, \]
and hence $\|g_n\|_\phi \leq \varepsilon/\phi^{-1}(1)$.

For $0 < \varepsilon < 1$, we have obtained, for $n \geq n_0$,
\[ \|f_n - f\|_\phi \leq \frac{1}{\phi^{-1}(1)} \frac{\varepsilon}{1 - \varepsilon}. \]
So
\[ \lim_{n \to +\infty} \|f_n - f\|_\phi = 0. \]

Hence, on $X$, the convergences in $L^\phi$-norm, in $L^1$-norm and in measure are equivalent. ■

**Remark.** Without the additional assumption on the Orlicz function $\phi$, Proposition 3 is no longer true, and $\phi \in \Delta_2$ does not suffice; indeed, one has the following example.

**Example.** There exists an Orlicz function $\phi$ such that $L^\phi(0, 1)$ is reflexive (so $\phi \in \Delta_2$ and $\psi = \phi^* \in \Delta_2$), but not isomorphic to any subspace of any $L^p$ space, $1 \leq p < \infty$.

This space was constructed by F. Hernández and V. Peirats in [5]. It is based on the construction by J. Lindenstrauss and L. Tzafriri ([9, Theorem 3]) of a reflexive Orlicz sequence space which contains no complemented subspace isomorphic to any $\ell_p$, $1 \leq p \leq \infty$ ([10, Theorem 3]). More precisely, for every $2 \leq \alpha \leq \beta < +\infty$, they constructed an Orlicz function on $[0, 1]$ such that $\ell_\phi$ contains a subspace isomorphic to $\ell_q$ for any $q$ such that $\alpha \leq q \leq \beta$ ([11, Theorem 1], or [12, Theorem 4.a.9]), but no complemented subspace isomorphic to any $\ell_p$. It is proved in [5] that the minimal (see [9, Definition 2]) Orlicz function $\phi$ constructed by Lindenstrauss and Tzafriri on $[0, 1]$ has an extension $\phi$ to a minimal Orlicz function defined on $[0, +\infty]$, and that the Orlicz function space $L^\phi(0, 1)$ contains a (complemented) subspace isomorphic to $\ell_\phi$, but no complemented subspace isomorphic to $\ell_p$ for $p \neq 2$.

This Orlicz space $L^\phi(0, 1)$ is reflexive (because $1 < \alpha_\phi = \alpha$ and $\beta_\phi = \beta < +\infty$; see [5]) and cannot be isomorphic to a subspace of any $L^p$ space. Indeed, if $\beta > \alpha$, then $\ell_\phi$, and hence $L^\phi(0, 1)$, contains a subspace isomorphic to $\ell_q$ for any $q \in [\alpha, \beta]$, and in particular with $q > 2$; hence $L^\phi(0, 1)$ cannot be isomorphic to a subspace of $L^p$ for $1 \leq p \leq 2$, since these latter spaces have cotype 2, whereas the cotype of $L^p$ is $p$. On the other hand, $L^\phi(0, 1)$ cannot be isomorphic to a subspace of any $L^p$ space for $p > 2$ since, by the Kadec–Pełczyński theorem (see [1, Theorem 6.4.8]), every non-Hilbertian reflexive subspace (which is the case of $L^\phi(0, 1)$) of such an $L^p$ space must
contain a complemented subspace isomorphic to $\ell_p$, and $L_\phi(0, 1)$ contains no such subspace.

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**REFERENCES**


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