KRULL–GABRIEL DIMENSION AND CANTOR–BENDIXSON RANK OF 1-DOMESTIC STRING ALGEBRAS

BY

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Abstract. We prove that the Krull–Gabriel dimension of the category of modules over any 1-domestic non-degenerate string algebra is 3.

1. Introduction. The Ziegler spectrum of a ring R, $_R\mathrm{Zg}$, was defined by Martin Ziegler in his seminal paper [33]. It is a (quasi-compact) topological space whose points are indecomposable pure injective (= algebraically compact) modules, and basic open sets are determined by morphisms between finitely presented modules. A standard topic in the model theory of modules is to describe this space, both the points and topology. One possibility to measure the complexity of $_R\mathrm{Zg}$ is to calculate its Cantor–Bendixson rank. In all known examples this rank equals another important characteristic of the category of modules over R, its Krull–Gabriel dimension; and $\mathrm{CB}(\mathrm{Zg}_R) \leq \mathrm{KG}(R)$ is always true. Furthermore, the KG-dimension of a ring coincides with the m-dimension of the lattice of finitely generated subfunctors of $\mathrm{Hom}(R, -)$.

By now there exist plenty of sources where this program has been successfully carried through for particular classes of rings. For instance, Eklof and Herzog [5] and Puninski [18] described the Ziegler spectra of serial rings, and Trlifaj [32] investigated this space for von Neumann regular rings.

The class of finite-dimensional algebras over a field is of particular importance. As was noted by Prest [17], if A is a finite-dimensional algebra, then the isolated points in $_A\mathrm{Zg}$ are exactly the indecomposable finite-dimensional modules. For instance, A is of finite representation type iff the CB-rank of A (or the KG-dimension of A) is zero. To describe the points of $_A\mathrm{Zg}$ we have to classify at least indecomposable finite-dimensional representations, therefore A should be tame. Another reason why wild algebras should be excluded from consideration is that the KG-dimension of any wild algebra is undefined.

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A classical example of tame (domestic) finite-dimensional algebras is the class of tame hereditary algebras. In this case the Ziegler spectrum was described by Prest [16] and Ringel [25]. It turned out that the CB-rank (and the KG-dimension) equals 2, and there is a standard (like over integers) partition of points into finite-dimensional, Prüfer, adics (parameterized by simple regular representations) and the (unique if A is connected) generic. In the case when the ground field is algebraically closed, this value of KG-dimension has been known since Geigle [7]. According to Krause [14] the KG-dimension never equals 1 for finite-dimensional algebras, and whether the CB-rank of the Ziegler spectrum could take 1 as a value is still an open problem.

The next step was made by Burke and Prest [1] and Schröer [28]: they found, for each $n \geq 2$, a finite-dimensional algebra whose KG-dimension (and therefore CB-rank) equals n. All algebras in these examples are string algebras, a celebrated class of finite-dimensional algebras whose finite-dimensional representations have been known since Gelfand–Ponomarev [8] and Butler–Ringel [2]. Whether there exists a finite-dimensional algebra of infinite KG-dimension is an open problem.

The progress on classification of indecomposable pure injective modules over string algebras is quite limited. If A is a domestic string algebra, the classification of points of $_A$Zg was conjectured by Ringel [26]: apart of indecomposable finite-dimensional modules and infinite-dimensional band modules (that is, Prüfer, adic, and generic modules corresponding to non-degenerate tubes) they are direct product, direct sum or mixed modules corresponding to 1-sided almost periodic or 2-sided biperiodic strings over A. Only recently Puninski [22] has verified Ringel's conjecture for 1-domestic string algebras (every domestic string algebra is n-domestic for some n). Besides this there are a few examples (see [1] and [21]) of domestic string algebras where points of $_A$Zg are classified, and no such example of a non-domestic string algebra is known.

If A is a non-domestic string algebra, then (see [28]) the KG-dimension of A is undefined, and the same is undoubtedly true for its CB-rank. For domestic string algebras there is a conjecture, due to Schröer [28], that the KG-dimension of A equals $n+2$, where n is the maximal length of a path in its bridge quiver. This conjecture was checked up for some particular classes of (domestic) string algebras (see for instance [1], [28]), but has remained open even in the simplest case of 1-domestic string algebras (see [21] for partial results). Note that the bridge quiver of any 1-domestic string algebra is either trivial (there are no bridges at all), or consists of bridges of length 1. In this paper we will show that the KG-dimension of A equals 2 or 3 respectively, verifying Schröer's conjecture for 1-domestic string algebras.

Of course, this result depends on the classification of indecomposable pure injective modules over 1-domestic string algebras (see Proposition 4.1), which is to be published yet.

Note that the KG-dimension of an algebra A is a purely combinatorial invariant whose definition involves only finite-dimensional modules and morphisms between them. However it seems to be hard to calculate this invariant straight from the definition. In our proof we will use a more advanced approach which is quite indirect.

Firstly, similar to [22] we will cover large open sets in $_A$Zg by a net of intervals freely generated by two chains (in particular, these intervals are distributive). The advantage of using such intervals is an easy receipt for calculating the CB-ranks of their points. Being calculated in open subsets of $_A$Zg, those ranks are global, but this is not the case for points in complementary closed subsets: although we know their relative CB-ranks, when coming back to the whole space, the points will usually 'jump', that is, increase their CB-ranks. We will show that this jump is not too high, for instance every Prüfer and every adic point has CB-rank at most 2; therefore the unique generic point is of CB-rank 3. However the precise value of CB-ranks of Prüfer and adic points is elusive, and this is the only drawback of our considerations.

Having calculated the CB-rank of A, we will use a standard trick to show that it equals the KG-dimension. Namely we will check the so-called isolation property (see [17]): any isolated point in a closed subset of $_A$Zg is isolated by a minimal pair.

Straightforward consequences are on the way: if A is a 1-domestic string algebra, then $_A$Zg is a T_0-space, and there is no superdecomposable pure injective module over A. Note that Harland [10] constructed topologically indistinguishable points in the Ziegler spectrum of a particular non-domestic string algebra, and the existence of superdecomposable pure injective modules over string algebras is a delicate problem (see [19] and [12] for recent progress).

In this paper we will assume that the ground field, F, is algebraically closed. Although we believe that all results are true for an arbitrary field, some proofs depend on this assumption, for instance we will use the description of morphisms between string and band modules, which seems to be available for algebraically closed fields only.

2. Preliminaries. This is a follow-up paper to [22] and the reader is referred to it for most definitions. We will recall some of them, but rather give some examples. Recall that a *string algebra* A is the (finite-dimensional) path algebra of a particular (finite) quiver with monomial relations (see [2]

or [28]). For instance, the Kronecker algebra $\widetilde{A}_1$ is the string algebra (with no relations)

$$\widetilde{A}_1 \qquad \circ \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \circ$$
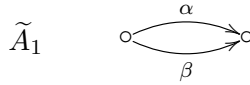
Also the path algebra R_1 of the following quiver with relations $\alpha^2 = \beta^2 = \alpha\beta = 0$ is a string algebra:

$$R_1 \qquad \alpha \, \circ \, \beta$$
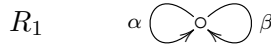
Suppose that s is a vertex of the quiver Q of a string algebra A. There is a standard (usually non-unique) way of separating arrows going in and out of s into two classes: $H_1(s)$ and $H_{-1}(s)$. Informally we introduce a borderline such that getting 'through s' one has to cross this line. For instance, for R_1 one could choose $\beta, \beta^{-1} \in H_1(s)$ and $\alpha, \alpha^{-1} \in H_{-1}(s)$.
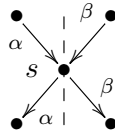
Usually (as in this diagram) we will consider strings from $H_1(s)$ to be pointed on the right, and strings from $H_{-1}(s)$ to be pointed on the left.

Then $H_1(s)$ is a chain under a standard ordering (a direct arrow is larger than an inverse arrow). For instance, $\beta^{-1} < 1_{s,1} < \beta\alpha^{-1}$ in $H_1(s)$. Similarly $H_{-1}(s)$ is a chain and $\alpha^{-1}\beta < 1_{s,-1} < \alpha$. Over R_1 (or in general, see [29]) it is not difficult to describe these chains completely.

Namely, in view of the relation $\alpha\beta = 0$, there is just one way to extend β^{-1} to the right: $\beta^{-1}\alpha\beta^{-1}\alpha\dots$. Furthermore, the strings whose first letter is β^{-1} form the chain

$$\beta^{-1} < \beta^{-1}\alpha\beta^{-1} < (\beta^{-1}\alpha)^2\beta^{-1} < \cdots \quad \cdots < (\beta^{-1}\alpha)^3 < (\beta^{-1}\alpha)^2 < \beta^{-1}\alpha$$

isomorphic to $\omega + \omega^*$, where ω^* denotes the ordering opposite to ω.

