COLLOQUIUM MATHEMATICUM

VOL. 127 2012 NO. 2

A HARTOGS TYPE EXTENSION THEOREM FOR GENERALIZED (N,k)-CROSSES WITH PLURIPOLAR SINGULARITIES

BY

MAŁGORZATA ZAJĘCKA (Kraków)

Abstract. The aim of this paper is to present an extension theorem for (N,k)-crosses with pluripolar singularities.

1. Introduction. Statement of the main result

1.1. Introduction. The topic of separately holomorphic functions have a long history in complex analysis. The problem was first investigated by W. F. Osgood [Osg 1899]. Seven years later F. Hartogs [Har 1906] proved his famous theorem stating that every separately holomorphic function is, in fact, holomorphic. Since then the interest switched to a more general problem: whether a function f defined on a product $D \times G$ of two domains, and separately holomorphic on some subsets $A \subset D$ and $B \subset G$, is holomorphic on the whole $D \times G$ (see for example papers of M. Hukuhara [Huk 1942] and T. Terada [Ter 1967]). This led to the question of possible holomorphic extension of a function separately holomorphic on objects called crosses.

In a recent paper [Lew 2012] A. Lewandowski introduces an object called a generalized (N,k)-cross $\mathbf{T}_{N,k}$, a generalization of the (N,k)-cross defined by M. Jarnicki and P. Pflug [JarPfl 2010], and proves an extension theorem for this new type of cross with analytic singularities. In this paper we will prove a similar extension theorem for $\mathbf{T}_{N,k}$ crosses with pluripolar singularities, generalizing Theorem 10.2.9 of [JarPfl 2011] and the Main Theorem of [JarPfl 2003]. We will also introduce another type of generalized (N,k)-crosses, called $\mathbf{Y}_{N,k}$ crosses, a natural object to consider in light of Theorem 3.6. This theorem will turn out to be a strong tool, allowing us to prove two Hartogs-type extension theorems for functions separately holomorphic on $\mathbf{X}_{N,k}$, $\mathbf{T}_{N,k}$ and $\mathbf{Y}_{N,k}$ crosses, including the Main Theorem of this paper.

The paper is divided into four sections. In the first section we define generalized (N,k)-crosses and we state the Main Theorem. Section 2 contains

2010 *Mathematics Subject Classification*: 32D15, 32U15.

Key words and phrases: crosses, generalized crosses, separately holomorphic functions, pluripolar sets, relative extremal function.

DOI: 10.4064/cm127-2-1

some useful definitions and facts. Section 3 is dedicated to (N,k)-crosses, their properties and recent cross theorems. It also contains the statement of Theorem 3.6 and the proof of the Main Theorem. In the last section we present a detailed proof of Theorem 3.6.

1.2. Generalized (N,k)-crosses and the main result. Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be locally pluriregular (see Definition 2.1), $j = 1,\dots,N$, where $N \geq 2$. For $\alpha = (\alpha_1,\dots,\alpha_N) \in \{0,1\}^N$ and $B_j \subset D_j$, $j = 1,\dots,N$, define

$$\mathcal{X}_\alpha := \mathcal{X}_{1,\alpha_1} \times \cdots \times \mathcal{X}_{N,\alpha_N}, \quad \mathcal{X}_{j,\alpha_j} := \begin{cases} D_j & \text{when } \alpha_j = 1, \\ A_j & \text{when } \alpha_j = 0, \end{cases} \quad j = 1,\dots,N,$$
$$B_0^\alpha := \prod_{j\in\{1,\dots,N\}:\,\alpha_j=0} B_j, \quad B_1^\alpha := \prod_{j\in\{1,\dots,N\}:\,\alpha_j=1} B_j.$$

For $\alpha \in \{0,1\}^N$ we merge $c_0 \in D_0^\alpha$ and $c_1 \in D_1^\alpha$ into $(\overleftrightarrow{c_0, c_1}) \in \prod_{j=1}^n D_j$ by putting variables in right places.

We also use the following convention: for $D \subset D_0^\alpha$, $G \subset D_1^\alpha$, $\alpha \in \{0,1\}^N$, define

$$\overleftrightarrow{D \times G} := \{(\overleftrightarrow{a, b}) : a \in D,\, b \in G\}.$$

To simplify notation define

$$\mathcal{T}_k^N := \{\alpha \in \{0,1\}^N : |\alpha| = k\}, \quad J := \{\alpha \in \{0,1\}^N : 1 \leq |\alpha| \leq k\}.$$

Definition 1.1. For $k \in \{1,\dots,N\}$ we define an *(N,k)-cross* $\mathbf{X}_{N,k}$ by

$$\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N) := \bigcup_{\alpha\in\mathcal{T}_k^N} \mathcal{X}_\alpha.$$

For $\alpha \in \mathcal{Y}_k^N$ let $\Sigma_\alpha \subset A_0^\alpha$ and put

$$\mathcal{X}_\alpha^\Sigma := \{z \in \mathcal{X}_\alpha : z_\alpha \notin \Sigma_\alpha\}, \quad a \in \mathcal{Y}_k^N,$$

where z_α denotes the projection of z on D_0^α.

Definition 1.2. We define a *generalized (N,k)-cross* $\mathbf{T}_{N,k}$ by

$$\mathbf{T}_{N,k} = \mathbb{T}_{N,k}((A_j,D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}) := \bigcup_{\alpha\in\mathcal{T}_k^N} \mathcal{X}_\alpha^\Sigma$$

and a *generalized (N,k)-cross* $\mathbf{Y}_{N,k}$ by

$$\mathbf{Y}_{N,k} = \mathbb{Y}_{N,k}((A_j,D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{Y}_k^N}) := \bigcup_{\alpha\in\mathcal{Y}_k^N} \mathcal{X}_\alpha^\Sigma.$$

Observe that always $\mathbf{T}_{N,k} \subset \mathbf{Y}_{N,k}$.

EXAMPLE 1.3. To see the difference between $\mathbf{T}_{N,k}$ and $\mathbf{Y}_{N,k}$ consider for example $N=3$, $k=2$, and let

$$\Sigma_{(1,1,0)}=\{z_3\}\subset A_3,\quad \Sigma_{(1,0,1)}=\{z_2\}\subset A_2,\quad \Sigma_{(0,1,1)}=\{z_1\}\subset A_1,$$
$$\Sigma_\alpha=\emptyset,\quad \alpha\in\mathcal{Y}_2^3\setminus\mathcal{T}_2^3.$$

Observe that if $\Sigma_\alpha=\emptyset$ for all $\alpha\in\mathcal{Y}_k^N$, then

$$\begin{aligned}\mathbb{T}_{N,k}((A_j,D_j)_{j=1}^N,(\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}) &= \mathbb{Y}_{N,k}((A_j,D_j)_{j=1}^N,(\Sigma_\alpha)_{\alpha\in\mathcal{Y}_k^N})\\ &= \mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N).\end{aligned}$$

Moreover, for $k=1$ we have $(\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}=(\Sigma_\alpha)_{\alpha\in\mathcal{Y}_k^N}=(\Sigma_j)_{j=1}^N$ and we use the simplified notation

$$\mathbf{T}_{N,1}=\mathbf{Y}_{N,1}=:\mathbb{T}((A_j,D_j,\Sigma_j)_{j=1}^N).$$

DEFINITION 1.4. For an (N,k)-cross $\mathbf{W}_{N,k}\in\{\mathbf{X}_{N,k},\mathbf{T}_{N,k},\mathbf{Y}_{N,k}\}$ we define its *center* as

$$c(\mathbf{W}_{N,k}):=\mathbf{W}_{N,k}\cap(A_1\times\cdots\times A_N).$$

DEFINITION 1.5. For a cross $\mathbf{X}_{N,k}=\mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N)$ we define its *hull*

$$\begin{aligned}\widehat{\mathbf{X}}_{N,k}&=\widehat{\mathbb{X}}_{N,k}((A_j,D_j)_{j=1}^N)\\ &:=\Big\{(z_1,\ldots,z_N)\in D_1\times\cdots\times D_N:\sum_{j=1}^N\mathbf{h}_{A_j,D_j}(z_j)<k\Big\},\end{aligned}$$

where $\mathbf{h}_{B,D}$ denotes the relative extremal function of B with respect to D (see Definition 2.1).

