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RELATIVE BUCHSBAUMNESS OF BIGRADED MODULES

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Abstract. We study finitely generated bigraded Buchsbaum modules over a standard bigraded polynomial ring with respect to one of the irrelevant bigraded ideals. The regularity and the Hilbert function of graded components of local cohomology at the finiteness dimension level are considered.

Introduction. Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over a field K with bigraded irrelevant ideals P generated by all elements of degree $(1, 0)$, and Q generated by all elements of degree $(0, 1)$. In other words, $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let M be a finitely generated bigraded S-module. Let $K^\bullet(Q, M)$ be the Koszul complex of M with respect to Q and $H^i(Q, M)$ its ith cohomology module. Note that the local cohomology modules $H^i_Q(M)$ and $H^i(Q, M)$ are naturally bigraded.

We say that M is a relative Buchsbaum module with respect to Q if the canonical maps $\lambda^i_M : H^i(Q, M) \to H^i_Q(M)$ are surjective for all $i < \operatorname{cd}(Q, M)$ where $\operatorname{cd}(Q, M)$ denotes the cohomological dimension of M with respect to Q. Note that in general the map λ^i_M is neither injective nor surjective. We observe that ordinary graded Buchsbaum modules are special cases of our definition. In fact, if we assume $P = 0$, then $m = 0$, and $Q = \mathfrak{m}$ is the unique graded maximal ideal of S and $\operatorname{cd}(Q, M) = \operatorname{cd}(\mathfrak{m}, M) = \dim M$.

In the preliminary section, we observe that there is an equivalent definition of relative Buchsbaum modules in terms of Ext functors. In fact, M is relative Buchsbaum with respect to Q if and only if the canonical maps $\varphi^i_M : \operatorname{Ext}^i_S(S/Q, M) \to H^i_Q(M)$ are surjective for all $i < \operatorname{cd}(Q, M)$. This in particular implies the known result that M is Buchsbaum if and only if the canonical maps $\varphi^i_M : \operatorname{Ext}^i_{K[y]}(K, M) \to H^i_{\mathfrak{m}}(M)$ are surjective for all $i < \dim M$ (see [7, Corollary 2.16]). By using this new characterization of relative Buchsbaum modules, we have the following result: First, we set $K[x] = K[x_1, \dots, x_m]$, $K[y] = K[y_1, \dots, y_n]$ and $M = M_1 \otimes_K M_2$ where

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M_1 is a non-zero finitely generated graded $K[x]$-module and M_2 a non-zero finitely generated graded $K[y]$-module. Then M is relative Buchsbaum with respect to Q if and only if M_2 is Buchsbaum. Secondly, we show that if M is relative Buchsbaum with respect to Q with $\operatorname{grade}(Q, M) = 0$, then $M/H^0_Q(M)$ is relative Buchsbaum with respect to Q too.

In Section 2, we give a characterization of all relative Buchsbaum modules with two non-vanishing local cohomology modules. In fact, we prove the following: let M be a finitely generated bigraded S-module, $r = \operatorname{grade}(Q, M) < \operatorname{cd}(Q, M) = d$ and $H^i_Q(M) = 0$ for all $i \neq r, d$; then M is relative Buchsbaum with respect to Q if and only if $QH^r_Q(M) = 0$. This generalizes the following known result: if M is finitely generated graded $K[y]$-module, $r = \operatorname{depth} M < \dim M = d$ and $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq r, d$, then M is Buchsbaum if and only if $\mathfrak{m}H^r_{\mathfrak{m}}(M) = 0$ (see [7, Corollary 3.6]). By using these results we give several examples to show that one cannot expect any connections between Buchsbaumness and relative Buchsbaumness with respect to the irrelevant bigraded ideals P and Q.

In the final section, we consider the jth components of $H^i_Q(M)$ as a finitely generated graded $K[x]$-module with grading $(H^i_Q(M)_j)_k = H^i_Q(M)_{(k,j)}$. We improve [6, Proposition 2.3] as follows: let M be a finitely generated bigraded S-module such that $f_Q(M) = \operatorname{cd}(Q, M)$ where $f_Q(M)$ is the finiteness dimension of M with respect to Q; then for $j \ll 0$, we have $-c \leq \operatorname{reg} H^q_Q(M)_j \leq c$ for some c. In particular, if M is relative Buchsbaum with respect to Q, then the regularity of $H^i_Q(M)_j$ is bounded for all i and $j \ll 0$. Finally, we give an explicit formula for the Hilbert function of $H^q_Q(M)_j$ where $q = f_Q(M) = \operatorname{cd}(Q, M)$. As a consequence, the Krull dimension of $H^q_Q(M)_j$ is constant for $j \ll 0$. This is a well known result originally proved in [1]. Here we obtain it by a different method.

