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THEORY OF COVERINGS IN THE STUDY OF RIEMANN SURFACES

BY

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Abstract. For a G-covering $Y \to Y/G = X$ induced by a properly discontinuous action of a group G on a topological space Y, there is a natural action of $\pi(X,x)$ on the set F of points in Y with nontrivial stabilizers in G. We study the covering of X obtained from the universal covering of X and the left action of $\pi(X,x)$ on F. We find a formula for the number of fixed points of an element $g \in G$ which is a generalization of Macbeath's formula applied to an automorphism of a Riemann surface. We give a new method for determining subgroups of a given Fuchsian group.

1. Introduction. It turns out that the general theory of coverings provides some new insights into the theory of Riemann surfaces. Every covering of a topological space X is isomorphic to one obtained from the universal covering of X and the action of the fundamental group $\pi(X,x)$ on some set T.

In particular, for a homomorphism $\theta : \pi(X,x) \to G$ of groups, we can turn G into a $\pi(X,x)$-set and obtain a G-covering of X. There is a natural action of $\pi(X,x)$ on the set F of points with nontrivial stabilizers in G. Gromadzki [5] studied such an action in the case when G was an automorphism group of a Riemann surface. We consider the covering obtained from the univeral covering of a topological space X and the action of $\pi(X,x)$ on F. This approach allows us to find a formula for the number of fixed points of any $g \in G$ which is a generalization of Macbeath's formula applied to an automorphism of a Riemannn surface.

There is a one-to-one correspondence between the set of homomorphisms $\theta : \pi(X,x) \to G$, up to conjugacy, and the set of G-coverings of X, up to isomorphism. In the case when X is an orientable surface, isomorphic G-coverings of X correspond to topologically equivalent actions, which in turn correspond to equivalent representations of G in the symmetric group on the set F.

The study of automorphism groups of Riemann surfaces of genera greater than 1 uses the theory of Fuchsian groups which are discrete subgroups of

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the group of orientation preserving homeomorphisms of the complex upper half-plane. There are well known useful methods for determining subgroups of a given Fuchsian group. They were obtained by Hoare, Karrass and Solitar [6], [7] and Bujalance [1]–[3], mainly by comparing fundamental regions of Fuchsian groups. Singerman [9] proved that a Fuchsian group Λ admits a subgroup Λ' of index n if and only if Λ acts transitively on a set T of n points and he found a presentation of Λ' by studying the action of the elliptic generators of Λ on T. We suggest a new method using the theory of branch coverings which provides the previously known methods as particular cases.

The paper is organized in the following way. The second section provides preliminary information about G-sets, coverings, Fuchsian groups and Riemann surfaces essential for understanding the paper. The third describes the main results which are applied later in Section 4 to finite group actions on Riemann surfaces.

2. Preliminaries

2.1. G-sets and G-coverings. This section is based on the book [4] and it provides preliminary information about coverings and G-sets. Since it is crucial for understanding the paper we provide the proofs although they are not new.

Let X and Y be topological spaces. A continuous, open and discrete map $p : Y \to X$ is called a *branch covering*. We say that the point $y \in Y$ is a *ramification point* if there is no neighborhood U of y such that $p|_U$ is injective. The image $p(y)$ of a ramification point is a *branch point*.

A branch covering is called a *smooth covering* if it has no branch points. In that case, each point $y \in Y$ has a neighborhood U such that $p|_U : U \to p(U)$ is a homeomorphism, with $p(U)$ open in X.

An *isomorphism* between coverings $p : Y \to X$ and $p' : Y' \to X$ is a homeomorphism $\varphi : Y \to Y'$ such that $p' \circ \varphi = p$.

A group H is said to *act discontinuously* on a topological space Y if each $y \in Y$ has a neighborhood U such that $h(U) \cap U = \emptyset$ for almost all $h \in H$. In addition, if arbitrary points y and y' belonging to different orbits have neighborhoods U and V, respectively, such that $h(U) \cap V = \emptyset$ for any $h \in H$ then H is said to act on Y *properly discontinuously*.

The *canonical projection* $\pi_H : Y \to Y/H$ onto the *orbit space* Y/H maps each point $y \in Y$ to its orbit Hy under the action of H. It is a branch covering, where the space Y/H is equipped with the quotient topology.

A covering $p : Y \to X$ is called an *H-covering* if it arises from a properly discontinuous action of H on Y. An isomorphism between H-coverings $p : Y \to X$ and $p' : Y' \to X$ is a homeomorphism $\varphi : Y \to Y'$ such that $p' \circ \varphi = p$ and $\varphi(hy) = h\varphi(y)$.