Let $\mathbf{W}_{N,k}\in\{\mathbf{T}_{N,k},\mathbf{Y}_{N,k}\}$ and let $M\subset\mathbf{W}_{N,k}$. For $\alpha\in\mathcal{Y}_k^N$ and $a\in A_0^\alpha$ let $M_{a,\alpha}$ denote the fiber

$$M_{a,\alpha}:=\{z\in D_1^\alpha:(\overleftrightarrow{a,z})\in M\}.$$

For $(z',z'')\in\prod_{j=1}^k D_j\times\prod_{j=k+1}^N D_j$, $k\in\{1,\ldots,N-1\}$, define

$$M_{(z',\cdot)}:=\Big\{b\in\prod_{j=k+1}^N D_j:(z',b)\in M\Big\},$$
$$M_{(\cdot,z'')}:=\Big\{a\in\prod_{j=1}^k D_j:(a,z'')\in M\Big\}.$$

DEFINITION 1.6. Let $M\subset\mathbf{T}_{N,k}$ be such that for all $\alpha\in\mathcal{T}_k^N$ and $a\in A_0^\alpha\setminus\Sigma_\alpha$ the set $D_1^\alpha\setminus M_{a,\alpha}$ is open. A function $f:\mathbf{T}_{N,k}\setminus M\to\mathbb{C}$ is called *separately holomorphic on* $\mathbf{T}_{N,k}\setminus M$ (written $f\in\mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k}\setminus M)$) if for all $\alpha\in\mathcal{T}_k^N$ and $a\in A_0^\alpha\setminus\Sigma_\alpha$, the function

$$(\dagger)\qquad D_1^\alpha\setminus M_{a,\alpha}\ni z\mapsto f((\overleftrightarrow{a,z}))=:f_{a,\alpha}(z)$$

is holomorphic.

For a generalized (N,k)-cross $\mathbf{Y}_{N,k}$ we give an analogous definition.

Definition 1.7. Let $M \subset \mathbf{Y}_{N,k}$ be such that for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the set $D_1^\alpha \setminus M_{a,\alpha}$ is open. A function $f : \mathbf{Y}_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $\mathbf{Y}_{N,k} \setminus M$ (written $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{Y}_{N,k} \setminus M)$) if for all $\alpha \in \mathcal{T}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$, the function (†) is holomorphic.

Remark 1.8. Observe that if $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{Y}_{N,k} \setminus M)$, then (†) is also holomorphic for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$.

Let $M \subset \mathbf{T}_{N,k}$. For $\alpha \in \mathcal{Y}_k^N$ and $b \in D_1^\alpha$ let $M_{b,\alpha}$ denote the fiber

$$M_{b,\alpha} := \{z \in A_0^\alpha : \overleftrightarrow{(z,b)} \in M\}.$$

The following class of functions plays an important role in the Main Theorem. It is a natural extension of the class $\mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k} \setminus M) \cap \mathcal{C}(\mathbf{T}_{N,k} \setminus M)$.

Definition 1.9. Let $M \subset \mathbf{T}_{N,k}$ be such that for all $\alpha \in \mathcal{T}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the set $D_1^\alpha \setminus M_{a,\alpha}$ is open. We denote by $\mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$ the space of all $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k} \setminus M)$ such that for all $\alpha \in \mathcal{T}_k^N$ and $b \in D_1^\alpha$, the function

$$A_0^\alpha \setminus (\Sigma_\alpha \cup M_{b,\alpha}) \ni z \mapsto f(\overleftrightarrow{(z,b)}) =: f_{b,\alpha}(z)$$

is continuous.

The following theorem is the main result of this paper. It is an analogue and a natural generalization of Theorem 10.2.9 of [JarPfl 2011]. It also extends the main result of [Lew 2012].

Main Theorem (Extension theorem for (N,k)-crosses with pluripolar singularities). *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. For $\alpha \in \mathcal{T}_k^N$ let $\Sigma_\alpha \subset A_0^\alpha$ be pluripolar. Let*

$$\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N), \qquad \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{T}_k^N}).$$

Let M be a relatively closed, pluripolar subset of $\mathbf{T}_{N,k}$ such that for all $\alpha \in \mathcal{T}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar. Let

$$\mathcal{F} := \begin{cases} \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k} \setminus M) & \text{if } \Sigma_\alpha = \emptyset \text{ for all } \alpha \in \mathcal{T}_k^N, \\ \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M) & \text{otherwise.} \end{cases}$$

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ and a generalized (N,k)-cross $\mathbf{T}'_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma'_\alpha)_{\alpha \in \mathcal{T}_k^N}) \subset \mathbf{T}_{N,k}$ with $\Sigma_\alpha \subset \Sigma'_\alpha \subset A_0^\alpha$, Σ'_α pluripolar, $\alpha \in \mathcal{T}_k^N$, such that:

- $\widehat{M} \cap (c(\mathbf{T}_{N,k}) \cup \mathbf{T}'_{N,k}) \subset M$,
- *for any $f \in \mathcal{F}$ there exists a function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $(c(\mathbf{T}_{N,k}) \cup \mathbf{T}'_{N,k}) \setminus M$,*

- *$\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$* (1),
- *if $M = \emptyset$, then $\widehat{M} = \emptyset$,*
- *if for all $\alpha \in \mathcal{T}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is thin in D_1^α, then $\widehat{M}$ is analytic in $\widehat{\mathbf{X}}_{N,k}$.*

The following remark shows that the Main Theorem can be stated analogously to Theorem 10.2.9 of [JarPfl 2011].

REMARK 1.10. Observe that for any relatively closed pluripolar set $M \subset \mathbf{T}_{N,k}$ and for all $\alpha \in \mathcal{T}_k^N$ there exists a pluripolar set $\Sigma_\alpha^0 \subset A_0^\alpha$ such that $\Sigma_\alpha \subset \Sigma_\alpha^0$ and for all $a \in A_0^\alpha \setminus \Sigma_\alpha^0$ the fiber $M_{a,\alpha}$ is pluripolar. Then the Main Theorem implies its version with $(\Sigma_\alpha)_{\alpha\in\mathcal{T}_k^N}$ and $\mathbf{T}_{N,k}$ replaced by $(\Sigma_\alpha^0)_{\alpha\in\mathcal{T}_k^N}$ and $\mathbf{T}_{N,k}^0 := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha^0)_{\alpha\in\mathcal{T}_k^N})$.

2. Preliminaries

2.1. Relative extremal function

DEFINITION 2.1 (Relative extremal function). Let D be a Riemann domain over $\mathbb{C}^n$ and let $A \subset D$. The *relative extremal function of A with respect to D* is the function

$$\mathbf{h}_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \le 1,\ u|_A \le 0\}.$$

For an open set $G \subset D$ we define $\mathbf{h}_{A,G} := \mathbf{h}_{A\cap G,G}$.

A set $A \subset D$ is called *pluriregular at a point $a \in \overline{A}$* if $\mathbf{h}^*_{A,U}(a) = 0$ for any open neighborhood U of a, where $\mathbf{h}^*_{A,U}$ denotes the upper semicontinuous regularization of $\mathbf{h}_{A,U}$.

We call A *locally pluriregular* if $A \ne \emptyset$ and A is pluriregular at every point $a \in A$.

2.2. N-fold crosses. Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be a nonempty set, $j = 1, \dots, N$, where $N \ge 2$. For $k = 1$, for historical reasons, we call $\mathbb{X}_{N,1}((A_j, D_j)_{j=1}^N)$ an *N-fold cross* $\mathbf{X}$ and we write

$$\mathbf{X} = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) = \mathbb{X}((A_j, D_j)_{j=1}^N) = \bigcup_{j=1}^N (A_j' \times D_j \times A_j''),$$

where

$$\begin{aligned} A_j' &:= A_1 \times \dots \times A_{j-1}, \quad j = 2, \dots, N,\\ A_j'' &:= A_{j+1} \times \dots \times A_N, \quad j = 1, \dots, N-1,\\ A_1' \times D_1 \times A_1'' &:= D_1 \times A_1'', \quad A_N' \times D_N \times A_N'' := A_N' \times D_N. \end{aligned}$$

(1) That is, for all $a \in \widehat{M}$ and every open neighborhood U_a of a there exists $f \in \mathcal{F}$ such that $\widehat{f}$ does not extend holomorphically to U_a. For more details see [JarPfl 2000, Chapter 3].