1. Preliminaries. Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over a field K with bigraded irrelevant ideals $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let M be a finitely generated bigraded S-module, and let $K^\bullet(Q, M)$ be the Koszul complex and $H^i(Q, M)$ its ith cohomology module. We denote by $\operatorname{cd}(Q, M)$ the *cohomological dimension* of M with respect to Q, i.e., the least integer i such that $H^j_Q(M) = 0$ for all $j > i$.

Definition 1.1. Let M be a finitely generated bigraded S-module. We say that M is *relative Buchsbaum* with respect to Q if the canonical maps

$$\lambda^i_M : H^i(Q, M) \to H^i_Q(M)$$

are surjective for all $i < \operatorname{cd}(Q, M)$.

By an ordinary Buchsbaum module we mean a relative Buchsbaum module with respect to the maximal ideal $\mathfrak{m} = P + Q$. We observe that ordinary

Buchsbaum modules are special cases of our definition. In fact, if we assume $P = 0$, then $m = 0$, and $Q = \mathfrak{m}$ is the unique graded maximal ideal of S with $\deg y_i = 1$ for $i = 1, \dots, m$ and $\operatorname{cd}(Q, M) = \operatorname{cd}(\mathfrak{m}, M) = \dim M$.

Note that, for $i = 0$ the map is indeed injective, simply because the isomorphism

$$H^0(Q, M) \cong H_n(Q, M) \cong \operatorname{Hom}(S/Q, M) \cong (0 :_M Q)$$

gives an embedding in $H^0_Q(M) = \{x \in M : xQ^k = 0 \text{ for some } k\}$. But in general the map λ^i_M is neither injective nor surjective. In the following we give a characterization of relative Buchsbaum modules by replacing cohomology modules with Ext functors.

Proposition 1.2. *Let M be a finitely generated bigraded S-module. Then M is relative Buchsbaum with respect to Q if and only if the canonical maps*

$$\varphi^i_M : \operatorname{Ext}^i_S(S/Q, M) \to H^i_Q(M)$$

are surjective for all $i < \operatorname{cd}(Q, M)$.

Proof. As $y_1, \dots, y_n$ is an S-sequence, the assertion follows from the well-known fact that $\operatorname{Ext}^i_S(S/Q, M) = H^i(Q, M)$ for all i. ∎

As an immediate consequence we obtain the following known result (see [7, Corollary 2.16]).

Corollary 1.3. *Let M be a finitely generated graded $K[y]$-module. Then M is Buchsbaum if and only if the canonical maps*

$$\varphi^i_M : \operatorname{Ext}^i_{K[y]}(K, M) \to H^i_{\mathfrak{m}}(M)$$

are surjective for all $i < \dim M$.

Proof. Assume $P = 0$. Then $m = 0$, $Q = \mathfrak{m}$ is the unique graded maximal ideal S with $\deg y_i = 1$ for $i = 1, \dots, n$ and $\operatorname{cd}(\mathfrak{m}, M) = \dim M$. Thus the assertion follows from Proposition 1.2. ∎

As another consequence we have

Corollary 1.4. *Let M be a finitely generated bigraded S-module which is relative Buchsbuam with respect to Q. Then $H^i_Q(M)$ is finitely generated for all $i < \operatorname{cd}(Q, M)$.*

Proof. Since M is a finitely generated S-module, $\operatorname{Ext}^i_S(S/Q, M)$ is a finitely generated S-module for all i. Hence the result follows from Proposition 1.2. ∎

Remark 1.5. Let M be a finitely generated bigraded S-module. We may define the *finiteness dimension* of M with respect to Q by

$$f_Q(M) = \inf\{i \in \mathbb{N} : H^i_Q(M) \text{ is not finitely generated}\}.$$

Thus, if M is relative Buchsbaum with respect to Q, then $f_Q(M) = \operatorname{cd}(Q, M)$.

Using Proposition 1.2 we have the following results:

PROPOSITION 1.6. *Let M_1 be a non-zero finitely generated graded $K[x]$-module and M_2 a non-zero finitely generated graded $K[y]$-module. Set $M = M_1 \otimes_K M_2$. Then:*

(a) *M is relative Buchsbaum with respect to Q if and only if M_2 is Buchsbaum;*
(b) *M is relative Buchsbaum with respect to P if and only if M_1 is Buchsbaum.*

