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ON TWISTED GROUP ALGEBRAS OF OTP REPRESENTATION TYPE

BY

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Abstract. Assume that S is a commutative complete discrete valuation domain of characteristic p, S^* is the unit group of S and $G = G_p \times B$ is a finite group, where G_p is a p-group and B is a p'-group. Denote by $S^\lambda G$ the twisted group algebra of G over S with a 2-cocycle $\lambda \in Z^2(G, S^*)$. We give necessary and sufficient conditions for $S^\lambda G$ to be of OTP representation type, in the sense that every indecomposable $S^\lambda G$-module is isomorphic to the outer tensor product $V \# W$ of an indecomposable $S^\lambda G_p$-module V and an irreducible $S^\lambda B$-module W.

0. Introduction. In [10], Brauer and Feit proved that if $G = G_p \times B$ and L is an algebraically closed field of characteristic p, then the group algebra LG is of OTP representation type. Blau [9] and Gudyvok [17, 18] independently showed that if L is an arbitrary field of characteristic p, then LG is of OTP representation type if and only if G_p is cyclic or L is a splitting field for B. Gudyvok [19, 20] also investigated a similar problem for group algebras RG, where R is a commutative complete discrete valuation domain. In particular, he proved that if R is of characteristic p and F is the quotient field of R, then RG is of OTP representation type if and only if $|G_p| = 2$ or F is a splitting field for B. In [3]–[6], the results of Blau and Gudyvok are generalized to twisted group algebras $L^\lambda G$, where L is either a field of characteristic p, or a commutative complete discrete valuation domain of characteristic p and $\lambda \in Z^2(G, L^*)$ satisfies a specific condition. Let L be a field of characteristic p. The main theorem in [4] asserts that, under suitable assumptions, $L^\lambda G$ is of OTP representation type if and only if $L^\lambda G_p$ is a uniserial algebra or L is a splitting field for $L^\lambda B$. In [5], necessary and sufficient conditions on G and L were given for G to be of OTP projective L-representation type, in the sense that there exists a cocycle $\lambda \in Z^2(G, L^*)$ such that $L^\lambda G$ is of OTP representation type. Let $L = K[[X]]$ be the ring of formal power series in the indeterminate X with coefficients in a field K of characteristic p. Twisted group algebras

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$L^\lambda G$ of OTP representation type with $\lambda \in Z^2(G, K^*)$ were described in [3, 6].

The reader is referred to [22, p. 66] for a definition of the twisted group algebra.

In the present work we determine new classes of twisted group algebras $S^\lambda G$ of OTP representation type, where $G = G_p \times B$ and S is a commutative complete discrete valuation domain of characteristic $p \geq 2$.

By [28, p. 307], S is isomorphic to the ring $K[[X]]$, where K is a field of characteristic p. Throughout this paper, S denotes $K[[X]]$ and T the quotient field of S. By a *principal unit* in S we understand an element $f(X) \in S$ such that $f(X) \equiv 1 \pmod{X}$. Denote by S_0^* the group of principal units of S. Then $S^* = K^* \times S_0^*$. Let q be a prime and $q \neq p$. Then $(S_0^*)^q = S_0^*$. Moreover S_0^* does not contain a primitive qth root of 1. By [22, Theorem 1.7, p. 11], every 2-cocycle $\sigma \in Z^2(B, S_0^*)$ is a coboundary. Hence each 2-cocycle $\tau \in Z^2(B, S^*)$ is cohomologous to a 2-cocycle $\nu \in Z^2(B, K^*)$. Therefore we shall use as a rule only K^*-valued 2-cocycles of B. We also assume that if G_p is non-abelian, then $[K(\xi) : K]$ is not divisible by p, where ξ is a primitive $(\exp B)$th root of 1.

Let us briefly present the main results obtained. Let $p \neq 2$, $G = G_p \times B$, Ω be the subgroup of S^* generated by K^* and $(S^*)^p$, $\mu \in Z^2(G_p, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. We prove in Theorem 3.5 that the algebra $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:

(i) G_p is abelian and $T^\mu G_p$ is a field;
(ii) K is a splitting field for $K^\nu B$.

Assume now that $p = 2$, $G = G_2 \times B$, $|G_2'| \neq 2$; Ω is the subgroup of S^* generated by K^* and $(S^*)^4$; $\mu \in Z^2(G_2, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. We show in Theorem 3.9 that the algebra $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:

(i) G_2 is abelian and $\dim_T(T^\mu G_2/\operatorname{rad} T^\mu G_2) \geq |G_2|/2$;
(ii) K is a splitting field for $K^\nu B$.

We obtain similar results also in the case when Ω is a subgroup of S^* generated by K^* and $f(X)$, where $f(X) \equiv 1 \pmod{X}$ and $f(X) \not\equiv 1 \pmod{X^2}$ (Theorem 3.10).

We note that the results derived are generalizations of Proposition 3 in [6], where $\mu \in Z^2(G_p, K^*)$.

Throughout the paper, we use the standard group representation theory notation and terminology introduced in the monographs by Alperin [1], Benson [7], Curtis and Reiner [11, 12, 13], and Karpilovsky [21, 22]. The books by Karpilovsky give a systematic account of the projective representation theory. For problems and solutions of the representation theory of orders in finite-dimensional algebras, we refer to the books by Curtis and Reiner. A background of the representation theory of finite-dimensional algebras can be found in the monographs by Assem, Simson and Skowroński [2], Drozd and Kirichenko [15], Simson [24], and Simson and Skowroński [27], where among other things the representation types (finite, tame, wild) of finite groups and algebras are discussed. Various aspects of representation types are considered also in the papers by Dowbor and Simson [14], Simson [25], and Simson and Skowroński [26].

1. Preliminaries. Throughout this paper, we use the following notation: K is a field of characteristic p; K^* is the multiplicative group of K, $S = K[[X]]$, $S^l = \{a^l : a \in S\}$, S^* is the unit group of S, $(S^*)^l = \{a^l : a \in S^*\}$, T is the quotient field of S; $G = G_p \times B$ is a finite group, where G_p is a Sylow p-subgroup and $|G_p| > 1$, $|B| > 1$; G'_p is the commutator subgroup of a group G_p; e is the identity element of a group H, $|h|$ is the order of $h \in H$; $\operatorname{soc} A$ is the socle of an abelian group A and $\exp H$ is the exponent of H. Unless stated otherwise, we assume that if G_p is non-abelian, then $[K(\xi) : K]$ is not divisible by p, where ξ is a primitive $(\exp B)$th root of 1. We assume also that all cocycle groups are defined with respect to the trivial action of the underlying group on S^*. An S-basis $\{u_h : h \in H\}$ of $S^\lambda H$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in H$ is called *canonical* (corresponding to $\lambda \in Z^2(H, S^*)$). We often identify γu_e with $\gamma \in S$. If D is a subgroup of H, then the restriction of $\lambda \in Z^2(H, S^*)$ to $D \times D$ will also be denoted by λ. We suppose that in this case $S^\lambda D$ is the S-subalgebra of $S^\lambda H$ consisting of all S-linear combinations of the elements $\{u_d : d \in D\}$, where $\{u_h : h \in H\}$ is a canonical S-basis of $S^\lambda H$ corresponding to λ. Given $\lambda \in Z^2(H, K^*)$, $K^\lambda H$ denotes the twisted group algebra of H over K and $\overline{K^\lambda H}$ the quotient algebra of $K^\lambda H$ by the radical $\operatorname{rad} K^\lambda H$.

Let $G_p = \langle a_1 \rangle \times \cdots \times \langle a_m \rangle$ be an abelian p-group of type $(p^{n_1}, \ldots, p^{n_m})$. For each cocycle $\mu \in Z^2(G_p, S^*)$, the algebra $S^\mu G_p$ is commutative. The algebra $S^\mu G_p$ has a canonical S-basis $\{v_g : g \in G\}$ satisfying the following conditions:

(1) if $g = a_1^{j_1} \cdots a_m^{j_m}$ and $0 \le j_i < p^{n_i}$ for every $i \in \{1, \ldots, m\}$, then
$$v_g = v_{a_1}^{j_1} \ldots v_{a_m}^{j_m};$$

(2) $v_{a_i}^{p^{n_i}} = \gamma_i v_e$, where $\gamma_i = \mu_{a_i,a_i} \mu_{a_i,a_i^2} \cdots \mu_{a_i,a_i^{r_i}}$, $r_i = p^{n_i} - 1$.