For $\Sigma_j \subset A'_j \times A''_j$, $j = 1, \ldots, N$ put

$$\mathcal{X}_j := \{(a'_j, z_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j\},$$

where

$$\begin{aligned} a'_j &:= (a_1, \ldots, a_{j-1}), \quad && j = 2, \ldots, N, \\ a''_j &:= (a_{j+1}, \ldots, a_N), \quad && j = 1, \ldots, N-1, \\ (a'_1, z_1, a''_1) &:= (z_1, a''_1), \quad && (a'_N, z_N, a''_N) := (a'_N, z_N). \end{aligned}$$

We call $\mathbb{T}_{N,1}((A_j, D_j, \Sigma_j)_{j=1}^N) = \bigcup_{j=1}^N \mathcal{X}_j$ a *generalized N-fold cross* $\mathbf{T}$.

For $(a'_j, a''_j) \in A'_j \times A''_j$, $j = 1, \ldots, N$, define the fiber

$$M_{(a'_j, \cdot, a''_j)} := \{z \in D_j : (a'_j, z, a''_j) \in M\}.$$

Our proof of the Main Theorem will be based on Theorem 3.6, which is a technically more complicated analogue of Theorem 2.2 below (the first inductive step in the proof of Theorem 3.6).

THEOREM 2.2 ([JarPfl 2007, Theorem 1.1]). *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular and let $\Sigma_j \subset A'_j \times A''_j$ be pluripolar, $j = 1, \ldots, N$. Put*

$$\mathbf{X} := \mathbb{X}((A_j, D_j)_{j=1}^N), \quad \mathbf{T} := \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N).$$

Let $\mathcal{F}$ be a collection of functions $f : c(\mathbf{T}) \setminus M \to \mathbb{C}$ and let $M \subset \mathbf{T}$ be such that:

- *for any $j \in \{1, \ldots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ the fiber $M_{(a'_j, \cdot, a''_j)}$ is pluripolar,*
- *for any $j \in \{1, \ldots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ there exists a closed pluripolar set $\widetilde{M}_{a,j} \subset D_j$ such that $\widetilde{M}_{a,j} \cap A_j \subset M_{(a'_j, \cdot, a''_j)}$,*
- *for any $a \in c(\mathbf{T}) \setminus M$ there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a, r))$ with $f_a = f$ on $\mathbb{P}(a, r) \cap (c(\mathbf{T}) \setminus M)$* (2),
- *for any $f \in \mathcal{F}$, any $j \in \{1, \ldots, N\}$, and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ there exists $\widetilde{f}_{a,j} \in \mathcal{O}(D_j \setminus \widetilde{M}_{a,j})$ such that $\widetilde{f}_{a,j} = f(a'_j, \cdot, a''_j)$ on $A_j \setminus M_{a,j}$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}$ such that:

- *$\widehat{M} \cap c(\mathbf{T}) \subset M$,*
- *for any $f \in \mathcal{F}$ there exists $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $c(\mathbf{T}) \setminus M$,*
- *$\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$,*
- *if for all $j \in \{1, \ldots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ we have $\widetilde{M}_{a,j} = \emptyset$, then $\widehat{M} = \emptyset$,*

---

(2) $\mathbb{P}(a, r)$ denotes the polydisc in the Riemann domain $D_1 \times \cdots \times D_N$ centered at a with radius r. For more details see [JarPfl 2000, Chapter 1].

- *if for all* $j \in \{1, \dots, N\}$ *and* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ *the set* $\widetilde{M}_{a,j}$ *is thin in* D_j*, then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}$.

3. (N, k)-crosses

3.1. Basic properties of (N, k)-crosses. The following properties will be implicitly used throughout the paper.

LEMMA 3.1 (Properties of (N, k)-crosses, see [JarPfl 2010, Remark 5]).

(i) $\mathbf{X}_{N,1} = \mathbb{X}((A_j, D_j)_{j=1}^N)$, $\widehat{\mathbf{X}}_{N,1} = \widehat{\mathbb{X}}((A_j, D_j)_{j=1}^N)$,
(ii) $\mathbf{X}_{N,k}$ *is arcwise connected,*
(iii) $\widehat{\mathbf{X}}_{N,k}$ *is connected,*
(iv) *if* $D_1, \dots, D_N$ *are Riemann domains of holomorphy, then* $\widehat{\mathbf{X}}_{N,k}$ *is a Riemann domain of holomorphy,*
(v) $\mathbf{X}_{N,k} \subset \mathbf{X}_{N,k+1}$ *and* $\widehat{\mathbf{X}}_{N,k} \subset \widehat{\mathbf{X}}_{N,k+1}$, $k = 1, \dots, N-1$,
(vi) $\mathbf{X}_{N,k} = \mathbb{X}(\mathbf{X}_{N-1,k-1}, A_N; \mathbf{X}_{N-1,k}, D_N)$, $k = 2, \dots, N-1$, $N > 2$.

The following technical lemmas will also be useful.

LEMMA 3.2 ([JarPfl 2010, Lemma 4]). *Let* D_j *be a Riemann domain of holomorphy over* $\mathbb{C}^{n_j}$ *and* $A_j \subset D_j$ *be locally pluriregular,* $j = 1, \dots, N$*. Then for all* $z = (z_1, \dots, z_N) \in \widehat{\mathbf{X}}_{N,k}$ *we have*

$$\mathbf{h}_{\widehat{\mathbf{X}}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}(z) = \max\Big\{0, \sum_{j=1}^{N} \mathbf{h}_{A_j, D_j}(z_j) - k + 1\Big\}.$$

LEMMA 3.3. *Let* D_j *be a Riemann domain of holomorphy over* $\mathbb{C}^{n_j}$ *and* $A_j \subset D_j$ *be locally pluriregular,* $j = 1, \dots, N$*. Then for* $z \in \widehat{\mathbf{X}}_{N,k}$,

$$\mathbf{h}_{\mathbf{X}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}(z) = \mathbf{h}_{\widehat{\mathbf{X}}_{N,k-1}, \widehat{\mathbf{X}}_{N,k}}(z).$$

Proof. The inequality "$\geq$" follows from properties of the relative extremal function (see [JarPfl 2011, Proposition 3.2.2]). To show the opposite inequality fix $u \in \mathcal{PSH}(\widehat{\mathbf{X}}_{N,k})$ such that $u \leq 1$ and $u|_{\mathbf{X}_{N,k-1}} = 0$. Then $u|_{\widehat{\mathbf{X}}_{N,k-1}} \in \mathcal{PSH}(\widehat{\mathbf{X}}_{N,k-1})$ and $u|_{\widehat{\mathbf{X}}_{N,k-1}} \leq \mathbf{h}_{\mathbf{X}_{N,k-1}, \widehat{\mathbf{X}}_{N,k-1}}$. Using a reasoning analogous (3) to that for Proposition 5.1.8(i) of [JarPfl 2011] we show that $\mathbf{h}_{\mathbf{X}_{N,k-1}, \widehat{\mathbf{X}}_{N,k-1}} \equiv 0$ on $\widehat{\mathbf{X}}_{N,k-1}$, which finishes the proof. ∎

3.2. Cross theorems for (N, k)-crosses. In this section we present some recent results on (N, k)-crosses which will be used in the proof of the Main Theorem. Observe that our main result generalizes both of them.

(3) Instead of the classical cross theorem for N-fold crosses we use the cross theorem for (N, k)-crosses (see Theorem 3.4).

Theorem 3.4 (Cross theorem for (N,k)-crosses, cf. [JarPfl 2011, Theorem 7.2.7]). *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$ and $A_j \subset D_j$ be locally pluriregular, $j = 1,\dots,N$. For $k \in \{1,\dots,N\}$ let $\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N)$. Then for every $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k})$ there exists a unique function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{X}_{N,k}$.*

The following result is a special case of Theorem 2.12 of [Lew 2012], which is a cross theorem without singularities for generalized (N,k)-crosses.

Theorem 3.5 (Cross theorem for generalized (N,k)-crosses). *Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be pluriregular, $j = 1,\dots,N$. For $\alpha \in \mathcal{T}_k^N$ let Σ_α be a subset of A_0^α. Then for every $f \in \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k})$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k}$.*

3.3. Extension theorem for generalized (N,k)-crosses with pluripolar singularities. Now we state the already mentioned main technical result, an analogue of Theorem 2.2 which is crucial for the proof of the Main Theorem. Its proof will be given in Section 4.