CONSTRUCTION 1. Given an H-covering $\pi_H : Y \to X$ and a left action of H on a set T, we can construct a new covering of X in the following way. Give T the discrete topology. The group H acts on $Y \times T$ by the rule $h \cdot (y \times t) = (h \cdot y \times h \cdot t)$. Let Y_T be the orbit space $(Y \times T)/H$ and let $\langle y \times t \rangle$ denote the orbit in Y_T containing $(y \times t)$. Then $p_T : Y_T \to X$ defined by the assignment $\langle y \times t \rangle \mapsto \pi_H(y)$ is a covering map.

By an *isomorphism of H-sets* T and T' we mean a bijection $\varphi : T \to T'$ such that $\varphi(h \cdot t) = h \cdot \varphi(t)$ for all $h \in H$ and $t \in T$. An isomorphism φ of two H-sets T and T' induces an isomorphism of the coverings Y_T and $Y_{T'}$ which maps $\langle y \times t \rangle$ to $\langle y \times \varphi(t) \rangle$. If Y is connected then two H-sets determine isomorphic coverings if and only if they are isomorphic.

CONSTRUCTION 2. Given an H-covering $\pi_H : Y \to X$ and a homomorphism of groups $\theta : H \to G$, we turn G into an H-set by $h \cdot g = g\theta(h)^{-1}$, the latter being the group composition in G. Then Construction 1 with $T = G$ provides a covering $p_G : Y_G \to X$ which can be made into a G-covering by defining a compatible left action of G on Y_G by $g' \cdot \langle y \times g \rangle = \langle y \times g'g \rangle$ for $g, g' \in G, y \in Y$.

THEOREM 2.1. *Let X be a connected, locally path-connected and semilocally simply connected topological space. Then there is a canonical bijection between the set of n-sheeted coverings of X, up to isomorphism, and the set of epimorphisms from $\pi_1(X, x)$ to finite permutation groups transitive on n points, up to conjugacy.*

Proof. The assumption about X ensures the existence of a universal covering $u : \widetilde{X} \to X$ which is an H-covering with H being the fundamental group $\pi_1(X, x)$ of X. Any connected covering of X is isomorphic to $\widetilde{X}/H' \to X$ for some subgroup $H' \leq H$. If n is the index of H' in H then the set $T = H/H'$ of left cosets is an H-set via the action $h' \cdot hH' = (h'h)H'$ and the covering $\widetilde{X}_T \to X$ is isomorphic to an n-sheeted covering $\widetilde{X}/H' \to X$ by identifying $\langle y \times hH' \rangle \in \tilde{X}_T$ with the H'-orbit of $h^{-1} \cdot y$.

Conversely, suppose that $H = \pi_1(X, x)$ acts transitively on a set T of n points via a homomorphism ρ. Let $K = \operatorname{Im} \rho$ and let $S \subset K$ be the stabilizer of a point $t \in T$. Then S has index n in K and so $H' = \rho^{-1}(S)$ has index n in H. There exists an isomorphism of H-sets T and H/H' which maps $h \cdot t \in T$ to $hH' \in H/H'$. Thus the n-sheeted covering $p_T : \tilde{X}_T \to X$ is isomorphic to $\widetilde{X}/H' \to X$.

A left action of the group H on the set T is the same as a homomorphism of H into the symmetric group $\Sigma(T)$ on T. Two such homomorphisms give isomorphic H-sets if and only if the homomorphisms are conjugate. ■

From the above proof we obtain

COROLLARY 2.2. *For any subgroup $H' \leq H = \pi(X,x)$ of index n, there is a transitive action of H on $T = H/H'$ that induces an n-sheeted covering $p_T : \widetilde{X}_T \to X$ which is isomorphic to $\tilde{X}/H' \to X$. In particular, if H' is a normal subgroup of H then p_T is a G-covering with $G = H/H'$.* ∎

We finish this section with two useful lemmas.

LEMMA 2.3. *Let $u : \widetilde{X} \to X$ be a universal covering of X and let H_1, H_2 be two subgroups of $H = \pi_1(X, x_0)$. Then the coverings $p_1 : \widetilde{X}/H_1 \to X$ and $p_2 : \widetilde{X}/H_2 \to X$ are isomorphic if and only if H_1 and H_2 are conjugate in H.*

Proof. The coverings $p_1 : \widetilde{X}/H_1 \to X$ and $p_2 : \widetilde{X}/H_2 \to X$ are isomorphic if and only if there is an isomorphism $\varphi : T_1 \to T_2$ of the H-sets $T_1 = H/H_1$ and $T_2 = H/H_2$. Let $t \in T_1$ and $t' \in T_2$. Since the action of H on T_2 is transitive, there exists $h \in H$ such that $t' = h \cdot \varphi(t)$ and so $\mathrm{Stab}_H(t')$ is conjugate to $\mathrm{Stab}_H(\varphi(t)) = \mathrm{Stab}_H(t)$ via h. Furthermore, $\mathrm{Stab}_H(t)$ is conjugate to H_1 and $\mathrm{Stab}_H(t')$ is conjugate to H_2, and so the lemma holds. ∎