We denote the algebra $S^\mu G_p$ also by $[G_p, S, \gamma_1, \ldots, \gamma_m]$. Similarly, if $\mu \in Z^2(G_p, K^*)$, then we denote the algebra $K^\mu G_p$ by $[G_p, K, \gamma_1, \ldots, \gamma_m]$ as well.

Let R be either a field of characteristic p, or a commutative complete discrete valuation domain of characteristic p, and $G = G_p \times B$. Given $\mu \in$

$Z^2(G_p, R^*)$ and $\nu \in Z^2(B, R^*)$, the map $\mu \times \nu\colon G \times G \to R^*$ defined by

$$(\mu \times \nu)_{x_1 b_1, x_2 b_2} = \mu_{x_1, x_2} \cdot \nu_{b_1, b_2},$$

for all $x_1, x_2 \in G_p$, $b_1, b_2 \in B$, belongs to $Z^2(G, R^*)$. Every cocycle $\lambda \in Z^2(G, R^*)$ is cohomologous to $\mu \times \nu$, where μ is the restriction of λ to $G_p \times G_p$ and ν is the restriction of λ to $B \times B$.

From now on, we assume that each cocycle $\lambda \in Z^2(G, R^*)$ under consideration satisfies the condition $\lambda = \mu \times \nu$, and all $R^\lambda G$-modules are assumed to be finitely generated left $R^\lambda G$-modules which are R-free.

Let $\lambda = \mu \times \nu \in Z^2(G, R^*)$ and $\{u_g : g \in G\}$ be a canonical R-basis of $R^\lambda G$. Then $\{u_h : h \in G_p\}$ is a canonical R-basis of $R^\mu G_p$ and $\{u_b : b \in B\}$ is a canonical R-basis of $R^\nu B$. Moreover, if $g = hb$, where $g \in G$, $h \in G_p$, $b \in B$, then $u_g = u_h u_b = u_b u_h$. It follows that $R^\lambda G \cong R^\mu G_p \otimes_R R^\nu B$.

Given an $R^\mu G_p$-module V and an $R^\nu B$-module W, we denote by $V \# W$ the $R^\lambda G$-module whose underlying R-module is $V \otimes_R W$ with $R^\lambda G$-module structure given by

$$u_{hb}(v \otimes w) = u_h v \otimes u_b w$$

for all $h \in G_p$, $b \in B$, $v \in V$, $w \in W$, and extended to $R^\lambda G$ and $V \otimes_R W$ by R-linearity. The module $V \# W$ is called the *outer tensor product* of V and W (see [22, p. 122]). The algebra $R^\lambda G$ is defined to be of *OTP representation type* if every indecomposable $R^\lambda G$-module is isomorphic to the outer tensor product $V \# W$, where V is an indecomposable $R^\mu G_p$-module and W is an irreducible $R^\nu B$-module.

Given an $R^\lambda H$-module V, we write $\mathrm{End}_{R^\lambda H}(V)$ for the ring of all $R^\lambda H$-endomorphisms of V, $\mathrm{rad}\,\mathrm{End}_{R^\lambda H}(V)$ for the Jacobson radical of $\mathrm{End}_{R^\lambda H}(V)$ and $\overline{\mathrm{End}_{R^\lambda H}(V)}$ for the quotient ring

$$\mathrm{End}_{R^\lambda H}(V)/\mathrm{rad}\,\mathrm{End}_{R^\lambda H}(V).$$

LEMMA 1.1. *Let R be either a field of characteristic p, or a commutative complete discrete valuation domain of characteristic p, $G = G_p \times B$, $\mu \in Z^2(G_p, R^*)$, $\nu \in Z^2(B, R^*)$ and $\lambda = \mu \times \nu$. The algebra $R^\lambda G$ is of OTP representation type if and only if the outer tensor product of any indecomposable $R^\mu G_p$-module and any irreducible $R^\nu B$-module is an indecomposable $R^\lambda G$-module.*

The proof is similar to that of the corresponding fact for a group algebra (see [9, p. 41], [20, p. 68] and [21, p. 658]).

LEMMA 1.2. *Let R be either a field of characteristic p, or a commutative complete discrete valuation domain of characteristic p, $G = G_p \times B$, $\mu \in Z^2(G_p, R^*)$, $\nu \in Z^2(B, R^*)$ and $\lambda = \mu \times \nu$. If V is an indecomposable $R^\mu G_p$-module and W is an irreducible $R^\nu B$-module, then*

$$\overline{\mathrm{End}_{R^\lambda G}(V \# W)} \cong \overline{\mathrm{End}_{R^\mu G_p}(V)} \otimes_{\overline{R}} \overline{\mathrm{End}_{R^\nu B}(W)},$$

where $\overline{R}$ is the residue class field of R.

Proof. See [6, p. 15]. ∎

LEMMA 1.3. *Let R be either a field of characteristic p, or a commutative complete discrete valuation domain of characteristic p, H a finite group, $\lambda \in Z^2(H, R^*)$ and V an $R^\lambda H$-module. Then V is indecomposable if and only if $\overline{\mathrm{End}_{R^\lambda H}(V)}$ is a skew field.*

Proof. Apply Proposition 6.10 of [12, p. 125]. ∎

LEMMA 1.4. *Let $S = K[[X]]$, H be a finite p-group, D a subgroup of H, $\lambda \in Z^2(H, S^*)$ and M an indecomposable $S^\lambda D$-module. Assume that $\overline{\mathrm{End}_{S^\lambda D}(M)}$ is isomorphic to a field F, $F \supset K$, and one of the following conditions is satisfied:*

(i) *H is abelian;*
(ii) *$[s(F) : K]$ is not divisible by p, where $s(F)$ is the separable closure of K in F.*

Then

$$M^H := S^\lambda H \otimes_{S^\lambda D} M$$

is an indecomposable $S^\lambda H$-module and $\overline{\mathrm{End}_{S^\lambda H}(M^H)}$ is isomorphic to a field that is a finite purely inseparable field extension of F.

The proof is similar to that of Lemma 2.2 in [3]. It uses the same idea as in Theorem 8 of [16].

LEMMA 1.5. *Let L be a finite separable field extension of K and H be a finite p-group. If $|H| > 2$ then there exists an indecomposable SH-module M such that $\overline{\mathrm{End}_{SH}(M)}$ is isomorphic to L.*

Proof. See [6, p. 12]. ∎

LEMMA 1.6. *Let K be an arbitrary field of characteristic p, $S = K[[X]]$, $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. If K is a splitting field for the K-algebra $K^\nu B$, then $S^\lambda G$ is of OTP representation type.*

Proof. See [6, p. 15]. ∎

Let H be a subgroup of G_p, $\mu \in Z^2(G_p, S^*)$ and $\tau \in Z^2(H, S^*)$. We say that $S^\tau H$ is a *μ-extended algebra* if there exists a subgroup D of G_p and a cocycle $\sigma \in Z^2(D, S^*)$ such that the following properties hold:

(i) $S^\mu D = S^\sigma D$ as S-algebras;
(ii) $H \subset D$ and τ is the restriction of σ to $H \times H$;

(iii) $S^\tau H$ is the S-subalgebra of $S^\sigma D$ consisting of all S-linear combinations of the elements $\{v_h : h \in H\}$, where $\{v_d : d \in D\}$ is a canonical S-basis of $S^\sigma D$ corresponding to σ.