Theorem 3.6. *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, and $A_j \subset D_j$ be locally pluriregular, $j = 1,\dots,N$. For $\alpha \in \mathcal{Y}_k^N$ let Σ_α be a pluripolar subset of A_0^α. Let $\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$, $M \subset c(\mathbf{W}_{N,k})$ and $\mathcal{F}$ a collection of functions $f : c(\mathbf{W}_{N,k}) \setminus M \to \mathbb{C}\}$ such that:*

(T1) *M is pluripolar* ([4]),
(T2) *for any $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar,*
(T3) *for any $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ there exists a closed pluripolar set $\widetilde{M}_{a,\alpha} \subset D_1^\alpha$ such that $\widetilde{M}_{a,\alpha} \cap A_1^\alpha \subset M_{a,\alpha}$* ([5]),
(T4) *for any $a \in c(\mathbf{W}_{N,k}) \setminus M$ there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ with $f_a = f$ on $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$,*
(T5) *for any $f \in \mathcal{F}$, $\alpha \in \mathcal{Y}_k^N$, and $a \in A_0^\alpha \setminus \Sigma_\alpha$ there exists an $\widetilde{f}_{a,\alpha} \in \mathcal{O}(D_1^\alpha \setminus \widetilde{M}_{a,\alpha})$ such that $\widetilde{f}_{a,\alpha} = f_{a,\alpha}$ on $A_1^\alpha \setminus M_{a,\alpha}$* ([6]).

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ such that:

- *$\widehat{M} \cap c(\mathbf{W}_{N,k}) \subset M$,*
- *for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $c(\mathbf{W}_{N,k}) \setminus M$,*
- *$\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$,*

([4]) Actually we can assume less: M is such that the set $\{a_j \in A_j : M_{(\cdot,a_j,\cdot)}$ is not pluripolar$\}$ is pluripolar for all $j \in \{1,\dots,N\}$.

([5]) When $k = N$ we assume that there exists an $\widetilde{M} \subset D_1 \times \dots \times D_N$ closed pluripolar such that $\widetilde{M} \cap c(\mathbf{W}_{N,k}) \subset M$.

([6]) When $k = N$ we assume that there exists an $\widetilde{f} \in \mathcal{O}(D_1 \times \dots \times D_N \setminus \widetilde{M})$ such that $\widetilde{f} = f$ on $c(\mathbf{W}_{N,k}) \setminus M$.

- *if for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ we have $\widetilde{M}_{a,\alpha} = \emptyset$, then $\widehat{M} = \emptyset$,*
- *if for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the set $\widetilde{M}_{a,\alpha}$ is thin in D_1^α, then $\widehat{M}$ is analytic in $\widehat{\mathbf{X}}_{N,k}$.*

Theorem 3.6 has one immediate and useful consequence, which might be called the main extension theorem for generalized (N,k)-crosses with pluripolar singularities (see analogous theorem for N-fold crosses, Theorem 10.2.6 of [JarPfl 2011]).

Proposition 3.7. *Let D_j, A_j and Σ_α be as in Theorem 3.6. Let*

$$\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}.$$

Let $M \subset \mathbf{W}_{N,k}$ and $\mathcal{F} \subset \mathcal{O}_{\mathcal{S}}(\mathbf{W}_{N,k} \setminus M)$ be such that:

(P1) *$M \cap c(\mathbf{W}_{N,k})$ is pluripolar,*
(P2) *for any $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar and relatively closed in D_1^α,*
(P3) *for any $a \in c(\mathbf{W}_{N,k}) \setminus M$ there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ with $f_a = f$ on $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ such that:

- *$\widehat{M} \cap c(\mathbf{W}_{N,k}) \subset M$,*
- *for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $c(\mathbf{W}_{N,k}) \setminus M$,*
- *$\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$,*
- *if for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ we have $M_{a,\alpha} = \emptyset$, then $\widehat{M} = \emptyset$,*
- *if for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is thin in D_1^α, then $\widehat{M}$ is analytic in $\widehat{\mathbf{X}}_{N,k}$.*

Proof. Define $M' := M \cap c(\mathbf{W}_{N,k})$ and $\mathcal{F} := \{f|_{c(\mathbf{W}_{N,k}) \setminus M} : f \in \mathcal{F}\}$. We show that they satisfy the assumptions of Theorem 3.6.

Indeed, for $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_a$ define $\widetilde{M}_{a,\alpha} := M_{a,\alpha}$ and $\widetilde{f}_{a,\alpha} := f_{a,\alpha}$. Then:

- M' is pluripolar and for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_a$ the fibers $M'_{a,\alpha}$ are pluripolar by (P1),
- for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_a$, the set $\widetilde{M}_{a,\alpha}$ is relatively closed and pluripolar,
- for all $f \in \mathcal{F}$, $\alpha \in \mathcal{Y}_k^N$, and $a \in A_0^\alpha \setminus \Sigma_a$, the function $\widetilde{f}_{a,\alpha}$ is holomorphic on $D_1^\alpha \setminus \widetilde{M}_{a,\alpha}$ (cf. (P2), Definitions 1.6, 1.7 and Remark 1.8),
- from (P3), for any $a \in c(\mathbf{W}_{N,k}) \setminus M$ there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ with $f_a = f$ on $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$.

Thus from Theorem 3.6 we get the conclusion. ∎

As we have already mentioned in Section 2, Theorem 3.6 or, to be more precise, Proposition 3.7 implies the Main Theorem. The idea of the proof is based on Lemmas 10.2.5, 10.2.7, and 10.2.8 of [JarPfl 2011].

Proof that Proposition 3.7 implies Main Theorem. Let D_j, A_j, Σ_α, $\mathbf{X}_{N,k}$, $\mathbf{T}_{N,k}$, M, and $\mathcal{F}$ be as in Theorem 1.2. We have to check the assumptions of Proposition 3.7. Because M is pluripolar, for all $\alpha \in \mathcal{Y}_k^N$ there exists a pluripolar set Σ_α such that for all $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar. Moreover, because M is relatively closed, all the fibers $M_{a,\alpha}$ are relatively closed. To check the last assumption we need the following lemma.

LEMMA 3.8. *Under the assumptions of Theorem* 1.2, *for every $a \in c(\mathbf{T}_{N,k}) \setminus M$ there exists an $r > 0$ such that for any $f \in \mathcal{F}$ there exists an $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ with $f_a = f$ on $\mathbb{P}(a,r) \cap (c(\mathbf{T}_{N,k}) \setminus M)$.*

Proof. Fix $a \in c(\mathbf{T}_{N,k}) \setminus M$. Let $\rho > 0$ be such that $\mathbb{P}(a,\rho) \cap M = \emptyset$ (7). Define new crosses

$$\mathbf{X}_{N,k}^{a,\rho} := \mathbb{X}_{N,k}((A_j \cap \mathbb{P}(a_j,\rho), \mathbb{P}(a_j,\rho))_{j=1}^N) \ (^8),$$
$$\mathbf{T}_{N,k}^{a,\rho} := \mathbb{T}_{N,k}((A_j \cap \mathbb{P}(a_j,\rho), \mathbb{P}(a_j,\rho))_{j=1}^N, (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho))_{\alpha\in\mathcal{T}_k^N}).$$

Fix $\alpha \in \mathcal{T}_k^N$ and $a \in (\prod_{j:\,\alpha_j=0}(A_j \cap \mathbb{P}(a_j,\rho))) \setminus (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho))$. Then

$$\Big(\prod_{j:\,\alpha_j=1} \mathbb{P}(a_j,\rho)\Big) \setminus M_{a,\alpha} = \prod_{j:\,\alpha_j=1} \mathbb{P}(a_j,\rho),$$

so for any $f \in \mathcal{F}$ the function $\prod_{j:\,\alpha_j=1}\mathbb{P}(a_j,\rho) \ni z \mapsto f_{a,\alpha}(z)$ is holomorphic and $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k}^{a,\rho})$. For $\mathcal{F} = \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$ we additionally fix $b \in \prod_{j:\,\alpha_j=1}\mathbb{P}(a_j,\rho)$. We have

$$\Big(\prod_{j:\,\alpha_j=1} \mathbb{P}(a_j,\rho)\Big)\setminus((\Sigma_\alpha\cap\mathbb{P}(a_\alpha,\rho))\cup M_{b,\alpha}) = \Big(\prod_{j:\,\alpha_j=1} \mathbb{P}(a_j,\rho)\Big)\setminus(\Sigma_\alpha\cap\mathbb{P}(a_\alpha,\rho))$$

and for any $f \in \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$ the function $(\prod_{j:\,\alpha_j=1}\mathbb{P}(a_j,\rho)) \setminus (\Sigma_\alpha \cap \mathbb{P}(a_\alpha,\rho)) \ni z \mapsto f_{b,\alpha}(z)$ is continuous. Thus $\mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M) \subset \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k}^{a,\rho})$. Using Theorem 3.4 for $\mathcal{F} = \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k} \setminus M)$ and Theorem 3.5 for $\mathcal{F} = \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$, we get

$$\forall f \in \mathcal{F}\ \exists \widehat{f}_a \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k}^{a,\rho}): \quad \widehat{f}_a = f \text{ on } \mathbf{T}_{N,k}^{a,\rho}\ (^9).$$

Choosing $r \in (0,\rho)$ so small that $\mathbb{P}(a,r) \subset \widehat{\mathbf{X}}_{N,k}^{a,\rho}$ finishes the proof. ■

(7) Recall that M is relatively closed.