LEMMA 2.4. *Let (T, ρ) be an H-set which splits into orbits T_i and let H_i be the stabilizer of a point $t_i \in T_i$ for $i \in I$. Then the kernel of $\rho : H \to \Sigma(T)$ is the largest normal subgroup of H contained in $\bigcap_{i\in I} H_i$.*

Proof. The orbits T_i are disjoint domains of transitivity, which means that the action of H on each T_i is transitive. The orbit T_i is isomorphic to the H-set H/H_i. The stabilizer in H of the coset $H_i \in H/H_i$ is H_i and so the stabilizer of hH_i is a conjugate subgroup H_i^h. Thus the kernel of ρ restricted to T_i is the group $\mathrm{Core}_H H_i = \bigcap_{h\in H} H_i^h$. Since $\mathrm{Core}_H H_i$ is the largest normal subgroup of H contained in H_i, it follows that the kernel of ρ is the largest normal subgroup of H contained in $\bigcap_{i\in I} H_i$. ∎

2.2. Fuchsian groups. A *Fuchsian group* Λ is a discrete subgroup of the group of linear fractional transformations

$$\mathrm{LF}(2,\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R},\ ad-bc = 1 \right\},$$

of the complex upper half-plane $\mathcal{H}$ onto itself with compact orbit space. This orbit space can be given an analytic structure such that the projection $\pi_\Lambda : \mathcal{H} \to \mathcal{H}/\Lambda$ is holomorphic. The algebraic structure of Λ is determined by the *signature*

$$\sigma(\Lambda) = (g; m_1, \ldots, m_r), \tag{1}$$

where g, m_i are integers satisfying $g \geq 0, m_i \geq 2$. The group with signature (1) has a canonical presentation given by

$$\begin{array}{ll} \text{generators:} & x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, \\ \text{relations:} & x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \ldots x_r [a_1, b_1] \ldots [a_g, b_g] = 1. \end{array} \tag{2}$$

Geometrically x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1, \ldots, m_r$ are called the *periods* of Λ and g is the genus of the orbit space $\mathcal{H}/\Lambda$ called the *orbit genus* of Λ. An element of Λ has a fixed point in $\mathcal{H}$ if and only if it has a finite order and it is conjugate to some power of precisely one x_i. A Fuchsian group which has no fixed points in $\mathcal{H}$ is called a *surface group* and has a signature $(g; -)$. The group Λ with presentation (2) has associated to it a fundamental region whose area $\mu(\Lambda)$, called the *area of the group*, is given by

$$\mu(\Lambda) = 2\pi\left(2g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right). \tag{3}$$

An abstract group with presentation (2) can be realized as a Fuchsian group if and only if the right hand side of (3) is greater than 0. If Γ is a subgroup of finite index in Λ, then the Riemann–Hurwitz formula states that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{4}$$

Each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Finally a finite group G is a group of automorphisms of a surface $X = \mathcal{H}/\Gamma$ if and only if it can be represented as $G = \Lambda/\Gamma$ for another Fuchsian group Λ.

3. G-coverings and the set of fixed points of G. In this section we assume that X is a connected, locally path-connected and semilocally simply connected topological space. We denote its fundamental group $\pi_1(X, x)$ by H. Every covering of X is isomorphic to one obtained by Construction 1 from the universal covering $u : \widetilde{X} \to X$ and a left action of H on some set T. This covering is connected if and only if the action on T is transitive. Two such coverings are isomorphic if and only if the H-sets are isomorphic.

A homomorphism of groups $\theta : H \to G$ gives the possibility to treat G as an H-set and Construction 2 provides a covering $p_G : \tilde{X}_G \to X$, where $\tilde{X}_G = (\tilde{X} \times G)/H$ and $p_G(\langle z \times g\rangle) = u(z)$. It is a G-covering, where the action of G on $\tilde{X}_G$ is defined by $g' \cdot \langle z \times g\rangle = \langle z \times g'g\rangle$. In order to underline that the construction involves θ, we shall write p_θ and $\tilde{X}_\theta$ instead of p_G and $\tilde{X}_G$, respectively.

There is a one-to-one correspondence between the set of homomorphisms $\theta : H \to G$ and the set of G-coverings of X. Two coverings $p_\theta : \tilde{X}_\theta \to X$ and $p_{\theta'} : \tilde{X}_{\theta'} \to X$ are isomorphic if and only if θ and θ' are conjugate in G, i.e., there is an element $g \in G$ such that $\theta'([\sigma]) = g\theta([\sigma])g^{-1}$ for any homotopy class $[\sigma]$ of a loop σ at x.