LEMMA 1.7. *Let $G = G_p \times B$, G_p be abelian, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. Assume that G_p has a subgroup H such that $|H| > 2$ and SH is a μ-extended algebra. Then $S^\lambda G$ is of OTP representation type if and only if K is a splitting field for $K^\nu B$.*

Proof. In view of Lemma 1.6, $S^\lambda G$ is of OTP representation type if K is a splitting field for $K^\nu B$. Suppose that K is not a splitting field for the K-algebra $K^\nu B$. Then there is an irreducible $S^\nu B$-module W such that $\Delta := \overline{\mathrm{End}_{S^\nu B}(W)}$ is a division K-algebra of dimension greater than one. Since $K^\nu B$ is a separable K-algebra, by Proposition 7.25 in [12, p. 157], there exists a splitting field L for $K^\nu B$, which is a finite separable field extension of K. Since SH is a μ-extended algebra, there exists a subgroup D of G_p and $\sigma \in Z^2(D, S^*)$ such that $H \subset D$, $S^\mu D = S^\sigma D$ and $SH = S^\sigma H$ as S-algebras. By Lemma 1.5, there is an indecomposable SH-module M such that $\overline{\mathrm{End}_{SH}(M)}$ is isomorphic to L. According to Lemma 1.4, the $S^\sigma D$-module

$$M^D := S^\sigma D \otimes_{SH} M$$

is indecomposable and $\overline{\mathrm{End}_{S^\sigma D}(M^D)}$ is isomorphic to a field L' that is a finite purely inseparable field extension of L. Applying again Lemma 1.4, we show that the $S^\mu G_p$-module

$$V := S^\mu G_p \otimes_{S^\mu D} M^D$$

is indecomposable and $\overline{\mathrm{End}_{S^\mu G_p}(V)}$ is isomorphic to a field F that is a finite purely inseparable field extension of L'. The F-algebra $F \otimes_K \Delta$ is not a skew field, because F is a splitting field for Δ. Applying Lemmas 1.2 and 1.3, we conclude that $V \# W$ is not an indecomposable $S^\lambda G$-module. Hence, by Lemma 1.1, $S^\lambda G$ is not of OTP representation type. ∎

LEMMA 1.8. *Let B be a finite p'-group. Assume that $[K(\xi) : K]$ is not divisible by p, where ξ is a primitive* $(\exp B)$*th root of* 1. *Then, for any K-algebra $K^\nu B$, there exists a splitting field L such that $[L : K]$ is not divisible by p.*

Proof. See [3, p. 548]. ∎

PROPOSITION 1.9. *Let $G = G_p \times B$, $|G_p'| > 2$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. The algebra $S^\lambda G$ is of OTP representation type if and only if K is a splitting field for the K-algebra $K^\nu B$.*

Proof. In view of Corollary 4.10 in [22, p. 42], the restriction of μ to $G_p' \times G_p'$ is a coboundary. By Lemma 1.8, there exists a splitting field L for $K^\nu B$ such that $[L : K]$ is not divisible by p. The field L is a separable extension of K. Arguing as in the proof of Lemma 1.7, we deduce that there is an indecomposable $S^{\mu}G_p$-module V such that $\overline{\mathrm{End}_{S^{\mu}G_p}(V)}$ is isomorphic to a field F that is a finite purely inseparable field extension of L. It follows that if K is not a splitting field for $K^{\nu}B$, then the algebra $S^{\lambda}G$ is not of OTP representation type. ∎

LEMMA 1.10. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. Denote by V an indecomposable $S^{\mu}G_p$-module such that $\overline{\mathrm{End}_{S^{\mu}G_p}(V)}$ is a finite purely inseparable field extension of K. The $S^{\lambda}G$-module $V \# W$ is indecomposable for any irreducible $S^{\nu}B$-module W.*

Proof. Let L be a finite purely inseparable field extension of K, and suppose L is K-isomorphic to $\overline{\mathrm{End}_{S^{\mu}G_p}(V)}$. Denote by $\widetilde{W}$ the factor module W/XW and by Δ the K-algebra $\overline{\mathrm{End}_{S^{\nu}B}(W)}$. By Proposition 5.22 in [12, p. 112] and Theorem 76.8 in [11, p. 532], we have

$$\overline{\mathrm{End}_{S^{\nu}B}(W)} \cong \mathrm{End}_{S^{\nu}B}(W)/X\,\mathrm{End}_{S^{\nu}B}(W) \cong \mathrm{End}_{K^{\nu}B}(\widetilde{W}).$$

The center of the division K-algebra Δ is a finite separable field extension of K, because $K^{\nu}B$ is a separable algebra (see [11, p. 485]). The index of Δ is not divisible by p (see [23]). This implies that $L \otimes_K \Delta$ is a skew field. In view of Lemmas 1.2 and 1.3, $V \# W$ is an indecomposable $S^{\lambda}G$-module. ∎

PROPOSITION 1.11. *Let G_p be abelian, $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. Assume that $T^{\mu}G_p$ is a field and $S^{\mu}G_p$ is the valuation ring in $T^{\mu}G_p$. Then $S^{\lambda}G$ is of OTP representation type.*

Proof. Let $\sigma \in Z^2(G_p, K^*)$ and $\sigma_{a,b} \equiv \mu_{a,b} \pmod{X}$ for all $a, b \in G_p$. Then $S^{\mu}G_p/XS^{\mu}G_p \cong K^{\sigma}G_p$. Any indecomposable $S^{\mu}G_p$-module is isomorphic to the regular $S^{\mu}G_p$-module. Since $\mathrm{End}_{S^{\mu}G_p}(S^{\mu}G_p) \cong S^{\mu}G_p$, we obtain

$$\overline{\mathrm{End}_{S^{\mu}G_p}(S^{\mu}G_p)} \cong (S^{\mu}G_p/XS^{\mu}G_p)/\mathrm{rad}(S^{\mu}G_p/XS^{\mu}G_p) \cong \overline{K^{\sigma}G_p}.$$

The K-algebra $\overline{K^{\sigma}G_p}$ is isomorphic to a field that is a finite purely inseparable field extension of K (see [22, p. 74]). By Lemmas 1.1 and 1.10, $S^{\lambda}G$ is of OTP representation type. ∎

2. Twisted group algebras of OTP representation type of a group $G_p \times B$ with cyclic G_p

PROPOSITION 2.1. *Let K be an arbitrary field of characteristic p, $S = K[[X]]$, T the quotient field of S, F a finite purely inseparable field extension of T, R the valuation domain in F, B a finite p'-group, $\nu \in Z^2(B, K^*)$ and $R^{\nu}B = R \otimes_S S^{\nu}B$.*

(i) *If W is an irreducible $S^{\nu}B$-module then $R \otimes_S W$ is an irreducible $R^{\nu}B$-module.*

(ii) *If U is an irreducible $R^\nu B$-module then there exists an irreducible $S^\nu B$-module W such that U is isomorphic to $R \otimes_S W$.*

Proof. The algebra $T^\nu B$ is separable. If A is a simple component of $T^\nu B$, then the center of A is a separable field extension of T and the index of A is not divisible by p (see [23]). It follows that:

(1) if M is a simple $T^\nu B$-module then $F \otimes_T M$ is a simple $F^\nu B$-module, where $F^\nu B = F \otimes_T T^\nu B$;
(2) if M' is a simple $F^\nu B$-module then there exists a simple $T^\nu B$-module M such that $M' \cong F \otimes_T M$.

Assume that W is an irreducible $S^\nu B$-module. Then, by Theorem 75.6 in [11], $T \otimes_S W$ is a simple $T^\nu B$-module. It follows that $F \otimes_T (T \otimes_S W)$ is a simple $F^\nu B$-module. Since

$$F \otimes_R (R \otimes_S W) \cong F \otimes_T (T \otimes_S W),$$

we deduce that $R \otimes_S W$ is an irreducible $R^\nu B$-module.