Proof. In order to prove (a), in view of [6, Lemma 1.4], for all i we have the isomorphisms of S-modules

$$\begin{aligned}\operatorname{Ext}^i_S(S/Q, M) &\cong \operatorname{Ext}^i_S(K[x] \otimes_K K[y]/Q, M_1 \otimes_K M_2)\\ &\cong \bigoplus_{s+t=i} \operatorname{Ext}^s_{K[x]}(K[x], M_1) \otimes_K \operatorname{Ext}^t_{K[y]}(K[y]/Q, M_2)\\ &\cong M_1 \otimes_K \operatorname{Ext}^i_{K[y]}(K[y]/Q, M_2).\end{aligned}$$

Here we note that Q is not an ideal of $K[y]$, but the extension of an ideal of $K[y]$, say $\mathfrak{q}$. On the other hand, by a similar argument as above we have

$$H^i_Q(M) \cong M_1 \otimes_K H^i_Q(M_2) \quad \text{for all } i \tag{1}$$

(see [6, Proposition 1.5]). Thus it immediately follows that $\operatorname{cd}(Q, M) = \dim M_2$ as we may consider M_1 and $H^i_Q(M_2)$ as K-vector spaces. Now let M_2 be a Buchsbaum module. Then M is relative Buchsbaum with respect to Q using Proposition 1.2, Corollary 1.3 and the above observation. The converse is of course the case, because we may consider M_1 as a free K-module. Part (b) is proved in the same way. ■

PROPOSITION 1.7. *Let M be a finitely generated bigraded S-module. Suppose that* $\operatorname{grade}(Q, M) = 0$ *and* $\operatorname{cd}(Q, M) > 0$. *If M is relative Buchsbaum with respect to Q, then so is $M/H^0_Q(M)$.*

Proof. Set $N = H^0_Q(M)$. We have to show that the maps $\operatorname{Ext}^i_S(Q, M/N) \to H^i_Q(M/N)$ are surjective for all $i < \operatorname{cd}(Q, M/N)$. The claim is clear for $i = 0$, because M/N is Q-torsion free, i.e., $H^0_Q(M/N) = 0$ and the map $\operatorname{Ext}^0_S(S/Q, M/N) \to H^0_Q(M/N)$ is always injective. Now let $0 < i < \operatorname{cd}(Q, M/N)$. The exact sequence $0 \to N \to M \to M/N \to 0$ yields the commutative diagram of S-modules

$$\begin{array}{ccccccccc} \longrightarrow & \operatorname{Ext}^i_S(S/Q, N) & \longrightarrow & \operatorname{Ext}^i_S(S/Q, M) & \longrightarrow & \operatorname{Ext}^i_S(S/Q, M/N) & \longrightarrow \\ & \downarrow{\scriptstyle \varphi^i_N} & & \downarrow{\scriptstyle \varphi^i_M} & & \downarrow{\scriptstyle \varphi^i_{M/N}} & \\ \longrightarrow & H^i_Q(N) & \longrightarrow & H^i_Q(M) & \longrightarrow & H^i_Q(M/N) & \longrightarrow \end{array}$$

Since $H^i_Q(N) = 0$ for $i > 0$, it follows that $H^i_Q(M) \cong H^i_Q(M/N)$ for $i > 0$ and hence $\mathrm{cd}(Q, M) = \mathrm{cd}(Q, M/N)$. Our assumption says that the maps φ^i_M are surjective for all $i < \mathrm{cd}(Q, M)$. In view of the above diagram, it therefore follows that the maps $\varphi^i_{M/N}$ are surjective for all $i < \mathrm{cd}(Q, M/N)$. ∎

2. Relative Buchsbuam modules with two non-vanishing local cohomology modules. Let M be a finitely generated bigraded S-module and $q \in \mathbb{Z}$. In [6] we call M *relative Cohen–Macaulay* with respect to Q if $H^i_Q(M) = 0$ for $i \neq q$, i.e., M has only one non-vanishing local cohomology module with respect to Q. Clearly, if M is a relative Cohen–Macaulay module with respect to Q, then M is relative Buchsbaum with respect to Q. In the following we give a characterization for Buchsbaumness modules with two non-vanishing local cohomology modules. First, we need the following lemma.

Lemma 2.1. *Let M be a finitely generated bigraded S-module which is relative Buchsbaum with respect to Q. Then $QH^i_Q(M) = 0$ for all $i < \mathrm{cd}(Q, M)$.*

Proof. As the maps λ^i_M are surjective for all $i < \mathrm{cd}(Q, M)$, we have $H^i_Q(M) \cong H^i(Q, M)/U$ for some S-submodule U of $H^i(Q, M)$ and for all $i < \mathrm{cd}(Q, M)$. Using the fact that $QH^i(Q, M) = 0$ for all i, we therefore have

$$QH^i_Q(M) \cong Q(H^i(Q, M)/U) \cong (QH^i(Q, M) + U)/U = 0$$

for all $i < \mathrm{cd}(Q, M)$, as required. ∎

Proposition 2.2. *Let M be a finitely generated bigraded S-module. Suppose $r = \mathrm{grade}(Q, M) < \mathrm{cd}(Q, M) = d$ and $H^i_Q(M) = 0$ for all $i \neq r, d$. Then the following statements are equivalent:*

(a) *M is relative Buchsbaum with respect to Q;*
(b) $QH^r_Q(M) = 0$.