(8) From the definition of a polydisc in a Riemann domain we obviously have $\mathbb{P}(a_j,\rho) \subset D_j$, $j = 1,\dots,N$.

(9) Recall that if $\Sigma_\alpha = \emptyset$ for all $\alpha \in \mathcal{T}_k^N$, then $\mathbf{T}_{N,k} = \mathbf{X}_{N,k}$ and $\mathbf{T}_{N,k}^{a,\rho} = \mathbf{X}_{N,k}^{a,\rho}$.

Now, it is clear that all necessary assumptions are satisfied and we can apply Proposition 3.7. We obtain a pluripolar relatively closed set $\widehat{M}$ such that for all $f \in \mathcal{F}$ there exists an $\widehat{f}$ with $\widehat{f} = f$ on $c(\mathbf{T}_{N,k}) \setminus M$ and $\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$.

Fix $\alpha \in \mathcal{T}_k^N$ and define $D_\alpha := D_0^\alpha$ and $G_\alpha := D_1^\alpha$. Then both D_α and G_α are Riemann domains and $\widehat{\mathbf{X}}_{N,k} \subset \overleftrightarrow{D_\alpha \times G_\alpha}$ is a Riemann domain of holomorphy. From Proposition 9.1.4 of [JarPfl 2011] there exists a pluripolar set $\Sigma'_\alpha \subset A_0^\alpha$ such that $\Sigma_\alpha \subset \Sigma'_\alpha$ and for all $a \in A_0^\alpha \setminus \Sigma'_\alpha$ the fiber $\widehat{M}_{a,\alpha}$ is singular with respect to $\{\widehat{f}_{a,\alpha} : f \in \mathcal{F}\}$. In particular, because every $\widehat{f}_{a,\alpha}$ is holomorphic on $(\widehat{\mathbf{X}}_{N,k})_{a,\alpha} \setminus \widehat{M}_{a,\alpha}$, we have $\widehat{M}_{a,\alpha} \subset M_{a,\alpha}$ for $a \in A_0^\alpha \setminus \Sigma'_\alpha$. Hence

$$\widehat{M} \cap \mathbf{T}'_{N,k} = \bigcup_{\alpha \in \mathcal{T}_k^N} \{z \in \widehat{M} \cap \mathcal{X}_\alpha : z_\alpha \notin \Sigma'_\alpha\} \subset M.$$

Now for all $\alpha \in \mathcal{T}_k^N$ and $a \in A_0^\alpha \setminus \Sigma'_\alpha$ the functions $\widehat{f}_{a,\alpha}$ and $f_{a,\alpha}$ are holomorphic on $D_1^\alpha \setminus M_{a,\alpha}$ (thanks to the inclusion $\widehat{M} \cap \mathbf{T}'_{N,k} \subset M$) and equal on $A_1^\alpha \setminus M_{a,\alpha}$, which is not pluripolar. Thus we have $\widehat{f}_{a,\alpha} = f_{a,\alpha}$ everywhere on $D_1^\alpha \setminus M_{a,\alpha}$, for every α and a. Hence finally $\widehat{f} = f$ on $\mathbf{T}'_{N,k} \setminus M$. ∎

4. Proof of Theorem 3.6. First we show that it is sufficient to prove Theorem 3.6 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.

LEMMA 4.1. *Theorem* 3.6 *with* $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$ *implies Theorem* 3.6 *with*

$$\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}.$$

Proof. Let D_j, A_j, Σ_α, $\mathbf{X}_{N,k}$, $\mathbf{T}_{N,k}$, $\mathbf{Y}_{N,k}$, $M \subset c(\mathbf{W}_{N,k})$ and a collection $\mathcal{F}$ of functions $f : \mathbf{W}_{N,k} \setminus M \to \mathbb{C}$, where $\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$, be as in Theorem 3.6. Assume that this theorem is true with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.

Observe that $c(\mathbf{Y}_{N,k}) = c(\mathbf{X}_{N,k}) \setminus \Delta$ and $c(\mathbf{T}_{N,k}) = c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}$, where

$$\Delta := \bigcap_{\alpha \in \mathcal{T}_k^N} \{a \in A_1 \times \cdots \times A_N : a_\alpha \in \Sigma_\alpha\},$$

$$\widetilde{\Delta} := \bigcap_{\alpha \in \mathcal{Y}_k^N} \{a \in A_1 \times \cdots \times A_N : a_\alpha \in \Sigma_\alpha\},$$

are pluripolar subsets of $c(\mathbf{X}_{N,k})$, where a_α denotes the projection of a on A_0^α.

Define $M' := M \cup \widetilde{\Delta} \subset c(\mathbf{X}_{N,k})$. Then $c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}) \setminus M = c(\mathbf{X}_{N,k}) \setminus M'$ and

$$(*) \qquad c(\mathbf{T}_{N,k}) \setminus M = (c(\mathbf{X}_{N,k}) \setminus \Delta) \setminus M \subset (c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}) \setminus M \quad \text{for } M \subset c(\mathbf{T}_{N,k}),$$

$$(**) \qquad c(\mathbf{Y}_{N,k}) \setminus M = (c(\mathbf{X}_{N,k}) \setminus \widetilde{\Delta}) \setminus M \quad \text{for } M \subset c(\mathbf{Y}_{N,k}).$$

Define $\mathcal{F}' := \{f|_{c(\mathbf{X}_{N,k})\setminus M'} : f \in \mathcal{F}\}$. Then M' is pluripolar and for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ we have $\widetilde{\Delta}_{a,\alpha} = \emptyset$, so $M'_{a,\alpha} = M_{a,\alpha}$. Thus M' and $\mathcal{F}'$ satisfy the assumptions of Theorem 3.6 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$. Then there exists $\widehat{M'} \subset \widehat{\mathbf{X}}_{N,k}$, relatively closed, pluripolar, and having all the properties of the conclusion. Properties $(*)$ and $(**)$ give us the conclusion for $\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$. ∎

Proof of Theorem 3.6 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.

Step 1. Theorem 3.6 is true for any N when $k = 1$ (Theorem 2.2) and when $k = N$ (by assumption).

Step 2. In particular, the theorem is true for $N = 2$, $k = 1, 2$. Assume we already have Theorem 3.6 for $(N-1, k)$, where $k \in \{1, \dots, N-1\}$, and for $(N, 1), \dots, (N, k-1)$, where $k \in \{2, \dots, N-1\}$. We need to prove it for (N, k).

Step 3. Fix $s \in \{1, \dots, N\}$ (to simplify notation let $s = N$). Let

$$Q_N := \{a_N \in A_N : M_{(\cdot, a_N)} \text{ is not pluripolar}\}.$$

Then Q_N is pluripolar. Define

$$\mathbf{X}_{N-1,k}^{(s)} := \mathbb{X}_{N-1,k}((A_j, D_j)_{j=1, j\neq s}^N), \qquad s = 1, \dots, N,$$

in particular

$$\mathbf{X}_{N-1,k}^{(N)} = \mathbf{X}_{N-1,k} := \mathbb{X}_{N-1,k}((A_j, D_j)_{j=1}^{N-1}).$$

Fix an $a_N \in A_N \setminus Q_N$ and consider the family $\{f(\cdot, a_N) : f \in \mathcal{F}\}$ of functions $f : c(\mathbf{X}_{N-1,k}) \to \mathbb{C}$. Then:

- $M_{(\cdot, a_N)} \subset c(\mathbf{X}_{N-1,k})$ is pluripolar.
- For any $\alpha' \in \mathcal{Y}_k^{N-1}$ and any $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$ ([10]) the fiber $(M_{(\cdot, a_N)})_{a', \alpha'}$ equals $M_{a,\alpha}$, where $a = (a', a_N)$ and $\alpha = (\alpha', 0)$, so it is pluripolar.
- For $\alpha' \in \mathcal{Y}_k^{N-1}$ and $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$ we define $\widetilde{M}_{a', \alpha'} := \widetilde{M}_{a,\alpha}$, where $a = (a', a_N)$ and $\alpha = (\alpha', 0)$. Then $\widetilde{M}_{a',\alpha'} \subset D_1^\alpha = D_1^{\alpha'}$ is closed, pluripolar and $\widetilde{M}_{a',\alpha'} \cap A_1^{\alpha'} \subset M_{a',\alpha'}$.
- For any $a' \in c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot, a_N)}$ there exists an $r > 0$ (the same as for $a = (a', a_N)$) such that for any $f \in \mathcal{F}$ there exists $f_{a'} \in \mathcal{O}(\mathbb{P}(a', r))$ such that $f_{a'} = f(\cdot, a_N)$ on $\mathbb{P}(a', r) \cap (c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot, a_N)})$.
- For $f \in \mathcal{F}$, for any $\alpha' \in \mathcal{Y}_k^{N-1}$ and any $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$, define $\widetilde{f}_{a',\alpha'} := \widetilde{f}_{a,\alpha} \in \mathcal{O}(D_1^\alpha \setminus \widetilde{M}_{a,\alpha}) = \mathcal{O}(D_1^{\alpha'} \setminus \widetilde{M}_{a',\alpha'})$, where $a = (a', a_N)$ and $\alpha = (\alpha', 0)$. Then $\widetilde{f}_{a',\alpha'} = f_{a',\alpha'}$ on $A_1^{\alpha'} \setminus (M_{(\cdot, a_N)})_{a',\alpha'}$.

([10]) By $A_0^{\alpha'}$ we denote the product $\prod_{j\in\{1,\dots,N-1\}:\, \alpha'_j = 0} A_j$, and analogously for $A_1^{\alpha'}$ and $D_1^{\alpha'}$.

From the inductive assumption we get a relatively closed pluripolar set $\widehat{M}_{a_N} \subset \widehat{\mathbf{X}}_{N-1,k}$ such that:

- $\widehat{M}_{a_N} \cap c(\mathbf{X}_{N-1,k}) \subset M_{(\cdot,a_N)}$,
- for any $f \in \mathcal{F}$ there exists an $\widehat{f}_{a_N} \in \mathcal{O}(\widehat{\mathbf{X}}_{N-1,k} \setminus \widehat{M}_{a_N})$ such that $\widehat{f}_{a_N} = f(\cdot, a_N)$ on $c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot,a_N)}$,
- $\widehat{M}_{a_N}$ is singular with respect to $\{\widehat{f}_{a_N} : f \in \mathcal{F}\}$,
- if for all $\alpha' \in \mathcal{Y}_k^{N-1}$ and $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$, we have $\widetilde{M}_{a',\alpha'} = \emptyset$, then $\widehat{M}_{a_N} = \emptyset$,
- if for all $\alpha' \in \mathcal{Y}_k^{N-1}$ and $a' \in A_0^{\alpha'} \setminus \Sigma_{\alpha'}$, the set $\widetilde{M}_{a',\alpha'}$ is thin in $D_1^{\alpha'}$, then $\widehat{M}_{a_N}$ is analytic in $\widehat{\mathbf{X}}_{N-1,k}$.

Define a new cross

$$\mathbf{Z}_N := \mathbb{X}(c(\mathbf{X}_{N-1,k}), A_N; \widehat{\mathbf{X}}_{N-1,k}, D_N).$$

Observe that $\mathbf{Z}_N$ with original M, $\Sigma_{(0,1)} := \Sigma_{(0,\dots,0,1)}$, $\Sigma_{(1,0)} := Q_N$, and the family $\mathcal{F}$ satisfy all the assumptions of Theorem 3.6 with $N = 2$, $k = 1$. Indeed:

- For all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ and $a_N \in A_N \setminus Q_N$ the fibers $M_{(a',\cdot)}$, $M_{(\cdot,a_N)}$ are pluripolar by (T1), (T2) and the definition of Q_N.
- For all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ from (T3) there exists an $\widetilde{M}_{a'} \subset D_N$ closed pluripolar such that $\widetilde{M}_{a'} \cap A_N \subset M_{(a',\cdot)}$. For $a_N \in A_N \setminus Q_N$ set $\widetilde{M}_{a_N} := \widehat{M}_{a_N}$. Then $\widetilde{M}_{a_N}$ is closed pluripolar in $\widehat{\mathbf{X}}_{N-1,k}$ and $\widetilde{M}_{a_N} \cap c(\mathbf{X}_{N-1,k}) \subset M_{(\cdot,a_N)}$.
- For all $(a', a_N) \in (c(\mathbf{X}_{N-1,k}) \times A_N) \setminus M$ from (T4) there exists an $r > 0$ such that for all $f \in \mathcal{F}$ there exists an $f_{(a',a_N)} \in \mathcal{O}(\mathbb{P}((a', a_N), r))$ such that
$$f_{(a',a_N)} = f \quad \text{on } \mathbb{P}((a', a_N), r) \cap (c(\mathbf{X}_{N,k} \setminus M)).$$
- For all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{\alpha=(0,\dots,0,1)}$ from (T5) there exists an $f_{a'} \in \mathcal{O}(D_N \setminus \widetilde{M}_{a'})$ such that $f_{a'} = f$ on $A_N \setminus M_{(a',\cdot)}$. For $a_N \in A_N \setminus Q_N$ define $f_{a_N} := \widehat{f}_{a_N}$. Then $f_{a_N} \in \mathcal{O}(\widehat{\mathbf{X}}_{N-1,k} \setminus \widetilde{M}_{a_N})$ and $f_{a_N} = f$ on $c(\mathbf{X}_{N-1,k}) \setminus M_{(\cdot,a_N)}$.

Then there exists an $\widehat{M}_N \subset \widehat{\mathbf{Z}}_N$ relatively closed pluripolar such that:

- $\widehat{M}_N \cap c(\mathbf{X}_{N,k}) \subset M$,
- for any $f \in \mathcal{F}$ there exists an $\widehat{f}_N \in \mathcal{O}(\widehat{\mathbf{Z}}_N \setminus \widehat{M}_N)$ such that $\widehat{f}_N = f$ on $c(\mathbf{X}_{N,k}) \setminus M$,
- $\widehat{M}_N$ is singular with respect to $\{\widehat{f}_N : f \in \mathcal{F}\}$,
- if for all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ we have $\widetilde{M}_{a'} = \emptyset$ and for all $a_N \in A_N \setminus Q_N$ we have $\widetilde{M}_{a_N} = \emptyset$, then $\widehat{M}_N = \emptyset$,

- if for all $a' \in c(\mathbf{X}_{N-1,k}) \setminus \Sigma_{(0,\dots,0,1)}$ the set $\widetilde{M}_{a'}$ is thin in D_N and for all $a_N \in A_N \setminus Q_N$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$, then $\widehat{M}_N$ is analytic in $\widehat{\mathbf{Z}}_N$.

We repeat the reasoning above for all $s = 1, \dots, N-1$, obtaining a family $\{\widehat{f}_s\}_{s=1}^N$ of functions such that for any $s \in \{1, \dots, N\}$ we have $\widehat{f}_s = f$ on $c(\mathbf{X}_{N,k}) \setminus M$. Define a new function by

$$F_f(z) := \begin{cases} \widehat{f}_1(z) & \text{for } z \in \widehat{\mathbf{Z}}_1 \setminus \widehat{M}_1, \\ \quad\vdots & \\ \widehat{f}_N(z) & \text{for } z \in \widehat{\mathbf{Z}}_N \setminus \widehat{M}_N. \end{cases}$$

Assume for a moment that we have the following lemma.