Suppose now that U is an irreducible $R^\nu B$-module. There exists an irreducible $S^\nu B$-module W such that the $F^\nu B$-modules $F \otimes_R U$ and $F \otimes_T (T \otimes_S W)$ are isomorphic. Since $F \otimes_T (T \otimes_S W) \cong F \otimes_R (R \otimes_S W)$, we see that $F \otimes_R U \cong F \otimes_R (R \otimes_S W)$. By Theorem 76.17 in [11], $U \cong R \otimes_S W$. ∎

Let G_p be an abelian p-group, $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. Assume that $H = \langle a \rangle$ is a cyclic group of order p^n, $G_p = A \times H$, $F := T^\mu A$ is a field and $R := S^\mu A$ is the valuation domain in F. The algebra $S^\mu G_p$ can be viewed as a twisted group algebra of H over R. Denote it by $R^\mu H$. Let $D = H \times B$. The algebra $S^\lambda G$ is a twisted group algebra $R^\sigma D$ of D over the ring R. We have $R^\sigma D \cong R^\mu H \otimes_R R^\nu B$.

Proposition 2.2. *$S^\lambda G$ is of OTP S-representation type if and only if $R^\sigma D$ is of OTP R-representation type.*

Proof. Let M be an $S^\lambda G$-module. Then M is a finitely generated R-module. Assume that $r \in R$, $v \in M$, $v \neq 0$ and $rv = 0$. If $p^l = \exp G_p$ then $r^{p^l} \in S$. Since $r^{p^l} v = 0$ and M is a free S-module, we find that $r^{p^l} = 0$, hence $r = 0$. Therefore M is a torsionfree module over the principal ideal ring R. It follows that M is a free R-module. This proves that M is an $R^\sigma D$-module. Conversely, if M is an $R^\sigma D$-module then M is a free R-module. Since R is a free S-module of finite rank, we see that M is an $S^\lambda G$-module. Moreover M is an indecomposable $S^\lambda G$-module if and only if M is an indecomposable $R^\sigma D$-module.

Suppose that $S^\lambda G$ is of OTP S-representation type. Let V be an indecomposable $R^\mu H$-module and U an irreducible $R^\nu B$-module. By Proposition 2.1, there exists an irreducible $S^\nu B$-module W such that U is isomorphic to

$R\otimes_S W$. By Lemma 1.1, $V\otimes_S W$ is an indecomposable $S^\lambda G$-module. Hence $V\otimes_S W$ is an indecomposable $R^\sigma D$-module. Since

$$V\otimes_R U\cong (V\otimes_R R)\otimes_S W\cong V\otimes_S W,$$

we deduce that $V\otimes_R U$ is an indecomposable $R^\sigma D$-module. In view of Lemma 1.1, $R^\sigma D$ is of OTP R-representation type.

Conversely, let $R^\sigma D$ be of OTP R-representation type. Assume that V is an indecomposable $S^\mu G_p$-module and W is an irreducible $S^\nu B$-module. By Proposition 2.1, $U:=R\otimes_S W$ is an irreducible $R^\sigma B$-module. Since V is an indecomposable $R^\mu H$-module Lemma 1.1 shows that $V\otimes_R U$ is an indecomposable $R^\sigma D$-module, hence $V\otimes_R U$ is an indecomposable $S^\lambda G$-module. It follows that $V\otimes_S W$ is an indecomposable $S^\lambda G$-module. By Lemma 1.1, $S^\lambda G$ is of OTP S-representation type. ∎

Let F be a field of characteristic 2 complete with respect to a discrete valuation, R the valuation domain in F, $\gamma\in R^*$ and $\gamma\notin R^2$. Denote by θ a root of the irreducible polynomial

$$Y^{2^n}-\gamma\in R[Y],\qquad n\geq 1,$$

and by $\widetilde{\theta}$ the matrix of multiplication by θ in the R-basis $1,\theta,\dots,\theta^{2^n-1}$ of the ring $R[\theta]$. For $n>1$, let $\rho=\theta^2$ and $\widetilde{\rho}$ be the matrix of multiplication by ρ in the R-basis $1,\rho,\dots,\rho^{2^{n-1}-1}$ of the ring $R[\rho]$. If $n=1$, we shall assume that $\rho=1$ and $\widetilde{\rho}$ is the identity matrix of order 1.

LEMMA 2.3. *Let $G_2=\langle a\rangle$ be a cyclic group of order 2^n $(n\geq 1)$ and $R^\mu G_2=[G_2,R,\gamma^{2^l}]$, where $l\in\{0,1\}$ and $\gamma\in R^*$.*

(i) *If $l=0$, $\gamma\notin R^2$ and $R[\theta]$ is the valuation domain in $F(\theta)$, then, up to equivalence, the algebra $R^\mu G_2$ has only one indecomposable matrix R-representation $\Gamma\colon u_a\mapsto\widetilde{\theta}$.*

(ii) *Let $l=1$, $\gamma\notin R^2$ for $n\geq 2$ and $\gamma=1$ for $n=1$. If $R[\rho]$ is the valuation domain in $F(\rho)$, then, up to equivalence, the indecomposable matrix R-representations of the algebra $R^\mu G_2$ are the following:*

$$\Gamma_1\colon u_a\mapsto\widetilde{\rho},\qquad \Gamma_{2j}\colon u_a\mapsto\begin{pmatrix}\widetilde{\rho} & \langle\pi^j\rangle\\ 0 & \widetilde{\rho}\end{pmatrix},\qquad j=0,1,2,\dots,$$

where π is a prime element of $R[\rho]$ and $\langle\pi^j\rangle$ is the matrix in which all columns but the last one are zero, and the last column consists of the coordinates of π^j in the R-basis $1,\rho,\dots,\rho^{2^{n-1}-1}$ of the ring $R[\rho]$.

Proof. If $l=0$ then $R^\mu G_2\cong R[\theta]$. Each $R^\mu G_2$-module M can be considered as a torsionfree module over the principal ideal ring $R[\theta]$, therefore if $M\neq 0$ then $M\cong R[\theta]\oplus\cdots\oplus R[\theta]$. Hence, up to equivalence, the algebra $R^\mu G_2$ has only one indecomposable R-representation $u_a\mapsto\widetilde{\theta}$.

Let $l = 1$, M be an arbitrary $R^\mu G_2$-module, $M \neq 0$ and

$$N = \{v \in M : (u_a^{2^{n-1}} - \gamma u_e)v = 0\}.$$

One can view the $R^\mu G_2$-module N as a module over the ring

$$R^\mu G_2/(u_a^{2^{n-1}} - \gamma u_e)R^\mu G_2 \cong R[\rho].$$

Since $R[\rho]$ is a principal ideal ring and N is an $R[\rho]$-torsionfree module, we get $N \cong R[\rho] \oplus \cdots \oplus R[\rho]$. The $R^\mu G_2$-module M/N can also be viewed as an $R[\rho]$-module. Since M is an R-torsionfree module, $N \cap \alpha M = \alpha N$ for every $\alpha \in R$. If $z \in R[\rho]$ then $z^{2^{n-1}} \in R$, and therefore the equality $z(v + N) = N$ yields $z = 0$ or $v \in N$. This means that M/N is a torsionfree module over $R[\rho]$. Hence in the case $N \neq M$ we have $M/N \cong R[\rho] \oplus \cdots \oplus R[\rho]$.

Since every R-basis of N can be extended to an R-basis of M (see [11, p. 100]), we deduce that a matrix R-representation Γ of the algebra $R^\mu G_2$ afforded by the $R^\mu G_2$-module M can be written in the form

$$\Gamma(u_a) = \begin{pmatrix} \widetilde{\rho}^{(s)} & * \\ 0 & \widetilde{\rho}^{(t)} \end{pmatrix},$$

where $\widetilde{\rho}^{(s)} = \mathrm{diag}[\widetilde{\rho}_1, \ldots, \widetilde{\rho}_s]$ and $\rho_1 = \cdots = \rho_s = \rho$. Using the technique of [8], we conclude that indecomposable matrix R-representations of the algebra $R^\mu G_2$ are Γ_1 and Γ_{2j}, where $j = 0, 1, 2, \ldots$. ∎

Lemma 2.4. *Keeping the notation of Lemma 2.3, assume that one of the following two conditions is satisfied:*

(i) *$l = 0$, $\gamma \notin R^2$ and $R[\theta]$ is the valuation domain in the field $F(\theta)$;*
(ii) *$l = 1$, $\gamma \notin R^2$ if $n \geq 2$, $\gamma = 1$ if $n = 1$, and $R[\rho]$ is the valuation domain in the field $F(\rho)$.*

Then for every indecomposable $R^\mu G_2$-module V, the $F^\mu G_2$-module $F \otimes_R V$ is also indecomposable.