Proof. (a)⇒(b): This follows from Lemma 2.1.

(b)⇒(a): Consider the Grothendieck spectral sequence

$$\mathrm{Ext}^j_S(S/Q, H^i_Q(M)) \Rightarrow \mathrm{Ext}^{j+i}_S(S/Q, M).$$

As $H^i_Q(M) = 0$ for all $i \neq r$ and $i < \mathrm{cd}(Q, M)$, the spectral sequence degenerates and one obtains for all j the isomorphisms $\mathrm{Ext}^j_S(S/Q, H^r_Q(M)) \cong \mathrm{Ext}^{j+r}_S(S/Q, M)$. Our assumption implies

$$\mathrm{Ext}^r_S(S/Q, M) = \mathrm{Hom}_S(S/Q, H^r_Q(M)) = (0 :_{H^r_Q(M)} Q) = H^r_Q(M).$$

Hence, the result follows from Proposition 1.2. ∎

This in particular generalizes the following known result (see [7, Corollary 3.6]):

COROLLARY 2.3. *Let M be a finitely generated graded $K[y]$-module. Suppose $r = \operatorname{depth} M < \dim M = d$ and $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq r, d$ where $\mathfrak{m} = (y_1, \dots, y_n)$ is the unique graded maximal ideal of $K[y]$. Then the following statements are equivalent:*

(a) *M is Buchsbaum;*
(b) $\mathfrak{m}H^r_{\mathfrak{m}}(M) = 0$.

Proof. In Theorem 2.2, we put $m = 0$. Then $Q = \mathfrak{m}$ is the unique graded maximal ideal S with $\deg y_i = 1$ for $i = 1, \dots, n$. Thus the assertion follows. ∎

In the following we give several examples to show that one cannot expect any relation between Buchsbaumness and relative Buchsbaumness with respect to the irrelevant ideals P and Q. We recall the following facts from [6]:

$$\operatorname{cd}(P, M) = \dim M/QM \quad \text{and} \quad \operatorname{cd}(Q, M) = \dim M/PM.$$

First we give a Buchsbaum K-algebra which is relative Buchsbaum with respect to both P and Q.

EXAMPLE 2.4. Let $R = K[x, y]/(xy, y^2)$. One has $\operatorname{depth} R = 0$, $\dim R = 1$, $\operatorname{grade}(P, R) = 0$ where $P = (x)$ and $\operatorname{cd}(P, R) = \dim R/(y)R = 1$. We claim that R is relative Buchsbaum with respect to P. By Proposition 2.2, we only need to show that $PH^0_P(R) = 0$. Let $g \in H^0_P(R)$ with $g \neq 0$. Thus $g = f + (xy, y^2)$ where $f \in K[x, y]$ with $f \notin (xy, y^2)$ and $gP^t = 0$ for some t. It follows that $gx^t = 0$ and hence $x^t f \in (xy, y^2)$. Therefore f contains y as a factor and hence $xg = 0$, as desired. The K-algebra R is relative Buchsbaum with respect to Q too, simply because $\operatorname{grade}(Q, R) = \operatorname{cd}(Q, R) = \dim R/(x)R = 0$. Now we claim that R is a Buchsbaum K-algebra. By Corollary 2.3 we only need to show that $\mathfrak{m}H^0_{\mathfrak{m}}(R) = 0$. This is equivalent to saying that $xg = 0$ and $yg = 0$ for all non-zero $g \in H^0_{\mathfrak{m}}(R)$. As before, let $g = f + (xy, y^2)$ where $f \in K[x, y]$ with $f \notin (xy, y^2)$ and $g\mathfrak{m}^t = 0$ for some t. Thus $x^t f \in (xy, y^2)$. Hence f contains y as a factor. Therefore, $xg = 0$ and $yg = 0$, as desired.

Next we give a Buchsbaum K-algebra which is relative Buchsbaum with respect to P but not relative Buchsbaum with respect to Q.