The strings with first letter β form a more complicated linear ordering. Namely, the strings $(\beta\alpha^{-1})^k$ and $(\beta\alpha^{-1})^k\beta$ are ordered into the following chain of type $\omega + \omega^*$:

$$1_{s,1} < \beta\alpha^{-1} < (\beta\alpha^{-1})^2 < (\beta\alpha^{-1})^3 < \cdots \quad \cdots < (\beta\alpha^{-1})^2\beta < \beta\alpha^{-1}\beta < \beta.$$

Now the interval $[\beta\alpha^{-1}, (\beta\alpha^{-1})^2]$ is refined to the chain

$$\beta\alpha^{-1} < (\beta\alpha^{-1})^2\beta^{-1} < (\beta\alpha^{-1})^2\beta^{-1}\alpha\beta^{-1} < \cdots < (\beta\alpha^{-1})^2\beta^{-1}\alpha < (\beta\alpha^{-1})^2$$

of type $\omega + \omega^*$, and we have a similar refinement for any interval $[(\beta\alpha^{-1})^n, (\beta\alpha^{-1})^{n+1}]$.

According to [2] there are two kinds of indecomposable finite-dimensional modules over a string algebra A: string and band modules. *String modules* are related to *strings*, that is, generalized walks in the quiver of A. Here is a typical example of a string module over $\widetilde{A}_1$ or R_1:
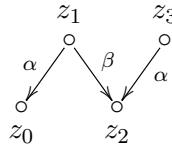
We denote this module by $M(\alpha\beta^{-1}\alpha)$ (note that we draw direct arrows from the upper right to the lower left). It follows from [2] that string modules $M(u)$ and $M(v)$ are isomorphic iff $u = v$ or $u = v^{-1}$.

Band modules are related to *bands*, that is, primitive unoriented cycles in the quiver of A. Here is an example of a (2-layer) band module $M(C, 2, \lambda)$, $0 \neq \lambda \in F$, over $\widetilde{A}_1$ constructed from the band $C = \alpha\beta^{-1}$:
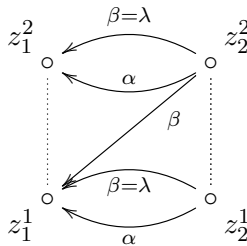
for instance $\beta(z_2^2) = \lambda z_1^2 + z_1^1$ (recall that we assume F to be algebraically closed).

We will require that a band C start with a direct arrow and end with an inverse arrow, that is, be of the form $\alpha D \beta^{-1}$. 1-layer band modules $M(C, 1, \lambda)$, $0 \neq \lambda \in F$, will be referred to as *simple band modules*. Again, by [2, Sect. 3], band modules with different data are non-isomorphic, and no band module is isomorphic to a string module.

For a general definition of a *domestic* algebra see [30, Sect. 14.4]. By [24, Prop. 2] a string algebra is domestic iff, for every α, there exists at most one band of A with first letter α. If the number of bands of A (up to a cyclic permutation and inversion) is n, then A is said to be *n-domestic* (again, this agrees with a general definition). Thus a string algebra A is 1-*domestic* if it has a unique band $C = \alpha D \beta^{-1}$. For instance, $\alpha\beta^{-1}$ is a unique band of $\widetilde{A}_1$ and R_1, therefore these algebras are 1-domestic.

Here is a more twisted example of a 1-domestic string algebra:

with relations $\beta^2 = \gamma^2 = \beta\alpha\gamma = 0$, whose unique band is $\alpha\gamma\alpha^{-1}\beta^{-1}$.

Note that this algebra is a *wind wheel algebra* in the terminology of Ringel [27] (in particular it is minimal representation infinite), and the reader could find more illuminating examples of 1-domestic string algebras in that paper. For instance, the path algebra of the quiver
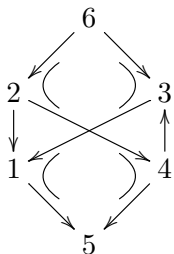
with 'long' relations $6 \to 2 \to 1 \to 5$ and $2 \to 4 \to 3 \to 1$ is 1-domestic (even minimal representation infinite).

1-*sided* infinite strings and 2-*sided* strings over a string algebra A are defined similarly to finite strings. For instance, $(\alpha\beta^{-1})^\infty$ is a 1-sided string over $\widetilde{A}_1$ which is *periodic*, and $\beta(\alpha\beta^{-1})^\infty$ is an *almost periodic* (non-periodic) string over R_1. Similarly $^\infty(\beta\alpha^{-1})\beta(\alpha\beta^{-1})^\infty$ is a *biperiodic* string over R_1 which is not periodic. By [24, Prop. 2], over a domestic string algebra A every 1-sided string is either periodic or almost periodic, and every 2-sided string is either periodic or biperiodic.

We say that a 1-domestic string algebra is *degenerate* if it has no 2-sided non-periodic strings, and *non-degenerate* otherwise. If A is non-degenerate, then (see [26] and [22, L. 5.3]) one can choose a representative $C = \alpha D\beta^{-1}$ of a unique band of A such that every 2-sided string is of the form $^{-\infty}CUC^\infty$, where the length of U is uniformly bounded (in particular there are just finitely many such strings). Furthermore, A has no 2-sided strings of the form $^\infty CVC^{-\infty}$.

Let s be a vertex of a string algebra A. Then one can 'complete' the chains $H_{\pm 1}(s)$ by taking into consideration 1-sided infinite strings. Namely, if $u = c_1 c_2 \ldots$ is an infinite string, then put u in $\widehat{H}_1(s)$ if $c_1 \in H_1(s)$, and similarly for $H_{-1}(s)$.

There is a natural extension of the linear ordering $<$ from $H_1(s)$ to $\widehat{H}_1(s)$, and from $H_{-1}(s)$ to $\widehat{H}_{-1}(s)$. Then every $\mathbb{N}$-string in $\widehat{H}_1(s)$ defines a cut on the set of finite strings: the lower part of this cut consists of all strings $C \in H_1(s)$ such that $C < u$, and its upper part consists of all strings $D \in H_1(s)$ such that $u < D$. For instance, let $H_1(s)$ for $A = \widetilde{A}_1$ be chosen beginning with α (or equal to $1_{s,1}$) and let $u = (\alpha\beta^{-1})^\infty$ be an $\mathbb{N}$-string. Then the lower part of the cut defined by u consists of the strings $(\alpha\beta^{-1})^m$, and its upper part is formed by the strings $(\alpha\beta^{-1})^n\alpha$.

3. Krull–Gabriel dimension. For the definition of a *pp-formula* $\varphi(x)$ (in one free variable) see [17, Ch. 1]. A pp-formula may also be thought of as a pointed module (M, m). For instance, the pointed (at z_0) string module $M(\beta\alpha\beta^{-1})$ over R_1

$$M(D)$$
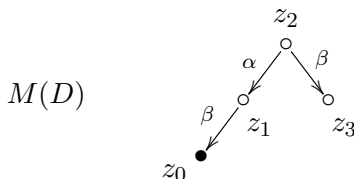
corresponds to the pp-formula $\varphi(x) \doteq \exists z_1, z_2, z_3 \ (\alpha x = 0 \wedge x = \beta z_1 \wedge z_1 = \alpha z_2 \wedge \beta z_2 = z_3 \wedge \alpha z_3 = 0)$, which claims just that $z_0 \in \beta\alpha M$, that is, $\varphi(x)$ is equivalent to the divisibility formula $\beta\alpha \,|\, x$. The module M as above is often called a *free realization* of φ (note that every pp-formula has a free realization).

The set of pp-formulae (or rather their equivalence classes with respect to logical equivalence) forms a modular lattice $L(A)$, where the meet is given by conjunction of formulae and the join $\varphi + \psi$ is defined to be the formula $\exists y \ (\varphi(y) \wedge \psi(x - y))$. For instance, if the string module $M(C) = M(\beta\alpha^{-1})$, pointed at the right end, corresponds to a pp-formula $\psi(x)$, and $M(D)$ is as above, then the conjunction $\varphi \wedge \psi$ has the following pushout $M(C.D)$ as a free realization:
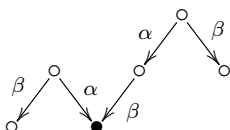
We will denote the corresponding pp-formula by $(C.D)$. Note that if $D, D' \in H_1(s)$ for some vertex s, then $(.D')$ implies $(.D)$ iff there is a pointed morphism from $M(D)$ to $M(D')$ iff $D \leq D'$ in $H_1(s)$. According to the way the ordering on $L(A)$ is defined, the map $(.D) \mapsto D$ from $L(A)$ to $H_1(s)$ reverses the ordering. Of course, similar considerations apply to $H_{-1}(s)$.

Having defined the lattice $L(A)$, we are ready to pick up a tool to measure its complexity.