Proof. Let V be an underlying $R^\mu G_2$-module of the representation Γ_{2j}, $s = 2^{n-1}$ and

$$\pi^j = \sum_{i=0}^{s-1} \alpha_i \rho^i, \quad \alpha_i \in R.$$

Denote by $(v_1, \ldots, v_{2s})$ an R-basis of V such that

$$u_a v_k = v_{k+1} \quad \text{for all } k \notin \{s, 2s\}, \quad u_a v_s = \gamma v_1, \quad u_a v_{2s} = \sum_{i=0}^{s-1} \alpha_i v_{i+1} + \gamma v_{s+1}.$$

Let $\hat{V} := F \otimes_R V$. We shall identify v with $1 \otimes v$. We set

$$w_1 = v_{s+1}, \ \ldots, \ w_s = v_{2s}, \quad w_{s+1} = \sum_{i=0}^{s-1} \alpha_i v_{i+1} + \gamma v_{s+1},$$

$$w_{s+k} = u_a^{k-1} w_{s+1} \quad \text{for } k = 2, \ldots, s.$$

Then $(w_1, \ldots, w_{2s})$ is an F-basis of $\hat{V}$, $u_a w_t = w_{t+1}$ for $t = 1, \ldots, 2s-1$ and $u_a w_{2s} = \gamma^2 w_1$. Consequently, the $F^\mu G_2$-module $\hat{V}$ is isomorphic to the regular $F^\mu G_2$-module.

If V is an underlying $R^\mu G_2$-module of the representation Γ_1, then $F \otimes_R V$ is a simple $F^\mu G_2$-module. ∎

PROPOSITION 2.5. *Let G_2 be a cyclic group of order 2^n $(n \geq 1)$, $G = G_2 \times B$, $\mu \in Z^2(G_2, R^*)$, $\nu \in Z^2(B, R^*)$, $\lambda = \mu \times \nu$, and ξ be a root of a polynomial*

$$Y^{2^n} - \gamma^{2^l}, \qquad \gamma \in R^*.$$

Assume that $R^\mu G_2 = [G_2, R, \gamma^{2^l}]$, where $l \in \{0, 1\}$, $\gamma \in R^*$, $\gamma \notin R^2$ if either $l = 0$, or $l = 1$ and $n \geq 2$. If $R[\xi]$ is the valuation domain in the field $F(\xi)$, then $R^\lambda G$ is of OTP representation type.

Proof. Since $F^\mu G_2$ is a uniserial algebra, by Theorem 3.1 in [3], $F^\lambda G$ is of OTP representation type. Let V be an indecomposable $R^\mu G_2$-module and W an irreducible $R^\nu B$-module. By Lemma 2.4, $F \otimes_R V$ is an indecomposable $F^\mu G_2$-module. In view of Theorem 75.6 in [11], $F \otimes_R W$ is a simple $F^\nu B$-module. It follows that $(F \otimes_R V) \otimes_F (F \otimes_R W)$ is an indecomposable $F^\lambda G$-module. Since

$$F \otimes_R (V \otimes_R W) \cong (F \otimes_R V) \otimes_F (F \otimes_R W),$$

the $R^\lambda G$-module $V \otimes_R W$ is indecomposable. By Lemma 1.1, $R^\lambda G$ is of OTP representation type. ∎

3. Twisted group algebras of OTP representation type of a group $G_p \times B$ with $|G'_p| \neq 2$. We recall that K is a field of characteristic p, $S = K[[X]]$, T is the quotient field of S and $G = G_p \times B$ is a finite group, where G_p is a p-group, B is a p'-group and $|G_p| \neq 1$, $|B| \neq 1$. We assume that if G_p is non-abelian then $[K(\xi) : K]$ is not divisible by p, where ξ is a primitive $(\exp B)$th root of 1.

Denote by l_B the product of all pairwise district prime divisors of $|B|$. It is not difficult to see that $[K(\xi) : K]$ is not divisible by p if and only if $[K(\varepsilon) : K]$ is not divisible by p, where ε is a primitive l_Bth root of 1. This condition is satisfied if K contains a primitive qth root of 1 for every prime q dividing $|B|$ such that the characteristic p divides $q - 1$.

PROPOSITION 3.1. *Let G_p be abelian and $\mu \in Z^2(G_p, S^*)$. If $S^\mu G_p / X S^\mu G_p$ is a field then $T^\mu G_p$ is also a field and $S^\mu G_p$ is the valuation domain in $T^\mu G_p$.*

Proof. The ring $S^\mu G_p$ is an integral domain. It follows that the algebra $T^\mu G_p$ is also an integral domain, hence $T^\mu G_p$ is a field. Denote by R the valuation domain in $F := T^\mu G_p$. It is well known that $[F : T] = e(F/T) \cdot f(F/T)$, where $e(F/T)$ is the ramification index and $f(F/T)$ is the residue class degree. Since $S^\mu G_p/XS^\mu G_p$ is a field extension of K of degree $|G_p|$ and $[F:T]=|G_p|$, we conclude that $e(F/T)=1$ and $f(F/T)=[F:T]$. Let $\{u_g : g\in G_p\}$ be a canonical S-basis of $S^\mu G_p$ corresponding to μ. The set $\Gamma := \{\sum_{g\in G_p}\alpha_g u_g : \alpha_g\in K\}$ is a full set of residue class representatives in R of the residue class field $\overline{R}=R/\mathfrak{m}$, where $\mathfrak{m}$ is the unique maximal ideal of R. Each $\rho\in R$ is uniquely expressible as

$$\rho = v_0 + v_1X + v_2X^2 + \cdots, \qquad v_i\in\Gamma.$$

Therefore $R=S^\mu G_p$. ∎

Proposition 3.2. *Let $G_p=\langle a_1\rangle\times\cdots\times\langle a_m\rangle$, $m\geq 2$, $H=\langle a_2\rangle\times\cdots\times\langle a_m\rangle$, $\overline{H}=\operatorname{soc} H$ and*

$$S^\mu G_p = [G_p, S, \gamma_1(1+X), \gamma_2(1+X)^{pr_2},\dots,\gamma_m(1+X)^{pr_m}],$$

where $\gamma_1,\dots,\gamma_m\in K^*$. If $K^\sigma\overline{H} := [\overline{H}, K, \gamma_2,\dots,\gamma_m]$ is a field, then $T^\mu G_p$ is a field and $S^\mu G_p$ is the valuation domain in $T^\mu G_p$.

Proof. Let $\overline{G}_p=\operatorname{soc} G_p$. Since $K^\sigma\overline{H}$ is a field, the T-algebra $T^\mu\overline{G}_p$ is a field. It follows that $T^\mu G_p$ is a field. Suppose that $a\in G_p$, $|a|=p^n$ and

$$u_a^{p^n} = \gamma(1+X)^{ip^t}u_e,$$

where $\gamma\in K^*$, $0\leq t<n$, $i\in\mathbb{Z}$ and $(i,p)=1$. There exist integers y and z such that $yi+zp^{n-t}=1$. We have

$$(u_a^y)^{p^n} = \gamma^y(1+X)^{p^t}(1+X)^{-zp^n}u_e.$$

Consequently, we may assume that

$$u_a^{p^n} = \gamma(1+X)^{p^t}u_e, \qquad \gamma\in K^*,\ 0\leq t<n.$$

We put

$$w=(1+X)^{-1}u_a^{p^{n-t}} \qquad \text{if } t>0.$$

Then $w^{p^t}=\gamma u_e$, $u_a^{p^{n-t}}=(1+X)w$.