EXAMPLE 2.5. We consider the hypersurface ring $R = K[x, y]/(xy^2)$. It is a Cohen–Macaulay $K[x, y]$-module of dimension 1. Thus it is a Buchsbaum K-algebra. Note that $\operatorname{grade}(P, R) = 0$ and $\operatorname{cd}(P, R) = 1$. By the same argument as in Example 2.4, one finds that R is relative Buchsbaum with respect to P. We claim that R is not relative Buchsbaum with respect to $Q = (y)$. Note that $\operatorname{grade}(Q, R) = 0$ and $\operatorname{cd}(Q, R) = 1$. Consider the non-zero element $x + (xy^2)$ in R. Since $Q^2(x + (xy^2)) = 0$, it follows that

$x+(xy^2) \in H^0_Q(R)$. Observe that $y(x+(xy^2)) \neq 0$ and hence $QH^0_Q(R) \neq 0$. Thus R is not relative Buchsbaum with respect to Q by Proposition 2.2.

In the following example the ring R is not Buchsbaum but relative Buchsbaum with respect to both P and Q.

EXAMPLE 2.6. Let $R = K[x,y]/(x^3,xy)$. One has $\operatorname{depth} R = 0$ and $\dim R = 1$. The K-algebra R is not Buchsbaum. In fact, consider the nonzero element $x+(x^3,xy)$ in R. Since $\mathfrak{m}^3(x+(x^3,xy)) = 0$ where $\mathfrak{m} = (x,y)$, it follows that $x+(x^3,xy) \in H^0_{\mathfrak{m}}(R)$. We observe that $x(x+(x^3,xy)) \neq 0$ and hence $\mathfrak{m}H^0_{\mathfrak{m}}(R) \neq 0$. Thus, R is not Buchsbaum by Corollary 2.3. Since $\operatorname{grade}(P,R) = \operatorname{cd}(P,R) = \dim R/(y)R = 0$, it follows that R is relative Buchsbaum with respect to P. We claim that R is relative Buchsbaum with respect to Q too. In fact, $\operatorname{cd}(Q,R) = \dim R/(x)R = 1$ and as $\operatorname{Ass}(R) = \{(x),(x,y)\}$ we have $\operatorname{grade}(Q,R) = 0$. By the same argument as in Example 2.4, we deduce that $QH^0_Q(R) = 0$. Thus R is relative Buchsbaum with respect to Q by Proposition 2.2.

Finally we give an example in which R is relative Buchsbaum with respect to Q with $\operatorname{grade}(Q,R) > 0$.

EXAMPLE 2.7. Let I be a homogeneous ideal of $K[x]$ and set $Q_1 = (y_1,\ldots,y_{n/2})$ and $Q_2 = (y_{n/2+1},\ldots,y_n)$ where n is even. Set $R = R_0 \otimes_K R_1$ where $R_0 = K[x]/I$ and $R_1 = K[y]/Q_1 \cap Q_2$. One has $\dim R_1 = n/2$ and $\operatorname{depth} R_1 = 1$. We consider the exact sequence

$$0 \to K[y]/Q_1 \cap Q_2 \to K[y]/Q_1 \oplus K[y]/Q_2 \to K[y]/(Q_1+Q_2) \to 0.$$

Applying the functor $H^i_Q(-)$ to this exact sequence yields

$$H^i_Q(R_1) = 0 \quad \text{for all } i \neq 1, n/2 \quad \text{and} \quad H^1_Q(R_1) \cong K[y]/Q.$$

By (1), we have

$$H^i_Q(R) \cong R_0 \otimes_K H^i_Q(R_1) = 0 \quad \text{for all } i \neq 1, n/2,$$

and

$$H^1_Q(R) \cong R_0 \otimes_K H^1_Q(R_1) \cong S/(I,Q).$$

Since $QH^1_Q(R) = 0$, it follows that R is relative Buchsbaum with respect to Q by Proposition 2.2. Note that if $m = 0$, then R is a Buchsbaum K-algebra.

3. On the graded components of local cohomology at the finiteness dimension level. Let M be a finitely generated bigraded S-module. Then the local cohomology modules $H^i_Q(M)$ are naturally bigraded S-modules and each graded component $H^i_Q(M)_j$ is a finitely generated graded $K[x]$-module with grading $(H^i_Q(M)_j)_k = H^i_Q(M)_{(k,j)}$. Let F be a finitely generated bigraded free S-module, i.e., $F = \bigoplus_{i=1}^t S(-a_i,-b_i)$. By using formula (1) in [4] we obtain

$$H_Q^n(F)_j = \bigoplus_{i=1}^{t} \bigoplus_{|a|=-n-j+b_i} K[x](-a_i)z^a. \tag{2}$$

Thus, we may consider $H_Q^n(F)_j$ as a finitely generated graded free $K[x]$-module.