Let L be an arbitrary modular lattice with 0 (the smallest element) and 1 (the largest element). Following Prest [17, Ch. 7], by induction on ordinals, we define an ascending chain $\sim_\eta$ of congruences on L, and the corresponding sequence of factor lattices $L_\eta = L/\sim_\eta$. Let $\sim_0$ be a trivial congruence, hence $L_0 = L/\sim_0 = L$. If $\sim_\eta$ and L_η have already been defined, let $\sim$ be the smallest congruence on L_η that identifies intervals of finite length, and let $\sim_{\eta+1}$ be the preimage of this congruence in L (with respect to the natural projection $L \to L_\eta$). Set $L_{\eta+1} = L/\sim_{\eta+1}$. For instance, $\sim_1$ identifies elements in any interval of finite length in L.

If η is a limit ordinal, then define $\sim_\eta = \bigcup_{\mu < \eta} \sim_\eta$ and set $L_\eta = L/\sim_\eta$. Since L has 0 and 1, it is easily seen that either $0 \nsim_\eta 1$ for all ordinals η, or there exists η such that $0 \nsim_\eta 1$ but $0 \sim_{\eta+1} 1$. In the former case we say that the *m-dimension* of L is undefined (or equals ∞); in the latter case we say that the m-dimension of L equals η. For instance, $\operatorname{mdim}(L) = 0$ iff L is a finite non-trivial lattice (it is customary to set $\operatorname{mdim}(L) = -1$ iff $0 = 1$ in L); and $\operatorname{mdim}(L) = \infty$ iff L contains a subchain isomorphic to the ordering of the rationals $(\mathbb{Q}, \leq)$.

Furthermore, the m-dimension of the chain $\omega + \omega^*$ is 1. Indeed, $\sim_1$ identifies all elements in ω, and all elements in ω^*, but nothing else. Thus $L_1 = L/\sim_1$ is a two-element lattice, therefore L_2 is a trivial lattice.

DEFINITION 3.1. The *Krull–Gabriel dimension* of A, $\operatorname{KG}(A)$, is defined to be the m-dimension of the lattice of all pp-formulae over A.

For a different (but equivalent) definition of the Krull–Gabriel dimension using Serre subcategories see Krause [15].

The pp-formulae as introduced above are usually called *left pp-formulae*, and a similar definition may be given for *right pp-formulae*. Fortunately (see [17, Prop. 13.1]) the lattices of left and right pp-formulae are anti-isomorphic, hence the definition of KG-dimension is left-right symmetric.

It is well known (see [17, Prop. 7.2.8]) that, for an arbitrary ring A, its Krull–Gabriel dimension equals 0 iff A is of finite representation type. On the other hand, if A is a non-domestic string algebra, then (see [28, Prop. 2]) $\operatorname{KG}(A)$ is undefined. Thus for string algebras we are left with the domestic case. It is also known (see [7]) that, if A is a hereditary tame finite-dimensional algebra which is not of finite representation type, then its Krull–Gabriel dimension equals 2. For instance, this applies to string algebras of type $\widetilde{A}_n$.

To give some idea of how the orderings $H_t(s)$ can be used when calculating the KG-dimension of string algebras, let us look at the Kronecker algebra $\widetilde{A}_1$. Suppose that $\alpha \in H_1(s)$ and consider the following subchain of $H_1(s)$:

$$1_{s,1} < \alpha\beta^{-1} < (\alpha\beta^{-1})^2 < (\alpha\beta^{-1})^3 < \cdots < (\alpha\beta^{-1})^2\alpha < \alpha\beta^{-1}\alpha < \alpha.$$
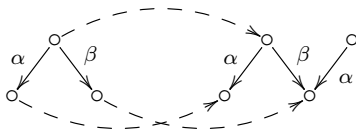
Now consider this chain as a set of pointed string modules $M(C)$, where we point the leftmost basis element (that is, z_0) of the canonical basis. For this,

$$M(\alpha\beta^{-1}) \qquad \overset{\alpha}{\swarrow} \overset{\circ}{\underset{\bullet}{}} \overset{\beta}{\searrow}_{\circ}$$
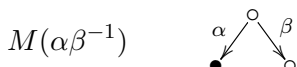
is an example. Clearly, if $C < D$ in $H_1(s)$, then there exists a pointed morphism $f_{c,d} : M(C) \to M(D)$, which is a graph map in the terminology of [3]. For instance, the morphism from $M(\alpha\beta^{-1})$ to $M(\alpha\beta^{-1}\alpha)$ is obtained

by just 'inserting' $\alpha\beta^{-1}$ at the beginning of $\alpha\beta^{-1}\alpha$.

Since the above modules $M(C)$ and $M(D)$ are indecomposable, finite-dimensional and not isomorphic, there is no pointed morphism from $M(D)$ to $M(C)$. Thus the pointed modules $(M(C), z_0)$, $C \in H_1(s)$, form a chain in $L(A)$ isomorphic to $\omega + \omega^*$. This chain has m-dimension 1, hence $\mathrm{KG}(\widetilde{A}_1) \geq 1$.

Similarly if $A = R_1$, then (see page 188) the subchain of $H_1(s)$ of strings starting with β has m-dimension 2, hence $\mathrm{KG}(R_1) \geq 2$. Both results are still short of the target (2 for $\widetilde{A}_1$ and 3 for R_1). To achieve the desired value we need some elaboration of this construction.

Recall that every ordinal η can be uniquely written in the *Cantor form* $\eta = \omega^{\eta_1} \cdot n_1 + \cdots + \omega^{\eta_k} \cdot n_k$, where $\eta_1 > \cdots > \eta_k$ are ordinals and $n_1, \ldots, n_k$ are natural numbers. If $\mu = \omega^{\eta_1} \cdot m_1 + \cdots + \omega^{\eta_k} \cdot m_k$ is another ordinal (we allow some n_i or m_i to be zero), then the *Cantor sum* of η and μ, $\eta \oplus \mu$, is the ordinal $\omega^{\eta_1} \cdot (n_1 + m_1) + \cdots + \omega^{\eta_k} \cdot (n_k + m_k)$. For instance $1 \oplus \omega = \omega \oplus 1 = \omega + 1$.

Let L_i, $i = \pm 1$, be chains such that the smallest element of L_i is 0_i, and the largest element of L_i is 1_i. We will denote by $L = L_{-1} \otimes L_1$ the modular lattice freely generated by L_{-1} and L_1 subject to the relations $0_{-1} = 0_1$ and $1_{-1} = 1_1$. For example, if L_1 is a two-element lattice, then $L_{-1} \otimes L_1 \cong L_{-1}$. The structure of L is well known (see [9, Thm. 13]), for instance L is distributive.

FACT 3.2 (see [17, Prop. 7.1.9]). $\mathrm{mdim}(L_{-1} \otimes L_1) = \mathrm{mdim}(L_{-1}) \oplus \mathrm{mdim}(L_1)$.

This kind of lattice occurs very naturally for string algebras. Namely, let A be a string algebra with a vertex s. For every $C \in H_1(S)$ we have already defined the formula $(.C)$. Furthermore, the set of pp-formulae $\{(.C) \mid C \in H_1(s)\}$ forms a chain anti-isomorphic to the chain $H_1(s)$. Some subsets of this chain will usually be used as L_1.

Similarly, the set of formulae $\{(D.) \mid D^{-1} \in H_{-1}(s)\}$ forms a chain anti-isomorphic to $H_{-1}(s)$, and we will use some subsets of this chain as L_{-1}.

If L is a sublattice of $L(A)$, then we denote by L' the lattice $L \cup \{0, 1\}$ obtained from L by adding the largest and the smallest elements.

The following proposition has a standard proof (see [23, L. 5.4] for this kind of argument).

PROPOSITION 3.3. *Let A be a string algebra with a vertex s. Suppose that L_1 is a chain of formulae $(.C)$, $C \in H_1(s)$, such that all C start with the same direct arrow, and L_{-1} is a chain of formulae $(D.)$, $D^{-1} \in H_{-1}(s)$, where all D end with the same inverse arrow (therefore DC is a string).*

Then the sublattice of $L(A)$ generated by the lattices $L'_{\pm 1}$ is isomorphic to $L'_{-1} \otimes L'_1$. In particular, the KG*-dimension of A is not less than* $\mathrm{mdim}(L'_{-1}) \oplus \mathrm{mdim}(L'_1)$.

Now we complete the calculations in the previous examples. Indeed, let $A = \widetilde{A}_1$ and let s denote the vertex where α ends. We may assume that $\alpha \in H_1(s)$ and $\beta \in H_{-1}(s)$. Let L_1 consist of all formulae $(.C)$ where C is a string with first letter α, and L_{-1} consist of all formulae $(D.)$ where the last letter of D is β^{-1}. Then both chains $L_{\pm 1}$ have m-dimension 1, therefore $\mathrm{KG}(\widetilde{A}_1) \geq 2$ by Proposition 3.3.

For $A = R_1$ recall (see page 188) that $\beta \in H_1(s)$ and $\alpha \in H_{-1}(s)$. Let L_1 be a chain of all formulae $(.C)$ where C is a string with first letter β; and let L_2 consist of all formulae $(D.)$ where D has last letter α^{-1} (hence D^{-1} will start with α). We know that $\mathrm{mdim}(L_1) = 2$ and it is easily seen that $\mathrm{mdim}(L_{-1}) = 1$. By Proposition 3.3 we conclude that $\mathrm{KG}(R_1) \geq 3$.