Theorem 3.1. *Let $p_\theta : \tilde{X}_\theta \to X$ be a G-covering corresponding to a homomorphism $\theta : H \to G$ and suppose that G is conjugate to a group G' via a homeomorphism of $\tilde{X}_\theta$ to itself. Then there exists a group H' of homeomorphisms of $\tilde{X}$, isomorphisms $\psi : H \to H'$, $\varphi : G \to G'$ and an epimorphism $\theta' : H' \to G'$ such that $\varphi\theta = \theta'\psi$. Furthermore, for the G'-covering $p_{\theta'} : \widetilde{X}_{\theta'} \to X$, there exists a homeomorphism $\eta : \widetilde{X}_\theta \to \widetilde{X}_{\theta'}$ such that $\eta g = \varphi(g)\eta$ for all $g \in G$.*

Proof. Assume that $G' = \{\tau g \tau^{-1} : g \in G\}$ for some homeomorphism τ of $\tilde{X}_\theta$. Then there exists a homeomorphism $\tilde{\tau}$ of $\widetilde{X}$ such that $\pi\tilde{\tau} = \tau\pi$, where $\pi : \widetilde{X} \to \widetilde{X}_\theta$ is defined by $\pi(z) = \langle z \times e\rangle$ for $z \in \tilde{X}$ and the identity element e of G. Let $H' = \{\tilde{\tau}[\sigma]\tilde{\tau}^{-1} : [\sigma] \in H\}$ and let $\psi : H \to H'$ be given by the rule $\psi([\sigma]) = \tilde{\tau}[\sigma]\tilde{\tau}^{-1}$. There is an epimorphism $\theta' : H' \to G'$ defined by $\theta'([\sigma']) = \varphi\theta\psi^{-1}([\sigma'])$, where $\varphi : G \to G'$ is given by $\varphi(g) = \tau g \tau^{-1}$. The epimorphism θ' induces the G'-covering $p_{\theta'} : \widetilde{X}_{\theta'} \to X$. Let us define $\tilde{\eta} : \widetilde{X} \times G \to \widetilde{X} \times G'$ by $\tilde{\eta}(z \times g) = (\tilde{\tau}(z) \times \varphi(g))$. Then for any $[\sigma] \in H$, we have

$$\begin{aligned}
\tilde{\eta}([\sigma]\cdot(z \times g)) &= \tilde{\eta}([\sigma]\cdot z \times g\theta([\sigma]^{-1}) \\
&= (\tilde{\tau}([\sigma]\cdot z) \times \varphi(g\theta([\sigma]^{-1})) \\
&= (\psi([\sigma])\cdot\tilde{\tau}(z) \times \varphi(g)\theta'(\psi([\sigma]^{-1})) \\
&= \psi([\sigma])\cdot(\tilde{\tau}(z) \times \varphi(g)) \\
&= \psi([\sigma])\cdot\eta(z \times g).
\end{aligned}$$

Thus the assignment $\langle z \times g\rangle \mapsto \langle\tilde{\tau}(z) \times \varphi(g)\rangle$ induces a homeomorphism $\eta : \widetilde{X}_\theta \to \tilde{X}_{\theta'}$ and it is easy to check that $\varphi(g)\eta = \eta g$ for any $g \in G$. ∎

We shall say that the actions of groups G and G' from Lemma 3.1 are *topologically equivalent.* Let $\tilde{F}$ be the set of points in $\tilde{X}$ with nontrivial stabilizers in H and let $\theta : H \to G$ be a homomorphism of groups. We shall show that the set F of points in $\tilde{X}_\theta$ with nontrivial stabilizers in G depend on $\tilde{F}$ and we shall consider a natural action of H on F. We shall show that topologically equivalent group actions correspond to equivalent representations of H in the symmetric group on F.

Lemma 3.2. *Assume that $\widetilde{F}$ splits into H-orbits $\tilde{F}_i$ for $i \in I$. Let H_i be the stabilizer in H of a point $z_i \in \tilde{F}_i$ and $S_i = \theta(H_i)$. Then the set F splits into G-orbits Gf_i for $i \in I$, were $f_i = \langle z_i \times e\rangle$ and e is the identity of G. Moreover, the stabilizer in G of any point $gf_i \in Gf_i$ is conjugate to S_i via g.*

Proof. Suppose that $\langle z \times g\rangle = \langle z \times g'g\rangle$ for $z \in \tilde{X}$ and $g, g' \in G$. Then $(h'\cdot z \times g'g\theta(h')^{-1}) = (z \times g)$ for some $h' \in H$. Thus z is a fixed point of h' and $g' = g\theta(h')g^{-1}$. Let $i \in I$ be an index such that $z \in \tilde{F}_i$. Then $z = h \cdot z_i$

for some $h \in H$ and so $\langle z \times g \rangle = \langle z_i \times g\theta(h) \rangle \in Gf_i$ for $f_i = \langle z_i \times e \rangle$. Consequently, F splits into G-orbits Gf_i and the stabilizer in G of gf_i, for $g \in G$, is conjugate to $\theta(H_i)$ via g. ∎