Let N be a finitely generated graded $K[x]$-module with graded minimal free resolution

$$\mathbb{F}:\ 0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to 0.$$

The *Castelnuovo–Mumford regularity* of N is the invariant

$$\operatorname{reg}(N) = \max\{b_i(\mathbb{F}) - i : i \geq 0\}$$

where $b_i(\mathbb{F})$ denotes the maximal degree of the generators of F_i. As a generalization of [6, Proposition 2.3] we have the following

PROPOSITION 3.1. *Let M be a finitely generated bigraded S-module such that $f_Q(M) = \operatorname{cd}(Q, M) = q$. Then for $j \ll 0$ we have*

$$-c \leq \operatorname{reg} H_Q^q(M)_j \leq c \quad \textit{for some } c.$$

Proof. Let

$$\mathbb{F}:\ 0 \to F_{m+n} \xrightarrow{\varphi_{m+n}} F_{m+n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} 0,$$

where $F_i = \bigoplus_{k=0}^{t_i} S(-a_{ik}, -b_{ik})$ is a bigraded free resolution of M. Applying the functor $H_Q^n(-)_j$ to this resolution yields a graded complex $H_Q^n(\mathbb{F})_j$ of free $K[x]$-modules

$$0 \to H_Q^n(F_{m+n})_j \xrightarrow{\psi_{m+n}} H_Q^n(F_{m+n-1})_j \to \cdots \xrightarrow{\psi_1} H_Q^n(F_0)_j \xrightarrow{\psi_0} 0, \tag{3}$$

where the maps $\psi_i : H_Q^n(F_i)_j \to H_Q^n(F_{i-1})_j$ are induced by φ_i for all i. By [4, Theorem 1.1] we have the graded isomorphisms of $K[x]$-modules

$$H_Q^{n-i}(M)_j \cong H_i(H_Q^n(\mathbb{F})_j). \tag{4}$$

By [4, Proposition 2.6], and by (2) and (4), we deduce that $\operatorname{reg} H_Q^q(M)_j = \operatorname{reg} H_{n-q}(H_Q^n(\mathbb{F})_j)$ is bounded below. Thus we only need to prove that $\operatorname{reg} \mathrm{H}_Q^q(M)_j$ is bounded above. Note that $H_i(H_Q^n(\mathbb{F})_j) = \operatorname{Ker}\psi_i / \operatorname{Im}\psi_{i+1} = 0$ for $i < n-q$ and $i > n$. Thus we get the following resolutions of free $K[x]$-modules:

$$0 \to \operatorname{Ker}\psi_{n-q} \to H_Q^n(F_{n-q})_j \to \cdots \xrightarrow{\psi_1} H_Q^n(F_0)_j \xrightarrow{\psi_0} 0, \tag{5}$$

$$0 \to H_Q^n(F_{m+n})_j \to \cdots \xrightarrow{\psi_{n+2}} H_Q^n(F_{n+1})_j \xrightarrow{\psi_{n+1}} \operatorname{Im}\psi_{n+1} \to 0. \tag{6}$$

From (5) and (6) by a similar argument to the proof of [6, Proposition 2.3] we infer that $\operatorname{reg}\operatorname{Ker}\psi_{n-q}$ and $\operatorname{reg}\operatorname{Im}\psi_{n+1}$ are bounded above. Next from

(3) and (4) for $i = n - q, \dots, n$ we have the exact sequences

$$0 \to \operatorname{Im} \psi_{i+1} \to \operatorname{Ker} \psi_i \to H_Q^{n-i}(M)_j \to 0, \tag{7}$$

$$0 \to \operatorname{Ker} \psi_i \to H_Q^n(F_i)_j \to \operatorname{Im} \psi_i \to 0. \tag{8}$$

We first assume that $i = n$. From the exact sequence $0 \to \operatorname{Im} \psi_{n+1} \to \operatorname{Ker} \psi_n \to H_Q^0(M)_j \to 0$, using the fact that $H_Q^i(M)_j = 0$ for $j \ll 0$ and $i < q$ and that $\operatorname{reg} \operatorname{Im} \psi_{n+1}$ is bounded above, we deduce that $\operatorname{reg} \operatorname{Ker} \psi_n$ is bounded above. Hence the exact sequence $0 \to \operatorname{Ker} \psi_n \to H_Q^n(F_n)_j \to \operatorname{Im} \psi_n \to 0$ shows that $\operatorname{reg} \operatorname{Im} \psi_n$ is bounded above. Assume $i = n - 1$. Using the first part, (7) and (8), we find that $\operatorname{reg} \operatorname{Ker} \psi_{n-1}$ and $\operatorname{reg} \operatorname{Im} \psi_{n-1}$ are bounded above. Continuing in this way, we conclude from the last two exact sequences $0 \to \operatorname{Ker} \psi_{n-q+1} \to H_Q^n(F_{n-q+1})_j \to \operatorname{Im} \psi_{n-q+1} \to 0$ and $0 \to \operatorname{Im} \psi_{n-q+1} \to \operatorname{Ker} \psi_{n-q} \to H_Q^q(M)_j \to 0$ that $\operatorname{reg} H_Q^q(M)_j$ is bounded above, as required. ∎