Note that for R_1 both chains $H_{\pm 1}(s)$ have m-dimension 2. However it is impossible to combine them to get 'independent dimensions' as in Proposition 3.3. A clear restriction on combining words on the left and right is given by the structure of 2-sided strings over R_1 (or rather by the relation $\alpha\beta = 0$).

For the definition of the bridge quiver of a string domestic algebra A the reader is referred to [29]. We just give a few examples.

For instance, let $A = R_1$ (see page 188) and recall that $C = \alpha\beta^{-1}$ is the unique band of A. Any nonperiodic string over A is a substring of either $^{\infty}C\alpha C^{-\infty}$ or $^{\infty}C\alpha^{-1}C^{-\infty}$. Thus the bridge quiver of A consists of two arrows $C \to C^{-1}$ labeled by α and α^{-1}. For instance, the maximal path in this bridge quiver is 1, and the same is true for any (non-degenerate) 1-domestic string algebra. Degenerate 1-domestic string algebras have a trivial bridge quiver.

Let us consider the following string algebra:

$$\Lambda_2 \qquad \circ \xleftarrow[\varepsilon]{\delta} \circ \xleftarrow{\gamma} \circ \xleftarrow[\beta]{\alpha} \circ$$
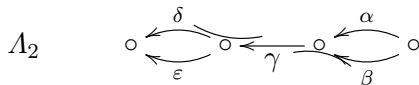
whose relations are shown by solid curves (that is, $\delta\gamma = 0$ and $\gamma\beta = 0$). Then $A = \Lambda_2$ has exactly two bands $C = \alpha\beta^{-1}$ and $D = \varepsilon\delta^{-1}$, therefore it is 2-domestic. The only 2-sided string that connects the bands of A is $^{\infty}D\varepsilon\gamma C^{\infty}$. Thus $D \xrightarrow{\varepsilon\gamma} C$ and (its inverse) $C^{-1} \xrightarrow{\gamma^{-1}\varepsilon^{-1}} D^{-1}$ are the only bridges over A. Again, the maximal path in the bridge quiver of A is 1.

The following conjecture is due to Schröer.

CONJECTURE 3.4 (see [29]). *Suppose that A is a domestic string algebra and let n be the length of a maximal path in the bridge quiver of A. Then the Krull–Gabriel dimension of A equals $n + 2$.*

According to this conjecture the KG-dimension of any 1-domestic string algebra A should be 3 if A is non-degenerate, and 2 if A is degenerate.

It follows from [29] that $\mathrm{KG}(A) \geq n + 2$, and this can also be proven using Proposition 3.3. To show that this estimate is sharp for 1-domestic algebras is the main goal of this paper.

Recall that Burke and Prest [1], for each $n \geq 2$, constructed a string algebra Λ_n whose KG-dimension equals n. A similar example was constructed by Schröer [28]. Furthermore, Prest and Puninski [21] verified that $\mathrm{KG} = 3$ for a special class of non-degenerate 1-domestic string algebras (whose bands have no self-intercepts).

Note that the KG-dimension of a domestic string algebra is a purely combinatorial invariant that is defined in terms of finite-dimensional modules and their morphisms. Thus it is quite plausible that one, especially advanced in combinatorics, could verify this conjecture 'with bare hands'. However, because of a very combinatorial nature, this kind of proof would be extremely difficult to write down and verify (maybe [28] is a good illustration for that). Thus one intention of the bypass we have chosen (through Ziegler spectrum and CB-ranks) is to give a 'structure' to this combinatorial proof.

4. Pure injective modules and the Ziegler spectrum. We will be brief on defining pure injective modules (more information can be found in [17] or [11]). A module M (over a finite-dimensional algebra A) is said to be *pure injective* if M is a direct summand of a direct product of finite-dimensional A-modules. If A is a string algebra, then those modules can be chosen to be string or band modules. For instance, every finite-dimensional module is pure injective. As was noticed by Ringel [24], if M is linearly compact as a (right) module over its endomorphism ring, then M is pure injective.
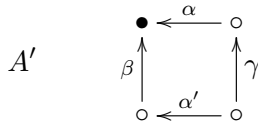
We will recall the structure of indecomposable pure injective modules over 1-domestic (non-degenerate) string algebras, as conjectured by Ringel [26] and confirmed in [22]. For a description of indecomposable pure injective modules over a tame hereditary finite-dimensional algebra A of type $\widetilde{A}_n$ the reader is referred to [25]. For instance, for every simple band A-module S, there exists a ray of irreducible monomorphisms $S = S(1) \to S(2) \to \cdots$ from a tube of A. The direct limit along this ray is an S-*Prüfer* module S_∞, which is pure injective and indecomposable.

Similarly the inverse limit along the coray of irreducible epimorphisms $\cdots \to S(2) \to S(1) = S$ produces an S-*adic* module $\widehat{S}$, which is also pure

injective and indecomposable. Finally, there exists an infinite-dimensional indecomposable *generic* module G of finite endolength. For instance for $\widetilde{A}_1$ the generic G is given by attaching to each vertex a copy of the rational field $k(x)$, where α acts as the identity and β acts by multiplication by x:

$$G \qquad k(x) \overset{\alpha=1}{\underset{\beta=x\times-}{\rightrightarrows}} k(x)$$

Suppose that A is a 1-domestic string algebra with a unique band C of length $n+1$ (that is, C has $n+1$ vertices). 'Recoiling C' we will obtain a tame hereditary algebra of type $\widetilde{A}_n$ (some vertices and arrows which are identified in C should be renamed). For instance, recoiling the band $C = \alpha\gamma\alpha^{-1}\beta^{-1}$ of R_2 (see page 189) we get an algebra A' of type $\widetilde{A}_3$ with the band $C' = \alpha\gamma\alpha'^{-1}\beta^{-1}$:

$$A' \qquad \begin{array}{ccc} \bullet & \overset{\alpha}{\longleftarrow} & \circ \\ \beta \uparrow & & \uparrow \gamma \\ \circ & \underset{\alpha'}{\longleftarrow} & \circ \end{array}$$

Then (see [6, p. 159]) there is a natural push-down functor F from the category of A'-modules to the category of A-modules which preserves pure injective indecomposable modules. Such modules in the image of this functor will be called *infinite-dimensional band modules*. We have not included in this list Prüfer and adic modules corresponding to simple regular A-modules which are strings, because we will count them separately.

Let $u = c_1 c_2 \ldots$ be a 1-sided string over A (or any domestic string algebra). Then u is almost periodic, therefore it can be uniquely written as $u = c_1 \ldots c_k D^\infty$, where D is a primitive cycle, and $c_k D^\infty$ is no longer periodic (or c_k is empty, hence u is periodic). If the last letter of D is a direct arrow, then we say that u is *contracting*. Then c_k is either empty, or an inverse arrow. For instance, the string $(\beta^{-1}\alpha)^\infty$ over $\widetilde{A}_1$ or R_1 is contracting:

$$\begin{array}{ccccccc} \circ & & \circ & & \circ & & \\ \beta \searrow & \nearrow \alpha & \beta \searrow & \nearrow \alpha & \beta \searrow & \\ & \circ & & \circ & & \circ & \end{array} \quad \ldots$$

As shown by Ringel [24], one could assign to this string the *direct sum module $M(u)$* which is pure injective and indecomposable. As an F-vector space, this module is a direct sum of countably many 1-dimensional spaces corresponding to circles in the above diagram, and the action is defined as for finite-dimensional string modules.

Similarly, we say that u is *expanding* if the last letter of D is an inverse arrow, therefore either c_k is empty, or c_k is a direct arrow. For instance, the

string $u = \beta(\alpha\beta^{-1})^\infty$ over R_1 is expanding, with $D = \alpha\beta^{-1}$:
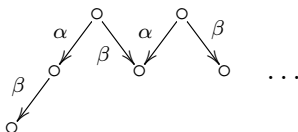
We will assign to u a *direct product* (or adic) module $\overline{M}(u)$, whose underlying vector space is the product $\prod_{i \in \omega} F$ of 1-dimensional vector spaces spanned by the z_i, therefore the z_i do not form a basis for $\overline{M}(u)$, and the action is defined similarly. Again, by Ringel [26, p. 50] (or rather Harland [10, Sect. 6.1]) this module is pure injective and indecomposable.

Applying a similar construction to a 2-sided string u we will obtain, depending on its shape, a *direct sum, direct product or mixed* pure injective indecomposable A-module.