THEOREM 3.3. *For $a \in G$ and $i \in I$, let $r_i(a)$ be the number of elements $g \in G$ for which $g\langle a \rangle g^{-1} \subset S_i$, where $\langle a \rangle$ is the cyclic group generated by a. Then the number $\mathrm{F}(a)$ of points in $\tilde{X}_\theta$ fixed by a is given by the formula*

$$\mathrm{F}(a) = \sum_{i \in I} r_i(a)/|S_i|. \tag{5}$$

Proof. Let us calculate the total number of fixed points of a in a single orbit Gf_i. By Lemma 3.2, gf_i is a fixed point of a if and only if $\langle a \rangle$ is conjugate to a subgroup of S_i via g. Since conjugate elements have the same number of fixed points we can assume that $\langle a \rangle \subset S_i$. There are $r_i(a)$ elements $g \in G$ such that gf_i is a fixed point of a, but $|S_i|$ of them correspond to the same point. Thus a preserves $r_i(a)/|S_i|$ points in Gf_i. ∎

COROLLARY 3.4. *For any $a \in G$, $\mathrm{F}(a) = \chi(a)$, where χ is the permutation character of the action of G on F.*

Proof. According to the decomposition of F as a disjoint union of transitive G-sets Gf_i for $i \in I$, the permutation character χ of the action of G on F decomposes as a disjoint union of transitive permutation characters χ_i which are the induced characters $1_{S_i}^G$. By Theorem 3.3, for any $a \in G$,

$$\mathrm{F}(a) = \sum_{i \in I} r_i(a)/|S_i| = \sum_{i \in I} 1_{S_i}^G(a) = \sum_{i \in I} \chi_i(a) = \chi(a). \ \blacksquare$$

THEOREM 3.5. *Let $L \leq G$ be a subgroup of G and let $\{g_j\}_{j \in J}$ be the right transversal of its normalizer $N_G(L)$. Let $J_0 \subset J$ be the maximal subset of indices j for which the orders of $S_i \cap L^{g_j}$ are different integers greater than 1. If J_0 is nonempty then for any $j \in J_0$, there are exactly*

$$n_{ij}[N_G(L) : L]/[S_i : S_i \cap L^{g_j}] \tag{6}$$

L-orbits of points with nontrivial stabilizers in L, where n_{ij} is the number of indices $j' \in J$ such that $|S_i \cap L^{g_{j'}}| = |S_i \cap L^{g_j}|$.

Proof. Let m_i denote the order of S_i for $i \in I$. We want to calculate the number of elements $g \in G$ for which $S_i \cap L^g$ are nontrivial subgroups of S_i. Since for any $g, g' \in G$, $L^g = L^{g'}$ if and only if $g^{-1}g' \in N_G(L)$, we need only consider the elements $\{g_j\}_{j \in J}$ of the right transversal of $N_G(L)$. For each $j \in J$, let m_{ij} be the order of $S_i \cap L^{g_j}$ and let n_{ij} be the number of indices $j' \in J$ for which $m_{ij} = m_{ij'}$. There are $n_{ij}|N_G(L)|/m_i$ points in the G-orbit of F_i with stabilizers in L of orders m_{ij}. They split up into L-orbits, each containing $|L|/m_{ij}$ points. Thus we obtain

$$\frac{n_{ij}|N_G(L)|/m_i}{|L|/m_{ij}} = \frac{n_{ij}[N_G(L):L]}{[S_i : S_i \cap L^{g_j}]}$$

L-orbits of points whose stabilizers have orders equal to m_{ij}. ■

COROLLARY 3.6. *Let L be a normal subgroup of G and suppose that $S_i \cap L$ is nontrivial for some $i \in I$. Then the points of Gf_i split into $[G : L]/[S_i : S_i \cap L]$ L-orbits and their stabilizers have orders $|S_i \cap L|$.*

PROPOSITION 3.7. *For any epimorphism $\theta : H \to G$ with kernel K, the G-coverings $p_\theta : \tilde{X}_\theta \to X$ and $\pi_K : \tilde{X}/K \to X$ are isomorphic.*