Let $|a_j|=p^{n_j}$ for $j=1,\dots,m$. The argument above yields the existence of a canonical S-basis in $S^\mu G_p$ such that

$$S^\mu G_p = [G_p, S, \gamma_1(1+X), \gamma_2(1+X)^{k_2p^{t_2}},\dots,\gamma_m(1+X)^{k_mp^{t_m}}],$$

where $k_j\in\{0,1\}$; $0\leq t_j<n_j$; $\gamma_1,\dots,\gamma_m\in K^*$. If $k_j=1$, $t_j>0$, then

$$u_{a_j}^{p^{n_j-t_j}} = (1+X)w_j, \quad \text{where} \quad w_j^{p^{t_j}}=\gamma_j u_e.$$

If $k_j=0$ or $t_j=0$, we set $w_j=\gamma_j u_e$. We suppose that $t_j=0$ if $k_j=0$.

Let $L=K(w_2,\dots,w_m)$ and $\hat{S}=L[[X]]$. Then

$$S^\mu G_p = [\hat{G}_p, \hat{S}, \gamma_1(1+X)u_e, w_2(1+X)^{k_2},\dots,w_m(1+X)^{k_m}],$$

where $\hat{G}_p=\langle b_1\rangle\times\cdots\times\langle b_m\rangle$ is an abelian group of type $(p^{n_1}, p^{n_2-t_2},\dots, p^{n_m-t_m})$. In the case $k_2+\cdots+k_m\neq 0$ we may assume that $k_2=1$,

$k_3 = 0, \dots, k_m = 0$. If $n_1 \geq n_2$ and $k_2 = 1$ then

$$S^\mu G_p = [\hat{G}_p, \hat{S}, \gamma_1(1+X)u_e, \gamma_1^{-1}w_2, w_3, \dots, w_m].$$

If $n_1 < n_2$ and $k_2 = 1$ then

$$S^\mu G_p = [\hat{G}_p, \hat{S}, \gamma_1 w_2^{-1}, w_2(1+X), w_3, \dots, w_m].$$

We restrict ourselves to the case $k_2 = 0, \dots, k_m = 0$.

Denote by F the field $[\hat{H}, L, w_2, \dots, w_m]$, where $\hat{H} = \langle b_2\rangle \times \dots \times \langle b_m\rangle$. Then $S^\mu H = F[[X]]$. Let R be the valuation domain in $T^\mu G_p$ and $R \neq S^\mu G_p$. Since

$$S^\mu G_p = \bigoplus_{i=0}^{p^{n_1}-1} S^\mu H u_{a_1}^i, \qquad u_{a_1}^{p^{n_1}} = \gamma_1(1+X)u_e,$$

the ring R contains a non-zero element

$$X^{-1}\Big(\sum_{i=0}^{p^{n_1}-1} \delta_i u_{a_1}^i\Big)$$

with $\delta_i \in F$ for every i. It follows that $F[[X]]$ contains the non-zero element

$$X^{-p^{n_1}} \sum_{i=0}^{p^{n_1}-1} \delta_i^{p^{n_1}} \gamma_1^i (1+X)^i,$$

a contradiction. Hence $R = S^\mu G_p$. ∎

Proposition 3.3. *Let G_p be abelian, $G = G_p \times B$, $\mu \in Z^2(G_p, S^*)$, $\nu \in Z^2(B, S^*)$ and $\lambda = \mu \times \nu$. If the K-algebra $S^\mu G_p / X S^\mu G_p$ is a field, then $S^\lambda G$ is of OTP representation type.*

Proof. By Proposition 3.1, $S^\mu G_p$ is the valuation domain in the field $T^\mu G_p$. Therefore, in view of Proposition 1.11, $S^\lambda G$ is of OTP representation type. ∎

Proposition 3.4. *Assume that $p \neq 2$, $G = G_p \times B$; Ω is a subgroup of S^*, generated by K^*, $(S^*)^p$ and $f(X)$, where $f(X) \equiv 1 \pmod X$ and $f(X) \not\equiv 1 \pmod{X^2}$. Let $\mu \in Z^2(G_p, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. If the algebra $S^\lambda G$ is of OTP representation type then one of the following conditions is satisfied:*

(i) *G_p is abelian and $T^\mu G_p$ is a field;*
(ii) *K is a splitting field for $K^\nu B$.*

Proof. If G_p is non-abelian then, by Proposition 1.9, $S^\lambda G$ is of OTP representation type if and only if condition (ii) holds. Suppose that G_p is abelian. Let $G_p = \langle a_1\rangle \times \dots \times \langle a_m\rangle$, $m \geq 2$ and $\overline{G}_p = \operatorname{soc} G_p$. We have $(f(X)-1)S = XS$, hence we may assume that $f(X) = 1 + X$ and

$$S^\mu G_p = [G_p, S, \gamma_1(1+X)^i f_1(X)^p, \gamma_2 f_2(X)^p, \dots, \gamma_m f_m(X)^p],$$

where $\gamma_1, \dots, \gamma_m \in K^*$ and $f_1(X), \dots, f_m(X)$ are principal units in S. Denote $A = \langle a_2\rangle \times \cdots \times \langle a_m\rangle$, $\overline{A} = \operatorname{soc} A$ and $K^\sigma\overline{A} = [\overline{A}, K, \gamma_2, \dots, \gamma_m]$. If $K^\sigma\overline{A}$ is not a field then $K^\sigma\overline{A} = K^\tau\overline{A} = [\overline{A}, K, \gamma_2, \dots, \gamma_{m-1}, 1]$. It follows that $S^\mu G_p$ contains a μ-extended group algebra of a group of order p over S. By Lemma 1.7, $S^\lambda G$ is of OTP representation type if and only if K is a splitting field for $K^\nu B$.

Suppose now that $K^\sigma\overline{A}$ is a field. By Proposition 3.1, $T^\mu A$ is a field and $S^\mu A$ is the valuation domain in $T^\mu A$. If $\gamma_1(1+X)^i u_e \notin (S^\mu A)^p$ then $T^\mu G_p$ is a field. If $\gamma_1(1+X)^i u_e \in (S^\mu A)^p$ then $S^\mu G_p$ contains a μ-extended group algebra of a cyclic group of order p over S. In view of Lemma 1.7, $S^\lambda G$ is of OTP representation type if and only if condition (ii) holds. ∎

THEOREM 3.5. *Let $p \neq 2$, $G = G_p \times B$; Ω be the subgroup of S^* generated by K^* and $(S^*)^p$; $\mu \in Z^2(G_p, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. The algebra $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *G_p is abelian and $T^\mu G_p$ is a field;*
(ii) *K is a splitting field for $K^\nu B$.*

Proof. The necessity part follows from Proposition 3.4. Let us prove the sufficiency. Keeping the notation of Proposition 3.4, assume that (i) holds. Then $[\overline{G}_p, K, \gamma_1, \dots, \gamma_m]$ is a field. It follows that $[G_p, K, \gamma_1, \dots, \gamma_m]$ is a field, hence the K-algebra $S^\mu G_p / X S^\mu G_p$ is a field. By Proposition 3.3, $S^\lambda G$ is of OTP representation type. If (ii) holds, we apply Lemma 1.6. ∎

PROPOSITION 3.6. *Let $p = 2$, $G = G_2 \times B$, where $|G_2'| \neq 2$; Ω is the subgroup of S^* generated by K^* and $(S^*)^2$; $\mu \in Z^2(G_2, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. If the algebra $S^\lambda G$ is of OTP representation type then one of the following conditions is satisfied:*

(i) *G_2 is abelian and $\dim_T \overline{T^\mu G_2} \geq |G_2|/2$;*
(ii) *K is a splitting field for $K^\nu B$.*