For instance, the string $u = {}^\infty(\beta\alpha^{-1})\beta(\alpha\beta^{-1})^\infty$
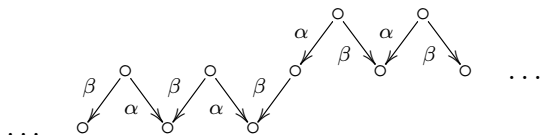
is contracting on the left and expanding on the right, therefore the corresponding 2-sided indecomposable pure injective module is $M^+(u)$, which is a direct product on the right and a direct sum on the left. The following result (Ringel's conjecture) was verified in [22].

PROPOSITION 4.1. *Let M be an infinite-dimensional indecomposable pure injective module over a 1-domestic (non-degenerate) string algebra A. Then M is either an infinite-dimensional band module, or a 1-sided or 2-sided direct sum, direct product, or mixed module corresponding to a 1-sided almost periodic or 2-sided biperiodic string.*

If A is degenerate (but not of finite representation type), then 2-sided (indecomposable pure injective) modules do not occur.

Let M be a module and $m \in M$. Let p^+ consist of all pp-formulae φ such that $M \models \varphi(m)$; and let p^- consist of all pp-formulae ψ such that $M \models \neg\psi(m)$. Then the collection $p = p^+ \cup \neg p^-$ is called the *pp-type* of m in M, $pp_M(m)$. Clearly p^+ is upward closed, and closed with respect to conjunctions, therefore it is a filter in the lattice of pp-formulae; and p^- is downward closed (but not always closed with respect to sums). Because p is uniquely determined by p^+, we will often identify p with its positive part, therefore with a filter in $L(A)$.

If M is indecomposable and pure injective, then (the isomorphism type of) M is uniquely determined by the pp-type p of any non-zero element

$m \in M$, in fact M is just the pure injective envelope, PE(p), of p. Recall that a pp-type is said to be *indecomposable* if the (pure injective) module PE(p) is indecomposable. There is a powerful syntactical criterion of Ziegler (see [33, Thm. 4.4]) for a given pp-type to be indecomposable. Thus one could classify indecomposable pure injective modules by describing the pp-types of their elements. In fact indecomposable pure injective modules are completely determined by their local behavior.

Namely, for pp-formulae φ and ψ, we will denote by $[\varphi/\psi]$ the interval $[\varphi \wedge \psi, \varphi]$ in the lattice $L(A)$. We say that a pp-type p *opens* this interval, written $p \in [\varphi/\psi]$, if $\varphi \in p^+$ and $\psi \in p^-$. In this case the restriction of p to $[\varphi/\psi]$ defines a (non-trivial) cut: if $\theta \in [\varphi/\psi]$, then we put θ in the upper part of this cut when $\theta \in p^+$; otherwise put θ in the lower part of the cut:
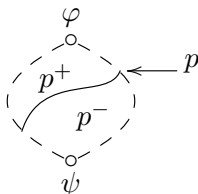

It follows from another result of Ziegler's (see [33, L. 7.10]) that the isomorphism type of an indecomposable pure injective module M is completely determined by any non-trivial cut given by a pp-type of a non-zero element of M.

Now we are ready for the main definition. The *Ziegler spectrum* of A is a topological space whose points are indecomposable pure injective modules, and a basis of open sets is given by $(\varphi/\psi) = \{M \in {}_A\mathrm{Zg} \mid \varphi(M)/(\varphi \wedge \psi)(M) \neq 0\}$, where φ and ψ are pp-formulae. The last condition says that there is $m \in M$ whose pp-type opens the interval $[\varphi/\psi]$, and we say that M *opens* $[\varphi/\psi]$ in this case.

It is known (see [17, Cor. 5.1.23]) that ${}_A\mathrm{Zg}$ is a quasi-compact space, whose basic open sets are also compact, but it is often the only nice topological property it enjoys. Though we defined the Ziegler spectrum using formulae in one variable, the use of arbitrary pp-formulae will lead to the same topology. Thus there is another useful definition of ${}_A\mathrm{Zg}$.

Let $f : M \to N$ be a (non-split) morphism between finite-dimensional modules and let a basic open set (f) consist of all $K \in {}_A\mathrm{Zg}$ such that there exists a morphism $g : M \to K$ which cannot be factored through f, that is, there is no morphism $h : N \to K$ with $hf = g$.
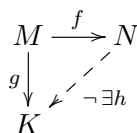
Now we will introduce a useful tool to measure the complexity of the Ziegler spectrum—the Cantor–Bendixson analysis. It runs as follows. At the first step remove from $_A\mathrm{Zg}$ all isolated points, that is, indecomposable finite-dimensional modules. What remains is a closed (hence compact) subset, Zg'_A, the first derivative of $_A\mathrm{Zg}$. Removing the isolated points again, we obtain the second derivative, and so on. Now we proceed by induction on ordinals, setting at limit stages $\mathrm{Zg}_A^{(\lambda)} = \bigcap_{\mu<\lambda} \mathrm{Zg}_A^{(\mu)}$. If this process reaches an empty set at some step μ then, by compactness, $\mu = \lambda + 1$ for some λ; and we define the *Cantor–Bendixson rank* of $_A\mathrm{Zg}$, $\mathrm{CB}(_A\mathrm{Zg})$, to be λ. For instance, $\mathrm{CB}(_A\mathrm{Zg}) = 0$ iff every point of A is isolated, therefore (by compactness) A is of finite representation type. Abusing notation slightly we will often write $\mathrm{CB}(A)$ for $\mathrm{CB}(_A\mathrm{Zg})$.

Otherwise we say that the CB-rank of $_A\mathrm{Zg}$ is undefined. If $\mathrm{CB}(A)$ is defined then, for every point M, there exists the least μ such that $M \in \mathrm{Zg}_A^{(\mu)} \setminus \mathrm{Zg}_A^{(\mu+1)}$. This ordinal μ is called the CB-*rank* of M, $\mathrm{CB}(M)$. Thus the CB-rank of $_A\mathrm{Zg}$ is the supremum of the CB-ranks of its points.

If V is an open set in $_A\mathrm{Zg}$, then the CB-rank of every point of V can be calculated inside V (endowed with the relative topology). We define the CB-rank of V, $\mathrm{CB}(V)$, as the supremum of the CB-ranks of points of V (in this paper we will be dealing only with finite ranks). If V is a closed subset of $_A\mathrm{Zg}$, then the CB-rank of a point calculated in the relative topology is often strictly less than its global CB-rank.

The usefulness of this notion comes from the following observation. Recall that the *isolation condition* for a ring A means that every isolated point in the closed subset of $_A\mathrm{Zg}$ is isolated by a minimal pair. For instance, by [17, Prop. 5.3.17] the isolation condition holds true if $\mathrm{KG}(A)$ is defined. It also follows from [17, Cor. 5.3.60] that $\mathrm{CB}(A) \leq \mathrm{KG}(A)$ for any ring A, and these values coincide if $\mathrm{KG}(A) < \infty$. For instance, if A is left pure semisimple, then $\mathrm{CB}(A) = 0$, and therefore the equality $\mathrm{CB}(A) = \mathrm{KG}(A)$ amounts to the famous pure semisimplicity conjecture (which most believe not to be true—see Simson [31] for the current state of affairs).

If A is a tame hereditary finite-dimensional connected algebra (of infinite representation type), we have already noticed that $\mathrm{KG}(A) = 2$. It follows that $\mathrm{CB}(A) = 2$ with a standard division of points of $_A\mathrm{Zg}$ into finite-dimensional (of CB-rank 0), Prüfer and adics (of CB-rank 1), and the unique generic point of CB-rank 2.

5. PP-types and their cuts. If p is an indecomposable pp-type then the CB-*rank of* p, $\mathrm{CB}(p)$, is the CB-rank of its pure injective envelope $\mathrm{PE}(p)$. There is also a corresponding notion of m-dimension.

Suppose that an indecomposable pp-type p opens an interval $[\varphi/\psi]$ with $\psi < \varphi$ and $M = \mathrm{PE}(p)$. Then (see [33, Cor. 4.16]) a basis of open neighborhoods for M can be chosen among open sets (φ'/ψ'), where $\varphi' \in p$, $\psi' \notin p$, and $\psi \leq \psi' < \varphi' \leq \varphi$. We define the *m-dimension* of p, $\mathrm{mdim}(p)$, as the infimum of the m-dimensions of intervals $[\varphi'/\psi']$ as above.

$$
\begin{array}{c}
\varphi \;\circ \\
\bullet\, \varphi' \\
\downarrow \\
p \longrightarrow \\
\uparrow \\
\bullet\, \psi' \\
\psi \;\circ
\end{array}
$$
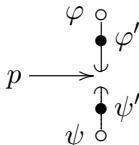
At least in some cases this definition is sound (does not depend on the interval chosen).

LEMMA 5.1. *Suppose that p is an indecomposable pp-type opening an interval $[\varphi/\psi]$. Then the m-dimension of the p measured in this interval is not less than $\mathrm{CB}(p)$.*

Furthermore, if $[\varphi/\psi]$ is a distributive interval, then $\mathrm{mdim}(p) = \mathrm{CB}(p)$, therefore $\mathrm{mdim}(p)$ does not depend on the interval. Also $\mathrm{mdim}[\varphi/\psi]$ is equal to the supremum of the CB-ranks of points opening this interval.