Proof. Let $\tilde{\mu} : \tilde{X} \times G \to \tilde{X}/K$ be induced by the assignment $(z \times g) \mapsto g \cdot \pi_K(z)$ for $z \in \tilde{X}$ and $g \in G$. Since $\tilde{\mu}(h \cdot z \times g\theta(h)^{-1}) = g \cdot \pi_K(z)$ for any homotopy class $h = [\sigma] \in H$ of a loop σ at x, we have a well defined mapping $\mu : \tilde{X}_\theta \to \tilde{X}/K$ given by $\mu(\langle z \times g\rangle) = g \cdot \pi_K(z)$. We shall show that μ is an isomorphism. For, suppose that $\mu(\langle z \times g\rangle) = \mu(\langle z' \times g'\rangle)$ for some $z, z' \in \tilde{X}$ and $g, g' \in G$ and let $h, h' \in H$ be homotopy classes of loops at x such that $g = \theta(h)$ and $g' = \theta(h')$. Then the equality $g \cdot \pi_K(z) = g' \cdot \pi_K(z')$ implies that $\pi_K(h \cdot z) = \pi_K(h' \cdot z')$. Thus there exists $k \in K$ such that $(h^{-1}kh') \cdot z' = z$. So for $h'' = h^{-1}kh'$, $h'' \cdot (z' \times g') = (z \times g'\theta(h'')^{-1}) = (z \times g)$, which means that $\langle z \times g\rangle = \langle z' \times g'\rangle$. Clearly μ is a surjection since $\mu(\langle z \times e\rangle) = \pi_K(z)$ for any $z \in \tilde{X}$. Finally, since $p\mu = p_G$, it follows that μ is an isomorphism of G-coverings. ■

For the G-covering $\pi_K : \tilde{X}/K \to X$, we have $g \cdot Kz = K(\tilde{g} \cdot z)$, where $z \in \tilde{X}$ and $\tilde{g} \in H$ is an element such that $\theta(\tilde{g}) = g$. Let $F = \bigcup_{i \in I} Gp_i$ for $p_i = Kz_i$. Then by Lemma 2.4, we have the following

THEOREM 3.8. *The kernel of the homomorphism $\varrho : H \to \Sigma(F)$ given by*

$$\varrho(h)(g \cdot p_i) = \theta(h)g \cdot p_i, \qquad h \in H,\, g \in G,$$

is the greatest normal subgroup of H contained in $\bigcap_{i \in I} H_iK$.

Let $p_F : \widetilde{X}_F \to X$ be a covering obtained by Construction 1 from the universal covering of X and the left action of H on F.

THEOREM 3.9. *The covering $p_F : \widetilde{X}_F \to X$ is a disjoint union of coverings $\widetilde{X}/H_iK \to X$, $i \in I$. Any bijection $\tau : F \to F'$ of sets induces an action of H on F' by the assignment $h \cdot f' = \tau(h \cdot \tau^{-1}(f'))$ and the covering $p_{F'} : \widetilde{X}_{F'} \to X$ is a disjoint union of coverings $\widetilde{X}/H_i'K \to X$, $i \in I$, for which there exists a permutation $\pi \in \Sigma(I)$ such that H_i' is conjugate to $H_{\pi(i)}$ via some $h_i \in H$.*

Proof. The covering $\widetilde{X}_F \to X$ is a disjoint union of coverings $\widetilde{X}_{F_i} \to X$ for $F_i = Hp_i$. The stabilizer in H of p_i is equal to H_iK and so by Corollary 2.2, the covering $\widetilde{X}_{F_i} \to X$ is isomorphic to $\widetilde{X}/H_iK \to X$.

The bijection τ induces an isomorphism of the coverings p_F and $p_{F'}$ defined by the assignment $\langle z \times f\rangle \mapsto \langle z \times \tau(f)\rangle$. Thus by Lemma 2.3, there exists a permutation π of the set I such that the covering $p_{F'}$ is a disjoint union of coverings $\widetilde{X}/H_i'K \to X$, where H_i' is conjugate to $H_{\pi(i)}$ via some $h_i \in H$. ∎

If $F' = \tau(F)$ for some homeomorphism τ of $\tilde{X}/K$ then F' is the set of fixed points of the group $G' = \{\tau g \tau^{-1} : g \in G\}$ and we have a homomorphism $\rho' : H \to \Sigma(F')$ defined by the formula $\rho'(h)(f') = \tau(\rho(h)(\tau^{-1}(f')))$. Thus topologically equivalent group actions give rise to equivalent representations of H in the symmetric group on F. This fact may be helpful in some situations when we have to decide if some actions are equivalent or not.

4. Coverings and Riemann surfaces. Here we study finite group actions on Riemann surfaces. Throughout the section, Λ will be a Fuchsian group with signature (1) and G will be a finite group. We say that a homomorphism $\theta : \Lambda \to G$ is *surface-kernel* if θ is surjective and its kernel is torsion free. We shall denote the orbit space $\mathcal{H}/\Lambda$ and the canonical projection $\mathcal{H} \to \mathcal{H}/\Lambda$ by $X(\Lambda)$ and π_Λ, respectively.