Lemma 4.2. *The function F_f is well defined on $(\bigcup_{s=1}^N \mathbf{Z}_s) \setminus (\bigcup_{s=1}^N \widehat{M}_s)$.*

Step 4. Define a 2-fold cross

$$\mathbf{Z} := \mathbb{X}(\mathbf{X}_{N-1,k-1}, A_N; \widehat{\mathbf{X}}_{N-1,k}, D_N) \subset \bigcup_{s=1}^N \mathbf{Z}_s,$$

a pluripolar set

$$\widetilde{M} := \Big(\bigcup_{s=1}^N \widehat{M}_s\Big) \cap (\mathbf{X}_{N-1,k-1} \times A_N)$$

and a family

$$\widetilde{\mathcal{F}} := \{\widetilde{f} := F_f|_{(\mathbf{X}_{N-1,k-1}\times A_N)\setminus\widetilde{M}} : f \in \mathcal{F}\}.$$

We show that $\mathbf{Z}$, $\widetilde{M}$, and $\widetilde{\mathcal{F}}$ satisfy the assumptions of Theorem 3.6 with $N = 1$ and $k = 1$:

- $\widetilde{M}$ is pluripolar in $\mathbf{X}_{N-1,k-1} \times A_N$, so there exist pluripolar sets $P \subset \mathbf{X}_{N-1,k-1}$, $Q \subset A_N$ such that for all $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ and $a_N \in A_N \setminus Q$, the fibers $\widetilde{M}_{(z',\cdot)}$, $\widetilde{M}_{(\cdot,a_N)}$ are pluripolar.
- Let $z' \in \mathbf{X}_{N-1,k-1} \setminus P$. Then there exists an $s \in \{1, \dots, N-1\}$ such that

$$(\star) \qquad \{z'\} \times D_N \subset \mathbf{X}_{N-1,k}^{(s)} \times A_s.$$

  Indeed, let $z' \in \mathbf{X}_{N-1,k-1}$. Then $z' = z'_\alpha$ for some $\alpha \in \{0,1\}^{N-1}$, $|\alpha| = k-1$, where $z'_\alpha = (z_{\alpha_1}, \dots, z_{\alpha_{N-1}})$ and $z_{\alpha_j} = a_j \in A_j$ when $\alpha_j = 0$, while $z_{\alpha_j} = z_j \in D_j$ otherwise. We may assume that $z' = (z_1, \dots, z_{k-1}, a_k, \dots, a_{N-1})$. Set $s = k$. Fix $z_N \in D_N$. Then $(z_1, \dots, z_{k-1}, a_k, \dots, a_{N-1}, z_N) \in \{z'\} \times D_N$ and

$$(z_1, \dots, z_{k-1}, a_k, \dots, a_{N-1}, z_N) \in \mathbf{X}_{N-1,k}^{(s)} \times A_s.$$

Define $\widetilde{M}_{z'} := (\widehat{M}_s)_{(z',\cdot)}$. Then $\widetilde{M}_{z'}$ is pluripolar relatively closed in D_N and $\widetilde{M}_{z'} \cap A_N \subset \widetilde{M}_{(z',\cdot)}$. For $a_N \in A_N \setminus Q$ define $\widetilde{M}_{a_N} := (\widehat{M}_N)_{(\cdot,a_N)}$ relatively closed pluripolar in $\widehat{\mathbf{X}}_{N-1,k}$ such that $\widetilde{M}_{a_N} \cap \mathbf{X}_{N-1,k-1} \subset \widetilde{M}_{(\cdot,a_N)}$.

- For any $(z', a_N) \in (\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M}$ there exist $s \in \{1, \dots, N-1\}$ and $r > 0$ such that $\mathbb{P}((z', a_N), r) \subset \widehat{\mathbf{Z}}_s \setminus \widehat{M}_s$. Then $\widehat{f}_s \in \mathcal{O}(P((z', a_N), r))$ and $\widehat{f}_s = F_f = \widetilde{f}$ on $P((z', a_N), r) \cap ((\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M})$.
- For $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ choose an s to have ($\star$) and define $\widetilde{f}_{z'} := \widehat{f}_s(z', \cdot)$. Then $\widetilde{f}_{z'}$ is holomorphic on $D_N \setminus \widetilde{M}_{z'}$ and equals $\widetilde{f}(z', \cdot)$ on $A_N \setminus \widetilde{M}_{(z',\cdot)}$. For $a_N \in A_N \setminus Q$ define $\widetilde{f}_{a_N} := \widehat{f}_s(\cdot, a_N)$. Then $\widetilde{f}_{a_N}$ is holomorphic on $\widehat{\mathbf{X}}_{N-1,k} \setminus \widetilde{M}_{a_N}$ and equals $\widetilde{f}(\cdot, a_N)$ on $\mathbf{X}_{N-1,k-1} \setminus \widetilde{M}_{(\cdot,a_N)}$.

Now from Theorem 3.6 there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\mathbf{Z}}$ such that:

- $\widehat{M} \cap (\mathbf{X}_{N-1,k-1} \times A_N) \subset \widetilde{M}$, in particular, $\widehat{M} \cap c(\mathbf{X}_{N,k}) \subset M$,
- for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{Z}} \setminus \widehat{M})$ such that $\widehat{f} = \widetilde{f}$ on $(\mathbf{X}_{N-1,k-1} \times A_N) \setminus \widetilde{M}$, in particular $\widehat{f} = f$ on $c(\mathbf{X}_{N,k}) \setminus M$,
- $\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$,
- if for all $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ we have $\widetilde{M}_{z'} = \emptyset$ and for all $a_N \in A_N \setminus Q$, $\widetilde{M}_{a_N} = \emptyset$, then $\widehat{M} = \emptyset$,
- if for all $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ the set $\widetilde{M}_{z'}$ is thin in D_N and for all $a_N \in A_N \setminus Q$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$, then $\widehat{M}$ is analytic in $\widehat{\mathbf{Z}}$.

Now assume that for any $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ we have $\widetilde{M}_{a,\alpha} = \emptyset$. Then for any $s \in \{1, \dots, N\}$ and $a_s \in A_s \setminus Q_s$ we have $\widehat{M}_{a_s} = \emptyset$, which implies that for all $s \in \{1, \dots, N\}$ we have $\widehat{M}_s = \emptyset$. Then from the definitions of $\widetilde{M}_{z'}$ and $\widetilde{M}_{a_N}$ we see that for any $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ we have $\widetilde{M}_{z'} = \emptyset$ and for all $a_N \in A_N \setminus Q$ we have $\widetilde{M}_{a_N} = \emptyset$, thus $\widehat{M} = \emptyset$.

Analogously if for all $\alpha \in \mathcal{Y}_k^N$ and $a \in A_0^\alpha \setminus \Sigma_\alpha$ the fiber $\widetilde{M}_{a,\alpha}$ is thin in D_1^α, then for any $s \in \{1, \dots, N\}$ and $a_s \in A_s \setminus Q_s$ the set $\widehat{M}_{a_s}$ is analytic (thus thin) in $\widehat{\mathbf{X}}_{N-1,k}^{(s)}$, so for all $s \in \{1, \dots, N\}$ the set $\widehat{M}_s$ is analytic in $\widehat{\mathbf{Z}}_s$. Because fibers of analytic sets are also analytic we infer that for any $z' \in \mathbf{X}_{N-1,k-1} \setminus P$ the set $\widetilde{M}_{z'}$ is thin in D_N and for any $a_N \in A_N \setminus Q$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$. Thus, finally, $\widehat{M}$ is analytic in $\widehat{\mathbf{Z}}$.

Now we show that $\widehat{\mathbf{X}}_{N,k} \subset \widehat{\mathbf{Z}}$. First observe that if $z = (z', z_N) \in \widehat{\mathbf{X}}_{N,k}$, then $z' \in \widehat{\mathbf{X}}_{N-1,k}$. From Lemma 3.3 for $(z_1, \dots, z_N) = (z', z_N) \in \widehat{\mathbf{X}}_{N,k}$ we get

$$(\ddagger) \quad \mathbf{h}_{\mathbf{X}_{N-1,k-1},\widehat{\mathbf{X}}_{N-1,k}}(z') + \mathbf{h}_{A_N,D_N}(z_N) = \mathbf{h}_{\widehat{\mathbf{X}}_{N-1,k-1}}(z') + \mathbf{h}_{A_N,D_N}(z_N).$$

For $z \in \widehat{\mathbf{X}}_{N-1,k-1} \subset \widehat{\mathbf{X}}_{N,k}$ we have $(\ddagger) = \mathbf{h}_{A_N,D_N}(z_N)$, which is less than 1 from properties of the relative extremal function, and for $z \in \widehat{\mathbf{X}}_{N,k} \setminus \widehat{\mathbf{X}}_{N-1,k-1}$ we use Lemma 3.2:

$$(\ddagger) = \Big(\sum_{j=1}^{N-1} \mathbf{h}_{A_j,D_j}(z_j)\Big) - k + 1 + \mathbf{h}_{A_N,D_N}(z_N) < k - k + 1 = 1.$$

To show the opposite inclusion take $(z_1, \dots, z_N) = (z', z_N) \in \widehat{\mathbf{Z}}$. From properties of the relative extremal function and Lemma 3.2 we get

$$\begin{aligned}\Big(\sum_{j=1}^{N-1} \mathbf{h}_{A_j,D_j}(z_j)\Big) + \mathbf{h}_{A_N,D_N}(z_N) &\le \mathbf{h}_{\widehat{\mathbf{X}}_{N-1,k-1}}(z') + k - 1 + \mathbf{h}_{A_N,D_N}(z_N)\\ &\le \mathbf{h}_{\mathbf{X}_{N-1,k-1},\widehat{\mathbf{X}}_{N-1,k}}(z') + \mathbf{h}_{A_N,D_N}(z_N) + k - 1 < 1 + k - 1 = k.\end{aligned}$$

Thus, it remains to prove Lemma 4.2:

Proof of Lemma 4.2. Fix s and p. We want to show that $\widehat{f_s} = \widehat{f_p}$ on $(\mathbf{Z}_s \cap \mathbf{Z}_p) \setminus (\widehat{M_s} \cup \widehat{M_p})$. To simplify notation we assume that $s = N - 1$ and $p = N$.