Proof. The Cantor–Bendixson analysis in $[\varphi/\psi]$ runs at least as fast as the m-dimension analysis, therefore the first part of the lemma holds true.

For the second part we notice that, because $[\varphi/\psi]$ is distributive, by [17, Cor. 5.3.28], the isolation condition holds true for points of $_A\mathrm{Zg}$ opening this interval. It follows that the CB-analysis in this interval coincides with the m-dimension analysis. Now the result is easily proved by induction. ∎

Here is a typical situation when this happens. Let A be a string algebra with a vertex s; $D^{-1} \in H_{-1}(s)$ is a non-maximal string ending at s with a direct arrow and $C \in H_1(s)$ is a string ending at s with a direct arrow. For instance, DC is a string, and $M(DC)$ is a free realization of the pp-formula $(D.C) = (D.) \wedge (.C)$. Recall that the string ^+D is obtained from D by adding an inverse arrow on the left (we assume for simplicity that such an arrow exists), and then as many direct arrows as possible: $^+D = \ldots \gamma\beta^{-1}D$. Clearly $(^+D.C) < (D.C)$ in the lattice of pp-formulae.

FACT 5.2 (see [29] and [20]). *The interval $[(D.C)/(^+D.C)]$ is a chain and each formula in this interval, except $(^+D.C)$, is equivalent to the formula $(^+D.C) + (D.E)$, where $C \leq E \in H_1(s)$.*

Furthermore, every indecomposable pure injective module $M = N(p)$ in this interval is uniquely determined by a 1-sided string $u \in \widehat{H}_1(s)$, $C \leq u$. A basis of open sets for M is given by the pairs $(D.E)/(^+D.C) + (D.F)$, where $E, F \in H_1(s)$ are such that $C \leq E \leq u < F$. Therefore $\mathrm{CB}(p)$ is equal to the m-dimension of the cut defined by p on the chain $H_1(s)$.

For instance, let $A = R_1$ and consider the following cut p (between the dots) on $H_1(s)$:

$$1_{s,-1} < \beta\alpha^{-1} < (\beta\alpha^{-1})^2 < \cdots \quad \cdots < \beta\alpha^{-1}\beta < \beta.$$

By what we have said, this cut defines an indecomposable pp-type (also denoted by p). Because (see page 188) every interval in the lower and upper part of this cut can be refined to a chain of type $\omega + \omega^*$, it follows that its m-dimension is 2, hence the CB-rank of $M = \mathrm{PE}(p)$ equals 2. Note that M can be easily identified with the direct product (adic) module $\overline{M}(\beta\alpha^{-1})^\infty$ from Ringel's list, because this module defines the same cut on $H_1(s)$:
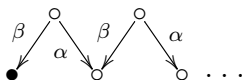
Similarly let us consider the following cut in $H_1(s)$ over R_1:

$$\beta\alpha^{-1}\beta < \beta\alpha^{-1}\beta\alpha\beta^{-1} < \cdots \quad \cdots < \beta\alpha^{-1}\beta\alpha\beta^{-1}\alpha < \beta\alpha^{-1}\beta\alpha.$$

Since every interval in the lower and upper parts of this cut is simple, its m-dimension is 1. If an indecomposable pp-type q corresponds to this cut and $M = \mathrm{PE}(q)$, then $\mathrm{CB}(M) = 1$. Clearly M is the direct product module $\overline{M}(\beta\alpha^{-1}\beta(\alpha\beta^{-1})^\infty)$:
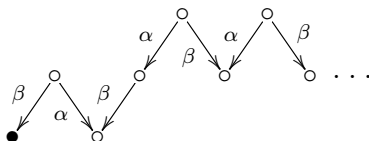
The explanation why the CB-rank of $\overline{M}(\beta\alpha^{-1})^\infty$ is 2 is the following: one could 'see ahead' one more bridge leading from this band to the band $\alpha\beta^{-1}$ on the right, and this is not the case for $(\alpha\beta^{-1})^\infty$. For a general receipt how to calculate the CB-ranks of such (1-sided) modules over domestic string algebras see [20].

If a module M opens a certain interval $[(D.C)/(^+D.C)]$, then M is called 1-*sided* (see [20]); otherwise M is said to be 2-*sided*. For instance, every finite-dimensional string module is 1-sided, but each band module is 2-sided. Furthermore, an infinite-dimensional string module from Ringel's list is 1-sided iff its defining string is 1-sided (see [10] for a proof). Using Fact 5.2 and calculations of Schröer [29], it is not difficult to find the CB-rank of any point in the interval $[(D.C)/(^+D.C)]$. Therefore, by Lemma 5.1, we know the m-dimension of this interval.

For instance, in this way it was shown in [20] that over a domestic string algebra A, the CB-rank of the (open set) of 1-sided points of $_A\mathrm{Zg}$ is $n+1$, where n is the maximal length of a path in the bridge quiver of A. From this it was derived that $\mathrm{CB}(A) \geq n+2$.

Suppose that $[\varphi/\psi]$ is an interval in $L(A)$ generated by two chains $L_{\pm 1}$. To conform with the above definitions, let L_i' be obtained from L_i by adding φ as a largest element (if φ is not already there), and adding ψ (or rather $\varphi \wedge \psi$) as a smallest element. Then there exists a natural surjection of lattices $L_{-1}' \otimes L_1' \to [\varphi/\psi]$. If this surjection is an isomorphism, we say that $[\varphi/\psi]$ is *freely generated* by $L_{\pm 1}$. Let p_i denote the cut on L_i defined by a pp-type $p \in [\varphi/\psi]$, that is, we put $\theta \in L_i$ in the upper part of the cut if $\theta \in p$, and in the lower part otherwise.

FACT 5.3 (see [22, Fact 8.4]). *Suppose that an interval $[\varphi/\psi]$ is freely generated by chains $L_{\pm 1}$ and let $p \in [\varphi/\psi]$ be an indecomposable pp-type. Then p is uniquely determined by its cuts $p_{\pm 1}$. Furthermore,* $\mathrm{mdim}(p) = \mathrm{mdim}(p_{-1}) \oplus \mathrm{mdim}(p_1)$ *and* $\mathrm{mdim}(p) = \mathrm{CB}(p)$.

Thus our main interest will be to 'catch' indecomposable pure injective modules in distributive intervals, making it easy to calculate their CB-ranks. But constructing such (distributive) intervals is hard. We will give a few examples.

Recall that every pp-formula $\varphi(x)$ defines a finitely generated subfunctor F_φ of the forgetful functor $\mathrm{Hom}(A, -)$, from the category of finite-dimensional left A-modules to the category of F-vector spaces, by the rule $M \mapsto \varphi(M)$; and the converse is also true. In what follows we will often identify pp-formulae and such functors.

EXAMPLE 5.4 (see [22, Prop. 8.5]). Let $S = M(\beta)$ be the following (simple regular) string module over R_1, pointed at the left end:
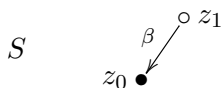
and consider the functor $\mathrm{Hom}(S, -)$. Since (S, z_0) is a free realization of the pp-formula $\varphi(x) \doteq \exists y \, (\beta y = x \wedge \alpha y = 0)$, this functor is isomorphic to φ. Then the lattice of finitely generated subfunctors of $\mathrm{Hom}(S, -)$ (therefore the interval $[\varphi/(x = 0)]$ in $L(A)$) is freely generated by two chains corresponding to extensions of β to the right and to the left in the following 2-sided string $^\infty(\beta\alpha^{-1})\beta(\alpha\beta^{-1})^\infty$:
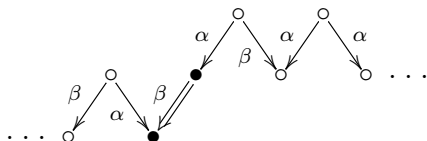
Namely, looking at the extension of β to the right we see a chain L_1 in $H_1(s)$ consisting of strings (or rather pointed string modules) with $\beta\alpha$ as

first letters, and β itself. Similarly, extending β to the left we see a chain L_{-1} in $H_{-1}(s)$ consisting of strings that start with α (that is, strings $\ldots\alpha^{-1}$ in the orientation of the above diagram); and these two chains generate our interval freely.