Proof. If G_2 is non-abelian then, by Proposition 1.9, $S^\lambda G$ is of OTP representation type if and only if condition (ii) holds. Assume that $G_2 = \langle a_1\rangle \times \cdots \times \langle a_m\rangle$, $\overline{G}_2 = \operatorname{soc} G_2$ and

$$S^\mu G_2 = [G_2, S, \gamma_1 f_1(X)^2, \dots, \gamma_m f_m(X)^2],$$

where $\gamma_1, \dots, \gamma_m \in K^*$ and $f_1(X), \dots, f_m(X)$ are the principal units in S. Choose a canonical S-basis of $S^\mu\overline{G}_2$ such that $S^\mu\overline{G}_2 = [\overline{G}_2, S, \gamma_1, \dots, \gamma_m]$. Let $K^\sigma\overline{G}_2 := [\overline{G}_2, K, \gamma_1, \dots, \gamma_m]$. If $\dim_K \overline{K^\sigma\overline{G}_2} \leq 2^{m-2}$ then (i) is not satisfied and we may assume that $S^\mu\overline{G}_2 = S^\tau\overline{G}_2 = [\overline{G}_2, S, \gamma_1, \dots, \gamma_{m-2}, 1, 1]$. According to Lemma 1.7, $S^\lambda G$ is of OTP representation type if and only if condition (ii) holds. Suppose now that $\dim_K(K^\sigma\overline{G}_2/\operatorname{rad} K^\sigma\overline{G}_2) = 2^{m-1}$. Let $H = \langle a_1\rangle \times \cdots \times \langle a_{m-1}\rangle$ and $\overline{H} = \operatorname{soc} H$. We may assume that $K^\sigma\overline{H} =$

$[\overline{H}, K, \gamma_1, \dots, \gamma_{m-1}]$ is a field. Then $S^\mu H/XS^\mu H$ is a field and, by Proposition 3.1, $S^\mu H$ is the valuation domain in the field $T^\mu H$. If $\dim_T \overline{T^\mu G_2} < |G_2|/2$ then $S^\mu G_2$ contains a μ-extended group algebra of a cyclic group of order 4 over S. By Lemma 1.7, $S^\lambda G$ is of OTP representation type if and only if condition (ii) holds. ∎

LEMMA 3.7. *Let $\alpha, \delta_1, \dots, \delta_r \in T$; ρ be a root of the polynomial $Y^{2^{d+1}} - \alpha$; θ_i a root of the polynomial $Y^{2^{n_i}} - \delta_i$ for $i = 1, \dots, r$; $L := T(\omega_1, \dots, \omega_r)$, where*

$$\omega_i = \begin{cases} \theta_i^{2^{n_i-2}} & \text{if } n_i \geq 2, \\ \theta_i & \text{if } n_i = 1. \end{cases}$$

Assume that $[T(\theta_1, \dots, \theta_r) : T] = 2^l$, where $l = n_1 + \dots + n_r$, and $\alpha = \beta^2$, where $\beta \in L$ and $\beta \notin L^2$. Then $[T(\theta_1, \dots, \theta_r, \rho) : T] = 2^{l+d}$.

Proof. Suppose that $n_i \geq 2$ for $i = 1, \dots, j$ and $n_i = 1$ for $i = j+1, \dots, r$. Let $F := T(\omega_1^2, \dots, \omega_j^2, \omega_{j+1}, \dots, \omega_r)$. We have

$$\beta = \sum_{k_1=0}^{1} \dots \sum_{k_j=0}^{1} \gamma_{k_1,\dots,k_j} \omega_1^{k_1} \dots \omega_j^{k_j},$$

where $\gamma_{k_1,\dots,k_j} \in F$. Then

$$\beta^2 = \sum_{k_1=0}^{1} \dots \sum_{k_j=0}^{1} \gamma_{k_1,\dots,k_j}^2 \omega_1^{2k_1} \dots \omega_j^{2k_j}.$$

Since $\gamma_{k_1,\dots,k_j}^2 \in T$, it follows that $\gamma_{k_1,\dots,k_j} = 0$ if $k_r = 1$ for some $r \in \{1, \dots, j\}$. Therefore $\beta \in F$. Denote by ω a root of the polynomial $Y^2 - \beta$. Then $[L(\omega) : L] = 2$, hence $[F(\omega_1, \dots, \omega_j, \omega) : F] = 2^{j+1}$. Since

$$[T(\theta_1, \dots, \theta_r, \rho) : F(\omega_1, \dots, \omega_j, w)] = 2^k,$$

where $k = n_1 + \dots + n_j - 2j + d - 1$, and $[F : T] = 2^r$, we obtain $[T(\theta_1, \dots, \theta_r, \rho) : T] = 2^{l+d}$. ∎

PROPOSITION 3.8. *Let $p = 2$, and Ω be the subgroup of S^* generated by K^*, $(S^*)^4$ and $f(X)$, where $f(X) \equiv 1 \pmod{X}$ and $f(X) \not\equiv 1 \pmod{X^2}$. Assume that $G = G_2 \times B$, $|G_2'| \neq 2$, $\mu \in Z^2(G_2, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. If the algebra $S^\lambda G$ is of OTP representation type then one of the following conditions is satisfied:*

(i) *G_2 is abelian and $\dim_T \overline{T^\mu G_2} \geq |G_2|/2$;*
(ii) *K is a splitting field for $K^\nu B$.*

Proof. If $|G_2'| > 2$, we apply Proposition 1.9. Let G_2 be abelian. We may assume that $f(X) = 1 + X$. Suppose also that $S^\lambda G$ is of OTP representation type and K is not a splitting field for $K^\nu B$. Let $G_2 = \langle a_1 \rangle \times \dots \times \langle a_m \rangle$,

$m \geq 2$ and $D = \{g \in G_2 : g^4 = e\}$. By Proposition 3.6, we may assume that

$$S^\mu G_2 = [G_2, S, \gamma_1(1+X)f_1(X)^4, \gamma_2(1+X)^i f_2(X)^4, \ldots, \gamma_m f_m(X)^4],$$

where $\gamma_1, \ldots, \gamma_m \in K^*$, $i \in \{0, 2\}$ and $f_1(X), \ldots, f_m(X)$ are principal units in S. In this case

$$S^\mu D = [D, S, \gamma_1(1+X), \gamma_2(1+X)^i, \gamma_3, \ldots, \gamma_m].$$

Let $H = \langle a_2 \rangle \times \cdots \times \langle a_m \rangle$ and $\overline{H} = \operatorname{soc} H$. If $K^\sigma \overline{H} := [\overline{H}, K, \gamma_2, \ldots, \gamma_m]$ is a field then, by Proposition 3.2, $T^\mu D$ is a field. It follows that $T^\mu G_2$ is a field. If $K^\sigma \overline{H}$ is not a field then, by Lemma 1.7, $\dim_K \overline{K^\sigma \overline{H}} = |\overline{H}|/2$. In view of Lemma 1.7 and Proposition 3.2, $\dim_T \overline{T^\mu D} = |D|/2$. Applying Lemma 3.7, we conclude that

$$\dim_T \overline{T^\mu G_2} = |G_2|/2. \ ■$$

THEOREM 3.9. *Let $p = 2$, $G = G_2 \times B$, $|G_2'| \neq 2$; Ω be the subgroup of S^* generated by K^* and $(S^*)^4$; $\mu \in Z^2(G_2, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. The algebra $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *G_2 is abelian and $\dim_T \overline{T^\mu G_2} \geq |G_2|/2$;*
(ii) *K is a splitting field for $K^\nu B$.*

Proof. The necessity follows from Proposition 3.8. Let us prove the sufficiency. Let $G_2 = \langle a_1 \rangle \times \cdots \times \langle a_m \rangle$ and $\overline{G}_2 = \operatorname{soc} G_2$. We have

$$S^\mu G_2 = [G_2, S, \gamma_1 f_1(X)^4, \ldots, \gamma_m f_m(X)^4],$$

where $\gamma_1, \ldots, \gamma_m \in K^*$ and $f_1(X), \ldots, f_m(X)$ are principal units in S. If $T^\mu G_2$ is a field then $S^\mu G_2 / X S^\mu G_2$ is also a field. By Proposition 3.3, $S^\lambda G$ is of OTP representation type.