STEP 1. Every connected component of $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$ contains part of the center.

From the definition of $\mathbf{Z}_{N-1}$ and $\mathbf{Z}_N$ we have

$$\begin{aligned}\mathbf{Z}_{N-1} \cap \mathbf{Z}_N = (A_1 \times \dots \times A_{N-2} \times D_{N-1} \times A_N) \cup (A_1 \times \dots \times A_{N-1} \times D_N)\\ \cup (\widehat{\mathbf{X}}_{N-2,k} \times A_{N-1} \times A_N).\end{aligned}$$

First take $B_1 := A_1 \times \dots \times A_{N-2} \times A_{N-1} \times D_N$. Since the product of a disconnected set with any set is not connected, connected components of B_1 are products of connected components of A_j, $j = 1, \dots, N-1$, and D_N. Since the last set is connected, every connected component of B_1 "contains" D_N (in the sense of the last coordinate in the product), thus it contains a part of the center $A_1 \times \dots \times A_N$.

The case of $B_2 := A_1 \times \dots \times A_{N-2} \times D_{N-1} \times A_N$ is similar.

Now take $B_3 := \widehat{\mathbf{X}}_{N-2,k} \times A_{N-1} \times A_N$. As in the previous cases, since $\widehat{\mathbf{X}}_{N-2,k}$ is connected, every connected component of B_2 "contains" the whole $\widehat{\mathbf{X}}_{N-2,k}$ in the product. Since $\widehat{\mathbf{X}}_{N-2,k}$ contains $A_1 \times \dots \times A_{N-2}$, every connected component of B_2 must contain part of the center.

STEP 2. One connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ contains the whole $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$.

The intersection $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ contains the cross $\mathbf{X}_{N,1}$, which is connected and contains the center. Thus the whole center must lie in one connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$. Now take any connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ which intersects $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$. From Step 1 it must contain part of the center, so there is only one connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ intersecting (thus containing) $\mathbf{Z}_{N-1} \cap \mathbf{Z}_N$.

STEP 3. Every connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$ with $\widehat{M}_{N-1} \cup \widehat{M}_N$ deleted is a domain, thus it is a connected component of $(\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$.

Take any connected component of $\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N$, name it Ω. Then Ω is a domain. The set $\widehat{M}_{N-1}$ is pluripolar and relatively closed in $\widehat{\mathbf{Z}}_{N-1}$, thus it is pluripolar and relatively closed in Ω, so $\Omega \setminus \widehat{M}_{N-1}$ is still a domain. Because $\widehat{M}_N$ is relatively closed and pluripolar in $\widehat{\mathbf{Z}}_N$, it is relatively closed and pluripolar in $\Omega \setminus \widehat{M}_{N-1}$. So $\Omega \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$ is a domain.

STEP 4. One connected component of $(\widehat{\mathbf{Z}}_{N-1} \cap \widehat{\mathbf{Z}}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$ contains the whole set $(\mathbf{Z}_{N-1} \cap \mathbf{Z}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$.

This follows immediately from Steps 2 and 3.

STEP 5. $\widehat{f}_{N-1} = \widehat{f}_N$ on $(\mathbf{Z}_{N-1} \cap \mathbf{Z}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$.

Let Ω be a connected component from Step 4. Then both $\widehat{f}_{N-1}$ and $\widehat{f}_N$ are defined on Ω. On the non-pluripolar center we have $\widehat{f}_{N-1} = \widehat{f}_N$. Since Ω is a domain and contains the center, $\widehat{f}_{N-1} = \widehat{f}_N$ on Ω. Moreover, Ω contains $(\mathbf{Z}_{N-1} \cap \mathbf{Z}_N) \setminus (\widehat{M}_{N-1} \cup \widehat{M}_N)$, which finishes the proof. ■

The proof of Theorem 3.6 is finished. ■

EXAMPLE 4.3. In the proof of Theorem 3.6 with $k = 1$ we do not need the cross $\widehat{\mathbf{Z}}$—it is sufficient to take $\widehat{\mathbf{Z}}_N$ (see [JarPfl 2010] for details), however for $k > 1$ Step 4 is necessary. Indeed, let $A_1 = A_2 = A_3 = (-1, 1)$, $D_1 = D_2 = D_3 = \mathbb{D}$, $\mathbf{X}_{3,2} := \mathbb{X}_{3,2}((A_j, D_j)_{j=1}^3)$, $\mathbf{Z}_3 := \mathbb{X}(A_1 \times A_2, A_3; \widehat{\mathbf{X}}_{2,2}, D_3)$. Then $\widehat{\mathbf{X}}_{2,2} = D_1 \times D_2$,

$$\begin{aligned}\widehat{\mathbf{Z}}_3 &:= \{z \in D_1 \times D_2 \times D_3 : \mathbf{h}_{A_1 \times A_2, D_1 \times D_2}(z_1, z_2) + \mathbf{h}_{A_3, D_3}(z_3) < 1\} \\ &= \{z \in D_1 \times D_2 \times D_3 : \max\{h_{A_j, D_j}(z_j),\ j = 1, 2\} + \mathbf{h}_{A_3, D_3}(z_3) < 1\},\end{aligned}$$

and $\mathbf{h}_{A_j, D_j}(\zeta) = \frac{2}{\pi}\left|\operatorname{Arg}\left(\frac{1+\zeta}{1-\zeta}\right)\right|$, $\zeta \in \mathbb{D}$, $j = 1, 2, 3$ (see Example 3.2.20(a) in [JarPfl 2011]). Take $z = (0, w, w)$, where $w = i/\sqrt{3}$. Then $z \in \mathbf{X}_{3,2}$ but $z \notin \widehat{\mathbf{Z}}_3$.

REFERENCES

[Har 1906] F. Hartogs, *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*, Math. Ann. 62 (1906), 1–88.

[Huk 1942] M. Hukuhara, *Extension of a theorem of Osgood and Hartogs*, Kansuhoteisiki ogobi Oyo-Kaiseki (1942), 48–49 (in Japanese).

[JarPfl 2000] M. Jarnicki and P. Pflug, *Extension of Holomorphic Functions*, de Gruyter Exp. Math. 34, de Gruyter, Berlin, 2000.

[JarPfl 2003] M. Jarnicki and P. Pflug, *An extension theorem for separately holomorphic functions with pluripolar singularities*, Trans. Amer. Math. Soc. 355 (2003), 1251–1267.

[JarPfl 2007] M. Jarnicki and P. Pflug, *A general cross theorem with singularities*, Analysis Munich 27 (2007), 181–212.

[JarPfl 2010] M. Jarnicki and P. Pflug, *A new cross theorem for separately holomorphic functions*, Proc. Amer. Math. Soc. 138 (2010), 3923–3932.

[JarPfl 2011] M. Jarnicki and P. Pflug, *Separately Analytic Functions*, EMS Publ. House, 2011.

[Lew 2012] A. Lewandowski, *An extension theorem with analytic singularities for generalized (N,k)-crosses*, Ann. Polon. Math. 103 (2012), 193–208.

[Osg 1899] W. F. Osgood, *Note über analytische Functionen mehrerer Veränderlichen*, Math. Ann. 52 (1899), 462–464.

[Ter 1967] T. Terada, *Sur une certaine condition sous laquelle une fonction de plusieurs variables complexes est holomorphe: Diminution de la condition dans le théorème de Hartogs*, Publ. Res. Inst. Math. Sci. Ser. A 2 (1967), 383–396.

Małgorzata Zajęcka
Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
and
Institute of Mathematics
Faculty of Physics, Mathematics and Computer Science
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: malgorzata.zajecka@gmail.com

Received 7 December 2011 (5589)