It follows that every indecomposable pure injective R_1-module M such that $\mathrm{Hom}(S, M) \neq 0$ is uniquely determined by a 1-sided or 2-sided string as above with β embedded in the middle. For instance, there exists a unique indecomposable pure injective module over R_1 that corresponds to the 2-sided string $u = {}^\infty(\beta\alpha^{-1})\beta(\alpha\beta^{-1})^\infty$ (see the diagram). Since u is contracting on the left and expanding on the right, this module is a mixed module $M^+(u)$ from Ringel's list. We could calculate its CB-rank. Indeed, the (right hand) cut p_1 defined by p on $H_1(s)$ is of the form

$$\beta < \beta\alpha\beta^{-1} < \cdots \quad \cdots < \beta\alpha\beta^{-1}\alpha < \beta\alpha,$$

hence of m-dimension 1. Furthermore, the cut p_{-1} defined by p on $H_{-1}(s)$,

$$\alpha\beta^{-1} < (\alpha\beta^{-1})^2 < \cdots \quad \cdots < \alpha\beta^{-1}\alpha < \alpha,$$

is also of m-dimension 1. By Fact 5.3 it follows that $\mathrm{mdim}(p) = 1 + 1 = 2$, therefore $\mathrm{CB}(p) = 2$ by the same fact.

By similar arguments one can show that, if S is any of the remaining simple regular R_1-modules (meaning simple regular as $\widetilde{A}_1$-modules), then the lattice of finitely generated subfunctors of $\mathrm{Hom}(S, -)$ is a chain of m-dimension 2. These chains may be considered as 'deformations' of the above free product of two chains.

However, there is a 1-domestic string algebra with a simple regular module T such that the lattice of finitely generated subfunctors of $\mathrm{Hom}(T, -)$ is not distributive.

EXAMPLE 5.5. Let S_2 be the following (non-degenerate) 1-domestic string algebra:
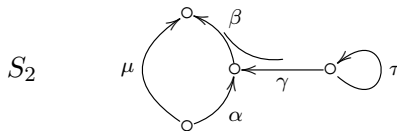
with relations $\beta\gamma = 0$, $\tau^2 = 0$, and (the unique) band $\beta\alpha\mu^{-1}$. Let T be the following (simple regular) string S_2-module:
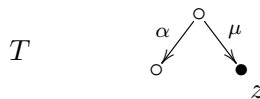
Then the lattice of finitely generated subfunctors of $\mathrm{Hom}(T, -)$ is not distributive.

Proof. Consider the graph maps that embed T in the following string module N such that z goes first to x, and then to y:
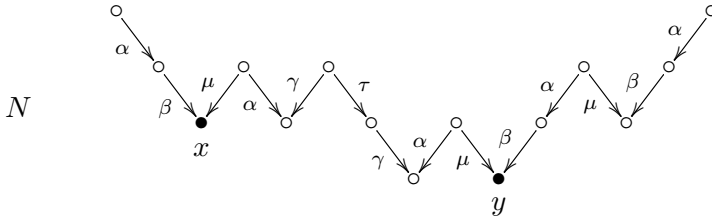
Clearly the pp-types of x and y in N are incomparable (equivalently, no endomorphism of N takes x to y or y to x). Indeed, we may assume that $\beta \in H_1(s)$ and $\mu \in H_{-1}(s)$. Then x has a word $\beta\alpha \in H_1(s)$ larger than the word $\beta\alpha\mu^{-1}\ldots \in H_1(s)$ of y, but y has a word $\mu\alpha^{-1}\gamma\tau\ldots \in H_{-1}(s)$ larger than the word $\mu\alpha^{-1}\gamma\tau^{-1}\ldots \in H_{-1}(s)$ of x. But this contradicts [21, Prop. 4.4]. ∎

However we do not know any example of a simple band module S (which is not a string module) over a 1-domestic algebra such that the functor $\mathrm{Hom}(S, -)$ is not distributive.

6. Main results. In this section we will prove the main result of the paper.

THEOREM 6.1. *Suppose that A is a non-degenerate 1-domestic string algebra. Then the Krull–Gabriel dimension and the Cantor–Bendixson rank of A are equal to* 3.

The proof of this result will occupy a few pages. First we will calculate the CB-rank of A. Recall that every non-periodic 2-sided string over A is of the form $^\infty CUC^{-\infty}$, where the length of U is uniformly bounded, and there is no string of the form $^{-\infty}CVC^\infty$.

As we have already mentioned, the points in $_A\mathrm{Zg}$ of CB-rank 0 are exactly indecomposable finite-dimensional modules, therefore each infinite-dimensional point has CB-rank ≥ 1. Furthermore (see [20]), the ranks of 1-sided points of $_A\mathrm{Zg}$ are also known. Namely, the Prüfer and adic modules corresponding to strings WC^∞ will have CB-rank 2; and the remaining 1-sided (infinite-dimensional) points are of CB-rank 1.

Similar considerations holds for 2-sided string modules.

LEMMA 6.2. *Let M be a 2-sided direct sum, direct product, or mixed module from Ringel's list. Then* $\mathrm{CB}(M) = 2$.

Proof. It follows from [22, proof of L. 9.2] that M opens an interval freely generated by two chains.

More precisely, there are strings $C \in H_1(s)$, $D^{-1} \in H_{-1}(s)$ starting with direct arrows such that M opens the interval $[(D.C)/\sum_{i=1}^k (D_i.C_i)]$, where $(D_i.C_i)$ are all 'proper factors' of $C.D$. Furthermore, this interval is freely generated by the chains $L_{\pm 1}$, where L_1 corresponds to the extensions of C to the right, and L_{-1} corresponds to the extensions of D to the left.

For an example, let $u = \beta\alpha^{-1}\beta.\alpha\beta^{-1}$ over R_1:
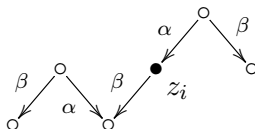
Then the complete list of proper factors of this string (not annihilating the pointed element) is $\alpha^{-1}\beta.\alpha\beta^{-1}$ or $.\alpha\beta^{-1}$ (factoring u on the left end), and $\beta\alpha^{-1}\beta.\alpha$ (factoring u on the right end). Thus L_1 corresponds to the extensions of $\alpha\beta^{-1}$ to the right (that is, either equals $\alpha\beta^{-1}$ or starts with $\alpha\beta^{-1}\alpha$), and L_{-1} corresponds to the extensions of $\beta\alpha^{-1}\beta$ to the left. For instance, the mixed module $M^+({}^\infty(\beta\alpha^{-1})\beta.(\alpha\beta^{-1})^\infty)$ opens this interval.

It is easily seen that the cut defined by the chain L_1 on $H_1(s)$ has m-dimension 1, and the same is true for the cut defined by L_{-1} on $H_{-1}(s)$. It follows from Fact 5.3 that $\mathrm{mdim}(M) = \mathrm{CB}(M) = 2$. $\blacksquare$

It remains to calculate the ranks of Prüfer and adic points corresponding to simple band modules S, and the rank of the generic point G. Let a simple band module S correspond to the band $C = \alpha E \beta^{-1}$, and point this module at a standard basis element z_i between β^{-1} and α (therefore, z_i is in the socle of S). In fact, by combinatorics of domestic string algebras there is just one position for z_i in the socle of S. Then (S, z_i) is a free realization of a pp-formula φ_s whose corresponding functor is isomorphic to $\mathrm{Hom}(S, -)$.

PROPOSITION 6.3. $\mathrm{CB}(S_\infty) \leq 2$.

Proof. We will check that $\mathrm{Hom}(S, -)$ provides a sufficient separation.

Clearly $\mathrm{Hom}(S, S_\infty) \neq 0$, therefore M opens the interval $[\varphi_s/x = 0]$. Furthermore $\mathrm{Hom}(S, \widehat{S}) = \mathrm{Hom}(S, G) = 0$ and $\mathrm{Hom}(S, T_\infty) = \mathrm{Hom}(S, \widehat{T}) = 0$ for any simple band module T which is not isomorphic to S, because this is true for the corresponding tame hereditary algebra. Thus we have separated S from these points (we need not care what their CB-ranks are).

By Proposition 4.1 it remains to show that this functor separates S_∞ from infinite-dimensional string points of CB-rank 2, that is, any morphism f from S to such a point will annihilate z_i.

Recall that there are just two kinds of (string) points of CB-rank 2: 2-sided points M_u corresponding to strings $u = {}^\infty CUC^{-\infty}$, and 1-sided points corresponding to strings WC^∞ (and each of them is Prüfer, adic or

mixed, depending on the shape of the string). We will give the proof only
for 2-sided points; arguments for 1-sided points of CB-rank 2 are similar.

Thus let $f \in \mathrm{Hom}(S, M)$ target a 2-sided module $M = M_u$. Consider
first the case when u is contracting, that is, M is a direct sum module (on
both ends). It follows that the image of h_j lies in a submodule $M' = M(v)$
of M defined by a finite substring $v = CVC^{-1}$ (when V is chosen long
enough).

We will use a description of morphisms between band and string modules
(possibly infinite-dimensional). For finite-dimensional modules this result is
due to Krause [13], but even in this case its interpretation given by Har-
land [10, Sect. 6.3.2] is more handy. Thus f first factors through a standard
morphism g from S to the periodic direct product module $\overline{M}(^\infty C^\infty)$, say
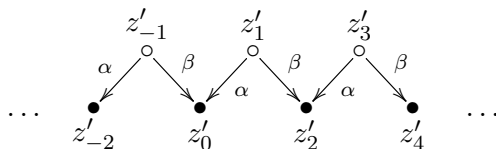$f = gh$:

$$S \xrightarrow{\ g\ } \overline{M}(^\infty C^\infty) \xrightarrow{\ h\ } M'$$
$$f$$

where g is a standard embedding, and h is a (finite) linear combination
$\sum_j \lambda_j h_j$ of graph maps.