COROLLARY 3.2. *Let M be a finitely generated bigraded S-module. If M is relative Buchsbaum with respect to Q, then for all i and $j \ll 0$ we have*

$$-c \le \operatorname{reg} H_Q^i(M)_j \le c \quad \textit{for some } c.$$

Proof. The assertion for $i = \operatorname{cd}(Q, M)$ follows from Proposition 3.1. For $i < \operatorname{cd}(Q, M)$ we note that $H_Q^i(M)_j = 0$ for $j \ll 0$. ∎

COROLLARY 3.3. *Let M be a finitely generated bigraded S-module. If M is relative Cohen–Macaulay with respect to Q with $\operatorname{cd}(Q, M) = q$, then for $j \ll 0$ we have*

$$-c \le \operatorname{reg} H_Q^q(M)_j \le c \quad \textit{for some } c.$$

Let M be a graded $K[x]$-module. We denote by $\mathrm{H}(M, j) = \dim_K M_j$ for all $j \in \mathbb{Z}$ the Hilbert function of M. The formal Laurent series

$$\mathrm{H}_M(t) = \sum_{j \in \mathbb{Z}} \mathrm{H}(M, j) t^j$$

is called the *Hilbert series* of M. In the following we give an explicit formula for the Hilbert function of the graded components of local cohomology at the finiteness dimension level.

PROPOSITION 3.4. *Let M be a finitely generated bigraded S-module such that $f_Q(M) = \operatorname{cd}(Q, M) = q$. Let*

$$\mathbb{F}:\ 0 \to F_{m+n} \xrightarrow{\varphi_{m+n}} F_{m+n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} 0$$

be a bigraded free resolution of M where $F_l = \bigoplus_{k=0}^{t_l} S(-a_{lk}, -b_{lk})$ for $l = 0, \dots, m + n$. Then for all i and $j \ll 0$ we have

$$\mathrm{H}(H_Q^q(M)_j, i) = \sum_{l=0}^{m+n} (-1)^{n-q+l} \sum_{k=0}^{t_l} \binom{m + i - a_{lk} - 1}{m - 1} \binom{-j + b_{lk} - 1}{n - 1}.$$

Proof. By using formula (1) in [4], we have

$$H^n_Q(F_l)_{(i,j)} = \bigoplus_{k=0}^{t_l} H^n_Q(S)_{(i-a_{lk},j-b_{lk})} = \bigoplus_{k=0}^{t_l} \bigoplus_{|a|=i-a_{lk}|b|=-n-j+b_{lk}} Kx^a y^b.$$

Thus

$$\dim_K H^n_Q(F_l)_{(i,j)} = \sum_{k=0}^{t_l} \binom{m+i-a_{lk}-1}{m-1}\binom{-j+b_{lk}-1}{n-1}. \tag{9}$$

From (5) and (6), it follows that

$$\mathrm{H}(\operatorname{Ker}\psi_{n-q}, i) = \sum_{l=0}^{n-q}(-1)^{n-q+l}\mathrm{H}(H^n_Q(F_l)_j, i) \tag{10}$$

and

$$\mathrm{H}(\operatorname{Im}\psi_{n+1}, i) = \sum_{l=n+1}^{m+n}(-1)^{n+l+1}\mathrm{H}(H^n_Q(F_l)_j, i). \tag{11}$$

Since $H^t_Q(M)_j = 0$ for $j \ll 0$ and $t < q$, it follows from (7) that

$$\mathrm{H}(\operatorname{Im}\psi_{l+1}, i) = \mathrm{H}(\operatorname{Ker}\psi_l, i) \quad \text{for } l = n-q+1, \dots, n \tag{12}$$

and

$$\mathrm{H}(H^q_Q(M)_j, i) = \mathrm{H}(\operatorname{Ker}\psi_{n-q}, i) - \mathrm{H}(\operatorname{Im}\psi_{n-q+1}, i). \tag{13}$$