THEOREM 4.1. *There is a one-to-one correspondence between the set of G-coverings of $X(\Lambda)$ by Riemann surfaces and the set of surface-kernel epimorphisms $\theta : \Lambda \to G$. Two such epimorphisms correspond to isomorphic coverings if and only if they are conjugate.*

Proof. If $\theta : \Lambda \to G$ is a surface-kernel epimorphism with kernel Γ then $\mathcal{H}/\Gamma$ is a Riemann surface and we have a G-covering $p : \mathcal{H}/\Gamma \to X(\Lambda)$, where the action of G on $\mathcal{H}/\Gamma$ is defined by the formula $(\lambda\Gamma)(\Gamma z) = \Gamma(\lambda z)$ for any $\lambda \in \Lambda$ and $z \in \mathcal{H}$.

Conversely, suppose that $p : Y \to X(\Lambda)$ is a G-covering, where Y is a Riemann surface. Then Y is isomorphic to the orbit space $\mathcal{H}/\Gamma$ for some Fuchsian surface group Γ. Let $\pi_\Gamma : \mathcal{H} \to \mathcal{H}/\Gamma$ be the canonical projection. For any $g \in G$, there exists $\tilde{g} \in \mathrm{Aut}(\mathcal{H})$ such that $g\pi_\Gamma = \pi_\Gamma\tilde{g}$. Let $\tilde{\Lambda}$ be the Fuchsian group generated by such lifts to $\mathcal{H}$ of all elements of G and let $\tilde{\theta} : \tilde{\Lambda} \to G$ be the homomorphism defined by $\tilde{\theta}(\tilde{g}) = g$. Then $\tilde{\theta}$ is a surface-kernel epimorphism with kernel Γ and there exists $\tilde{h} \in \mathrm{Aut}(\mathcal{H})$ such that $\Lambda = \{\tilde{h}\tilde{g}\tilde{h}^{-1} : \tilde{g} \in \tilde{\Lambda}\}$. Thus there exists a surface-kernel epimorphism $\theta : \Lambda \to G$, up to $\mathrm{Aut}(\mathcal{H})$-conjugation. ∎

By Proposition 3.7 we have the following

REMARK 4.2. The G-covering $p_\theta : \mathcal{H}_\theta \to X(\Lambda)$ obtained from a Λ-covering $\pi_\Lambda : \mathcal{H} \to X(\Lambda)$ and a surface-kernel epimorphism $\theta : \Lambda \to G$ with kernel Γ is isomorphic to the G-covering $p : \mathcal{H}/\Gamma \to \mathcal{H}/\Lambda$.

Now we determine the periods in the signature of any, not necessarily normal, subgroup Λ' of Λ. Recall that the periods $m_1, \dots, m_r$ in the signature of Λ are the orders of maximal cyclic subgroups of Λ generated by the elliptic elements $x_1, \dots, x_r$. Any element of Λ has a fixed point in $\mathcal{H}$ if and only if it is conjugate to some power of an elliptic element x_i. The orbit genus of Λ' can be calculated by the Riemann–Hurwitz formula (4).

THEOREM 4.3. *Assume that $\theta : \Lambda \to G$ is a surface-kernel epimorphism and $L = \theta(\Lambda')$ is the image of a subgroup $\Lambda' \le \Lambda$. Let $\{g_1, \dots, g_s\}$ be a right transversal of $\mathrm{N}_G(L)$ in G. For i and j in the range $1 \le i \le r$ and $1 \le j \le s$, let $m_{ij} = m_i/l_{ij}$, where l_{ij} is the smallest positive integer such that $g_j^{-1}\theta(x_i)^{l_{ij}} g_j \in L$. Let n_{ij} be the sum of all indices $1 \le j' \le s$ for which $m_{ij} = m_{ij'}$, and let $p_{ij} = n_{ij}[N_G(L) : L]/l_{ij}$. If j runs over all indices for which l_{ij} are different integers not equal to 1 and i changes from 1 to r then $\{m_{ij}, \overset{p_{ij}}{\dots}, m_{ij}\}$ is the set of all periods in the signature of Λ' .*

Proof. The points of $\mathcal{H}$ with nontrivial stabilizers in Λ split into r Λ-orbits $\tilde{F}_1, \dots, \tilde{F}_r$ such that every point belonging to $\tilde{F}_i$ has a cyclic stabilizer of order m_i generated by a conjugation of the elliptic generator x_i of Λ via some element of Λ. The projection $\pi_\Gamma : \mathcal{H} \to \mathcal{H}/\Gamma$ maps fixed points of Λ to fixed points of G while θ maps stabilizers to stabilizers preserving their orders. Thus the set of fixed points of G splits into orbits $F_1, \dots, F_r$ such that a point in F_i has a cyclic stabilizer of order m_i conjugate to $S_i = \langle \theta(x_i) \rangle$ via an element of G. Thus for any g_j belonging to the right transversal of $\mathrm{N}_G(L)$, the order of $S_i \cap L^{g_j}$ is equal to $m_{ij} = m_i/l_{ij}$, where l_{ij} is the smallest positive integer such that $g_j^{-1}\theta(x_i)^{l_{ij}} g_j \in L$. Thus the statement follows from Theorem 3.5. ∎

As a corollary we obtain the well known result of Bujalance.