To see what this morphism g is, suppose that $A = \widetilde{A}_1$, $C = \alpha\beta^{-1}$, and
let S be the simple band module determined by $0 \neq \lambda \in F$:

$$S \qquad z_1 \circ \overset{\alpha=1}{\underset{\beta=\lambda\times}{\rightleftarrows}} \bullet z_0$$
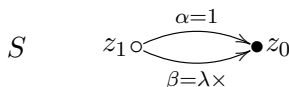
Let $\overline{M}(^\infty C^\infty)$ be the direct product module

Then $g(z_0) = \sum_i \lambda^i z'_{2i}$ and $g(z_1) = \sum_i \lambda^i z'_{2i+1}$.

Now each h_j is obtained by first factoring the string $w = {}^\infty C^\infty$ on both
ends, and then inserting the resulting finite string (or rather the resulting
string module) into M' (and hence in M). For instance in the above ex-
ample, one possibility for h_j would be to factor $\overline{M}(^\infty C^\infty)$ on the left of z'_{-2}
(inclusive) and on the right of z'_4 (inclusive) to get the following string
module:
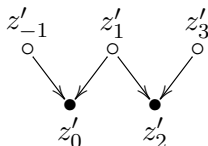
Clearly we can disregard the h_j which annihilate z_i. Now consider a certain h_j which does not annihilate z_i. The corresponding factor of $w = {}^\infty C^\infty$ can be written as $\not{\gamma} UC^m.C^n V\not{\delta^{-1}}$, where γU is the right end of C^2 and $V\delta^{-1}$ is the left end of C^2. Here γ is a direct arrow such that the standard basis element on the left of γ is the rightmost element annihilated by h_j; and δ^{-1} is an inverse arrow such that the standard basis element on the right of it is the leftmost element annihilated by h_j.

For instance, in the above example, γ is α connecting z'_{-2} and z'_{-1}, and δ^{-1} is β^{-1}, connecting z'_3 and z'_4.

Clearly U ends with β^{-1} (on the right), hence is non-empty, and V starts with α (on the left), hence is also non-empty. However the case $m = 0$ or $n = 0$ is quite possible.

By symmetry we may assume that UC^mC^nV is embedded in u with the 'same orientation' as $\ldots \varepsilon^{-1} UC^{m+n} V\pi \ldots$. Without loss of generality we may assume that $m, n \geq 1$: otherwise we can insert copies of C (in w and in u) in between U and V, without violating the following arguments.

The left end of u is ${}^\infty C \ldots$, therefore the part of u between ${}^\infty C$ and C^m is C^l for some l. It follows that $\varepsilon^{-1} U$ will be on the right end of C^l. But the same is true for γU, therefore $\varepsilon^{-1} = \gamma$, a contradiction.

Now suppose that u is mixed, so $u = w\gamma D^\infty$, where D is a cyclic permutation of C or C^{-1}. Then $N = M_w$ is a submodule of $M = M_u$ such that M/N is a direct product module $\overline{M}(D^\infty)$. Since M/N is an adic module corresponding to the simple regular string module $M(D)$, it follows that $\mathrm{Hom}(S, M/N) = 0$, therefore the image of $f : S \to M$ lies in the string submodule $N = M(w)$ (here $M_w = M(w)$, because the other end of u is contracting). Now, as in the previous case, we conclude that $f = 0$.

The case when u is expanding (that is, M is a direct product module) is considered similarly. ∎

We will derive a similar result for adic points.

COROLLARY 6.4. $\mathrm{CB}(\widehat{S}) \leq 2$.

Proof. Here is the only place where we will use the right Ziegler spectrum, Zg_A, of A, whose definition is essentially the same, but one has to use right A-modules.

Recall (see [17, Sect. 1.3.1]) that there is an anti-isomorphism D between lattices of left and right pp-formulae over A (for instance, the left divisibility formula $a \,|\, x$, $a \in A$, will go to the right annihilator pp-formula $xa = 0$). It was shown by Herzog (see [17, Cor. 5.12]) that D induces an

anti-isomorphism between lattices of open (and closed) subsets of Ziegler spectra. Under this anti-isomorphism, the basic open set (φ/ψ) will go to the basic open set $(D\psi/D\varphi)$.

In general it is not clear how to define this map pointwise, but it can be arranged for certain points (called reflexive) in Ziegler spectra (see [17, p. 272] or [15, Sect. 4.3] for the definition). For instance, the adic point $\widehat{S}$ will be sent (consistent with topology) to the Prüfer point $(DS)_\infty$, where $DS = \mathrm{Hom}_F(S, F)$ is a simple band right A-module, the dual of S. From what we have proved (or rather applying the same arguments on the right side) it follows that $(DS)_\infty$ has CB-rank not more than 2; therefore $\mathrm{CB}(\widehat{S}) \leq 2$ considering topology. ∎

Now we are in a position to calculate the CB-rank of A. Namely (as we already mentioned), at level 2 of the CB-analysis we have removed all 1-sided and 2-sided points, and also (by Proposition 6.3 and Corollary 6.4) all Prüfer and adic points. By the classification of points (see Proposition 4.1) only the generic point G remains. It will certainly get isolated at level 3, therefore $\mathrm{CB}(_A\mathrm{Zg}) \leq 3$. On the other hand, from the description of the Ziegler spectrum of the corresponding algebra $\widetilde{A}_n$, it is known that G cannot be separated from the adic point $M = \overline{M}(C^\infty)$ (because G is a direct summand of a direct limit of copies of M), therefore the same is true over A. Since $\mathrm{CB}(M) = 2$, it follows that $\mathrm{CB}(G) = 3$.

To complete the proof of Theorem 6.1, by [17, Sect. 5.3], it suffices to check that A satisfies the isolation condition.

LEMMA 6.5. *The isolation condition holds true for the* (*left*) *Ziegler spectrum of a* (*non-degenerate*) *1-domestic string algebra.*

Proof. By [17, Prop. 5.3.16] it suffices to check the isolation condition locally: each point isolated in some closed subset of $_A\mathrm{Zg}$ is isolated by a minimal pair in its closure. Furthermore, by [17, Prop. 7.3.16] it is enough to show that every point of $_A\mathrm{Zg}$ opens a distributive interval. By Fact 5.2 this is the case for 1-sided points, and [22, proof of L. 9.2] (see the proof of Lemma 6.2) shows that the same is true for points corresponding to 2-sided strings.

Thus we may assume that $M \in {}_A\mathrm{Zg}$ is either an S-Prüfer or an S-adic module corresponding to a simple band module S, or M is the generic module G. By [16, Cor. 2.10], M has a definable structure of a (pure injective indecomposable) module over a noetherian serial ring A_Σ (which is a universal localization of A—see [4] for this theory). If e_s is a basic idempotent of A and $\overline{e}_s$ denotes its image in A_Σ, then the projective module $A\overline{e}_s$ is a direct sum of isomorphic copies of an indecomposable (projective) A_Σ-module. It follows from [18, L. 11.4] that the interval $[\overline{e}_s x = 0/x = 0]$ in

$L(A_\Sigma)$ is a chain, hence distributive. Then the same is true for the interval $[e_s x = 0/x = 0]$ in $L(A)$, when evaluated at M. ∎

The case when A is degenerate is much easier.

PROPOSITION 6.6. *Suppose that A is a degenerate 1-domestic string algebra. Then the Krull–Gabriel dimension and the Cantor–Bendixson rank of A equal 2.*

Proof. Since A is degenerate, every non-periodic string over A is 1-sided. It follows that every infinite-dimensional string point of $_A\mathrm{Zg}$ is 1-sided and has CB-rank 1. An analysis similar to Proposition 6.3 and Corollary 6.4 shows that the CB-rank of any S-Prüfer and any S-adic point for a simple band module S equals 1. Again, the remaining point G is generic of CB-rank 2.

Now the equality $\mathrm{CB}(A) = \mathrm{KG}(A)$ is checked as above. ∎

The following corollary is a straightforward consequence of the existence of KG-dimension.

COROLLARY 6.7. *Suppose that A is a 1-domestic string algebra. Then $_A\mathrm{Zg}$ is a T_0-space, that is, it has no topologically indistinguishable points. Furthermore, A has no superdecomposable pure injective module.*

However we are still a bit short of completely describing the topology of $_A\mathrm{Zg}$. Here is the main question.

QUESTION 6.8. *Suppose that A is a non-degenerate 1-domestic string algebra and S is a simple band module. Is it true that $\mathrm{CB}(S_\infty) = \mathrm{CB}(\widehat{S}) = 2$?*

It is not difficult to see that this question has an affirmative answer for $A = R_1$.

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