Applying (8) for $i = n-q+1, \dots, n$ and using (12) and (11), we have

$$\begin{aligned}
&\mathrm{H}(\operatorname{Im}\psi_{n-q+1}, i)\\
&\quad= \sum_{l=n-q+1}^{n}(-1)^{n-q+l+1}\mathrm{H}(H^n_Q(F_l)_j, i) + (-1)^{2n-q+2}\mathrm{H}(\operatorname{Im}\psi_{n+1}, i)\\
&\quad= \sum_{l=n-q+1}^{m+n}(-1)^{n-q+l+1}\mathrm{H}(H^n_Q(F_l)_j, i).
\end{aligned}$$

Now the assertion follows from (9), (10) and (13). ∎

Proposition 3.5. *Let M be a finitely generated bigraded S-module. If $f_Q(M) = \operatorname{cd}(Q, M) = q$, then the Krull dimension of $H^q_Q(M)_j$ is constant for $j \ll 0$. In particular, this result holds for relative Cohen–Macaulay modules and also for relative Buchsbaum modules with respect to Q.*

Proof. We have

$$\begin{aligned}
\mathrm{H}_{H^q_Q(M)_j}(s) &= \sum_{i=0}^{\infty} \dim_K H^q_Q(M)_{(i,j)} s^i\\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{m+n}(-1)^{n-q+l}\sum_{k=0}^{t_l}\binom{m+i-a_{lk}-1}{m-1}\binom{-j+b_{lk}-1}{n-1}s^i
\end{aligned}$$

$$= \sum_{l=0}^{m+n} (-1)^{n-q+l} \sum_{k=0}^{t_l} \binom{-j+b_{lk}-1}{n-1} \sum_{i=0}^{\infty} \binom{m+i-a_{lk}-1}{m-1} s^i$$
$$= \frac{1}{(1-s)^m} \Big\{ \sum_{l=0}^{m+n} (-1)^{n-q+l} \sum_{k=0}^{t_l} \binom{-j+b_{lk}-1}{n-1} s^{a_{lk}} \Big\}$$
$$= \frac{Q_j(s)}{(1-s)^m},$$

where $Q_j(s) = \sum_{i=0}^{m} B_r(j)s^r$ and $B_r(j)$ is a polynomial with coefficients in $\mathbb{Q}$ of degree at most $n-1$. Here we used the fact that

$$\frac{1}{(1-s)^t} = \sum_{i=0}^{\infty} \binom{t+i-1}{t-1} s^i \quad \text{for all } t > 0.$$

We proceed in the same way as at the end of the proof of [5, Theorem 1.9] to find an integer c such that $Q_j(s) = (1-s)^c \widetilde{Q}_j(s)$ for $j \ll 0$ where $\widetilde{Q}_j(s)$ is a polynomial in s with $\widetilde{Q}_j(1) \neq 0$. Therefore by [2, Corollary 4.1.8] we have $\dim H^q_Q(M)_j = m - c$ for $j \ll 0$, as desired. ∎

Let R be a graded ring and N a graded R-module. The R-module N is called *tame* if there exists an integer j_0 such that either

$$N_j = 0 \quad \text{for all } j \le j_0, \quad \text{or} \quad N_j \neq 0 \quad \text{for all } j \le j_0.$$

Corollary 3.6. *Let M be a finitely generated bigraded S-module such that $f_Q(M) = \operatorname{cd}(Q,M)$. Then $H^i_Q(M)$ is tame for all i.*

Proof. The assertion for $i = \operatorname{cd}(Q,M)$ follows from Proposition 3.5. For $i < \operatorname{cd}(Q,M)$ we note that $H^i_Q(M)_j = 0$ for $j \ll 0$. ∎

Remark 3.7. Let M be a finitely generated bigraded S-module and let $f = f_Q(M)$ be the finiteness dimension of M relative to Q. By [1, Proposition 5.6] we know that $\operatorname{Ass}_{K[x]}(H^f_Q(M)_j)$ is asymptotically stable for $j \ll 0$, i.e., there exists an integer j_0 such that $\operatorname{Ass} H^f_Q(M)_j = \operatorname{Ass} H^f_Q(M)_{j_0}$ for all $j \le j_0$. This, in particular implies Proposition 3.5 and of course Corollary 3.6. We remark that here we obtained this result by a different method. Finally, we recall that if R is a Noetherian local ring and I an ideal of R, then the R-module N is said to be I-*cofinite* if $\operatorname{Supp} N \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for all $i \ge 0$. Now let M be a finitely generated bigraded S-module with $f_Q(M) = \operatorname{cd}(Q,M)$. Thus $H^i_Q(M)$ is Q-cofinite for all $i < \operatorname{cd}(Q,M)$, and hence by [3, Proposition 2.5] it is Q-cofinite for all i. In particular, if M is relative Cohen–Macaulay with respect to Q or relative Buchsbaum with respect to Q, then $H^i_Q(M)$ is Q-cofinite for all i.

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