Let $\dim_T \overline{T^\mu G_2} = |G_2|/2$ and $m \geq 2$. Up to renumbering $a_1, \ldots, a_m$, we may assume that $F := T^\mu A$ is a field, where $A = \langle a_1 \rangle \times \cdots \times \langle a_{m-1} \rangle$. Then $S^\mu A / X S^\mu A$ is also a field. By Proposition 3.1, $R := S^\mu A$ is the valuation domain in F. Let $|a_i| = 2^{n_i}$, $\overline{a}_i = a_i^{2^{n_i - 1}}$ for $i = 1, \ldots, m$ and $\overline{A} = \operatorname{soc} A$. Then $\overline{A} = \langle \overline{a_1} \rangle \times \cdots \times \langle \overline{a}_{m-1} \rangle$. Denote by $\{u_g : g \in G_2\}$ the canonical S-basis of $S^\mu G_2$ corresponding to μ. We put

$$v_i = f_i(X)^{-2} u_{\overline{a}_i}.$$

Then $v_i^2 = \gamma_i u_e$, $L := K[v_1, \ldots, v_{m-1}]$ is a twisted group algebra of $\overline{A}$ over the field K, and L is a subfield of $S^\mu A$. We have $\gamma_m u_e = v_m^2$, where $v_m \in L$ and $v_m \notin F^2$ if $|a_m| \geq 4$. The ring R is a twisted group algebra of the factor group $A/\overline{A}$ over the ring $\hat{S} = L[[X]]$. Assume that $A \neq \overline{A}$ and $n_m \geq 2$. Then

$$R = [A/\overline{A}, \hat{S}, v_1 f_1(X)^2, \ldots, v_t f_t(X)^2],$$

where t is the number of invariants of $A/\overline{A}$.

Let w be a root of the irreducible polynomial

$$Y^{2^{n_m-1}} - v_m f_m(X)^2 \in \hat{S}[Y].$$

By Proposition 3.1, $R[w]$ is the valuation domain in the field $F(w)$. Let $D = \langle a_m \rangle \times B$. The algebra $S^\lambda G$ is a twisted group algebra $R^\sigma D$ of D over the ring R. By Proposition 2.5, $R^\sigma D$ is of OTP R-representation type. It follows, by Proposition 2.2, that $S^\lambda G$ is of OTP S-representation type. The case when $m = 1$ is treated similarly. ∎

THEOREM 3.10. *Let $G = G_p \times B$, $|G'_p| \neq 2$; Ω be the subgroup of S^* generated by K^* and $f(X)$, where $f(X) \equiv 1 \pmod X$ and $f(X) \not\equiv 1 \pmod{X^2}$; $\mu \in Z^2(G_p, \Omega)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. The algebra $S^\lambda G$ is of OTP representation type if and only if one of the following conditions is satisfied:*

(i) *G_p is abelian and $T^\mu G_p$ is a field;*
(ii) *$p = 2$, G_2 is abelian and $\dim_T \overline{T^\mu G_2} = |G_2|/2$;*
(iii) *K is a splitting field for $K^\nu B$.*

Proof. The necessity follows from Propositions 3.4 and 3.8. Let us prove the sufficiency. We may assume that $f(X) = 1+X$. Let $G_p = \langle a_1 \rangle \times \dots \times \langle a_m \rangle$, $|a_j| = p^{n_j}$ for $j = 1, \dots, m$, $m \geq 2$ and

$$S^\mu G_p = [G_p, S, \gamma_1(1+X)^{i_1}, \gamma_2(1+X)^{pi_2}, \dots, \gamma_m(1+X)^{pi_m}],$$

where $\gamma_1, \dots, \gamma_m \in K^*$. If $T^\mu G_p$ is a field then, by Propositions 1.11, 3.1 and 3.2, $S^\lambda G$ is of OTP representation type.

Now, let $p = 2$ and $\dim_T \overline{T^\mu G_2} = |G_2|/2$. If i_1 is divisible by 2, argue as in the proof of Theorem 3.9. Let $i_1 = 1$, $[K(\sqrt{\gamma_2}, \dots, \sqrt{\gamma_{m-1}}) : K] = 2^{m-2}$, $2i_j = k_j \cdot 2^{t_j}$, where $k_j \in \{0, 1\}$, $0 \leq t_j < n_j$ and $t_j = 0$ if $k_j = 0$, for any $j = 2, \dots, m$. Denote

$$c_j = a_j^{2^{n_j - t_j}},$$

$C = \langle c_2 \rangle \times \dots \times \langle c_{m-1} \rangle$, $D = \langle a_1 \rangle \times \dots \times \langle a_{m-1} \rangle$, $\widetilde{G}_2 = G_2/C$, $\widetilde{D} = D/C$; let w_j be a root of the irreducible polynomial

$$Y^{2^{t_j}} - \gamma_j \in K[Y], \quad j \in \{2, \dots, m-1\},$$

$L = K(w_2, \dots, w_{m-1})$, $\hat{S} = L[[X]]$, and let $\hat{T}$ be the quotient field of $\hat{S}$. By Proposition 3.2, $F := T^\mu D$ is a field and $R := S^\mu D$ is the valuation domain in F.

We have

$$S^\mu G_2 = [\widetilde{G}_2, \hat{S}, \gamma_1(1+X), w_2(1+X)^{k_2}, \dots, w_{m-1}(1+X)^{k_{m-1}}, \gamma_m(1+X)^{k_m \cdot 2^{t_m}}].$$

If $k_2+\cdots+k_{m-1} \neq 0$ then we may assume that $k_2 = 1, k_3 = 0, \ldots, k_{m-1} = 0$. Therefore

$$R = [\widetilde{D}, \hat{S}, \gamma_1(1+X), w_2(1+X)^{k_2}, w_3, \ldots, w_{m-1}]$$

and $\gamma_m = w_m^2$, where $w_m \in L$. If $n_m = 1$ then, by Propositions 2.2 and 2.5, $S^\lambda G$ is of OTP representation type. Let $n_m \geq 2$. Denote by v a root of the irreducible polynomial

$$Y^{2^{n_m-1}} - w_m(1+X)^{k_m \cdot 2^{t_m-1}} \in R[Y]$$

with $t_m \geq 1$. Let

$$C_1 = C \times \langle a_m^{2^{n_m-1}} \rangle \quad \text{and} \quad \hat{G}_2 = G_2/C_1.$$

Then

$$\begin{aligned} R[v] &\cong \hat{S}^{\hat{\mu}} \hat{G}_2 \\ &= [\hat{G}_2, \hat{S}, \gamma_1(1+X), w_2(1+X)^{k_2}, w_3, \ldots, w_{m-1}, w_m(1+X)^{k_m 2^{t_m-1}}]. \end{aligned}$$

If $n_1 \geq n_2 - t_2$ then we may suppose that $k_2 = 0$. If $n_1 < n_2 - t_2$ and $k_2 = 1$, then choose a canonical $\hat{S}$-basis of $\hat{S}^{\hat{\mu}} \hat{G}_2$ such that

$$\hat{S}^{\hat{\mu}} \hat{G}_2 = [\hat{G}_2, \hat{S}, \gamma_1 w_2^{-1}, w_2(1+X), w_3, \ldots, w_{m-1}, w_m(1+X)^{k_m \cdot 2^{t_m-1}}].$$

Let $t_m \geq 2$. Then $w_m \notin F^2$. By Proposition 3.2, $\hat{S}^{\hat{\mu}} \hat{G}_2$ is the valuation domain in the field $\hat{T}^{\hat{\mu}} \hat{G}_2$. Hence, by Propositions 2.2 and 2.5, $S^\lambda G$ is of OTP representation type. Now let $t_m = 1$ and $k_m = 1$. Then

$$R[v] \cong [\hat{G}_2, \hat{S}, \rho_1(1+X), \rho_2, \ldots, \rho_m],$$

where $\rho_1, \ldots, \rho_m \in L^*$. In view of Propositions 2.2, 2.5 and 3.2, $S^\lambda G$ is of OTP representation type. ■

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