COROLLARY 4.4. *Let Λ be a Fuchsian group with signature* (1) *and let $\Lambda' \subset \Lambda$ be its normal subgroup of a finite index N. Then the proper periods in the signature of Λ' are: $m_1/l_1, \overset{N/l_1}{\dots}, m_1/l_1, \dots, m_r/l_r, \overset{N/l_r}{\dots}, m_r/l_r$, where l_i denotes the order of $x_i\Lambda' \in \Lambda/\Lambda'$ and the quotients equal to 1 are dropped out.*

A small modification of the proof of Theorem 2.1 provides Singerman's method of determining the signature of a subgroup Λ' of a group Λ. We describe it for the convenience of the reader.

THEOREM 4.5. *The existence of a homomorphism ρ from Λ onto a finite permutation group acting transitively on a set of n points is equivalent to the* *existence of a subgroup* $\Lambda' \subset \Lambda$ *of index* n. *Furthermore, if for the elliptic generator* $x_i \in \Lambda$, *the permutation* $\rho(x_i)$ *is a product of* s_i *cycles of lengths* $k_{i_1}, \ldots, k_{is_i}$ *then* $\{m_i/k_{ij_i}\}$ *for* $i = 1, \ldots, r$ *and* $j_i = 1, \ldots, s_i$ *is the set of all periods in* Λ'.

Proof. If Λ' is a subgroup of Λ of index n then Λ acts transitively, by left multiplication, on the set Λ/Λ' of left cosets. Conversely, if Λ acts transitively on the left on n points then the stabilizer of a point is a subgroup Λ' of index n in Λ. Let us consider the action of an elliptic generator x_i of Λ on Λ'-cosets. If it has a cycle of length k then there exist cosets $\lambda_1\Lambda', \ldots, \lambda_k\Lambda'$ such that $x_i\lambda_j\Lambda' = \lambda_{j+1}\Lambda'$ for $j = 1, \ldots, k-1$ and $x_i\lambda_k\Lambda' = \lambda_1\Lambda'$. Thus $\lambda_1\Lambda' = x_i^k\lambda_1\Lambda'$ and so $x_{i1} = \lambda_1^{-1}x_i^k\lambda_1 \in \Lambda'$. This last element is an elliptic generator of Λ' with period m_i/k. If x_i has another cycle of length k then similarly we conclude that there exists $\lambda_2 \in \Lambda$ such that $x_{i2} = \lambda_2^{-1}x_i^k\lambda_2 \in \Lambda'$. The elements x_{i1} and x_{i2} provide the same period in Λ' if and only if they are conjugate via some $\gamma \in \Lambda'$ what implies that $\lambda_1\Lambda'$ lies in the same cycle as $\lambda_2\Lambda'$. Thus distinct cycles give rise to different periods in the signature of Λ'. ∎

The representation of an automorphism group of a Riemann surface in the symmetric group on the set of fixed points was studied by Gromadzki [5] while the number of fixed points of an automorphism of a Riemann surface was calculated by Macbeath [8]. Their results given below are particular cases of Theorems 3.8 and 3.3.

THEOREM 4.6. *Let* $X = \mathcal{H}/\Gamma$ *be a Riemann surface with automorphism group* $G = \Lambda/\Gamma$ *and let* $x_1, \ldots, x_r$ *be elliptic canonical generators of* Λ *with periods* $m_1, \ldots, m_r$ *respectively. Let* $\theta : \Lambda \to G$ *be a surface-kernel epimorphism with kernel* Γ *and for* $1 \neq g \in G$ *let* $\varepsilon_i(g)$ *be* 1 *or* 0 *according as* g *is or is not conjugate to a power of* $\theta(x_i)$.

(i) *Then the number* $\mathrm{F}(g)$ *of points of* X *fixed by* g *is given by the formula*

$$\mathrm{F}(g) = |\mathrm{N}_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i. \tag{7}$$

(ii) *If* $F \subset X$ *is the set of points with nontrivial stabilizers in* G *and* Λ_i *denote subgroups* $\langle x_i \rangle \Gamma$ *of* Λ, $i = 1, \ldots, r$, *then the kernel of the homomorphism* $\rho : \Lambda \to \Sigma(F)$ *defined by the rule* $\rho(\lambda)(f) = \theta(\lambda)(f)$ *is the greatest normal subgroup of* Λ *contained in* $\Lambda_1 \cap \cdots \cap \Lambda_r$.

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