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ON AFFINITY OF PEANO TYPE FUNCTIONS

BY

TOMASZ SŁONKA (Katowice)

Abstract. We show that if n is a positive integer and $2^{\aleph_0} \le \aleph_n$, then for every positive integer m and for every real constant $c > 0$ there are functions $f_1, \dots, f_{n+m} : \mathbb{R}^n \to \mathbb{R}$ such that $(f_1, \dots, f_{n+m})(\mathbb{R}^n) = \mathbb{R}^{n+m}$ and for every $x \in \mathbb{R}^n$ there exists a strictly increasing sequence $(i_1, \dots, i_n)$ of numbers from $\{1, \dots, n+m\}$ and a $w \in \mathbb{Z}^n$ such that

$$(f_{i_1}, \dots, f_{i_n})(y) = y + w \quad \text{for } y \in x + (-c, c) \times \mathbb{R}^{n-1}.$$

According to Theorem 1 of [2] by M. Morayne the Continuum Hypothesis implies the existence of functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $(f_1, f_2)(\mathbb{R}) = \mathbb{R}^2$ and for each $x \in \mathbb{R}$ at least one of f_1, f_2 is differentiable at x. In [3] M. Morayne gave a more general result: If $n \in \mathbb{N}$ and $2^{\aleph_0} \le \aleph_n$, then there are functions $f_1, \dots, f_{n+1} : \mathbb{R}^n \to \mathbb{R}$ such that $(f_1, \dots, f_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$ and for each point of $\mathbb{R}$ at least n of those functions are differentiable at that point. J. Cichoń and M. Morayne [1] generalized this result as follows. If $2^{\aleph_0} \le \aleph_n$, then for any $m \in \mathbb{N}$ there are functions $f_1, \dots, f_{n+m} : \mathbb{R}^n \to \mathbb{R}$ such that $(f_1, \dots, f_{n+m})(\mathbb{R}^n) = \mathbb{R}^{n+m}$ and at each point of $\mathbb{R}^n$ at least n of them are analytic at that point. It is the aim of this paper to strengthen this statement, replacing analyticity at the point by affinity in some of its neighbourhoods. More exactly we will prove the following theorem.

THEOREM. *If $2^{\aleph_0} \le \aleph_n$ and $m \in \mathbb{N}$, then for every $c \in (0, \infty)$ there exist functions $f_1, \dots, f_{n+m} : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

(i) *for every $x \in \mathbb{R}^n$ there exist a strictly increasing sequence $(i_1, \dots, i_n)$ of numbers from $\{1, \dots, n+m\}$ and a $w \in \mathbb{Z}^n$ such that*

$$(f_{i_1}, \dots, f_{i_n})(y) = y + w \quad \text{for } y \in x + (-c, c) \times \mathbb{R}^{n-1}.$$

(ii) $(f_1, \dots, f_{n+m})(\mathbb{R}^n) = \mathbb{R}^{n+m}$.

We begin by proving several lemmas. The first one is a special case of [5, Theorem] by R. Sikorski, which is a variant of the well-known Sierpiński decomposition of the plane.

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LEMMA 1. *If X is a nonvoid set with $|X| \leq \aleph_n$, then there exists a collection*

$$\{A_{x_1,\ldots,x_n}\colon (x_1,\ldots,x_n) \in X^n\} \tag{1}$$

of countable sets such that

$$X^{n+1} = \bigcup_{k=1}^{n+1} \bigcup_{(x_1,\ldots,x_n)\in X^n} \bigcup_{a\in A_{x_1,\ldots,x_n}} \{(x_1,\ldots,x_{k-1},a,x_k,\ldots,x_n)\}. \tag{2}$$

Lemma 2 is probably folklore. For case $n = 1$ it boils down to the well-known fact (see [4, Ch. I, Prop. P_2]) that the square of a set of cardinality $\aleph_1$ is the countable union of graphs of functions and their inverses (as relations). Hence we only sketch its proof.

LEMMA 2. *Assume $|\mathbb{R}| \leq \aleph_n$. Then for every $k \in \mathbb{N}$ there are functions $f_{1,k},\ldots,f_{n+1,k}\colon [0,1]^n \to [0,1]$ with the following properties:*

(iii) *for every $k \in \mathbb{N}$ there is an $i \in \{1,\ldots,n+1\}$ such that*

$$(f_{1,k},\ldots,f_{i-1,k},f_{i+1,k},\ldots,f_{n+1,k}) = \mathrm{id}_{[0,1]^n}; \tag{3}$$

(iv) $\bigcup_{k\in\mathbb{N}}(f_{1,k},\ldots,f_{n+1,k})([0,1]^n) = [0,1]^{n+1}$.

Proof. Applying Lemma 1 for $X = [0,1]$ we obtain a collection (1) of countable sets such that (2) holds. Let $h\colon [0,1]^n \to \bigcup_{(x_1,\ldots,x_n)\in[0,1]^n} A^{\mathbb{N}}_{x_1,\ldots,x_n}$ be a function such that $h(x_1,\ldots,x_n)$ maps $\mathbb{N}$ onto $A_{x_1,\ldots,x_n}$ for any $x_1,\ldots,x_n \in [0,1]$. We define $f_{1,k},\ldots,f_{n+1,k}\colon [0,1]^n \to [0,1]$ by

$$\begin{aligned}(f_{1,(n+1)(k-1)+l},\ldots,f_{n+1,(n+1)(k-1)+l})(x_1,\ldots,x_n)\\ = (x_1,\ldots,x_{l-1},h(x_1,\ldots,x_n)(k),x_l,\ldots,x_n)\end{aligned}$$

for $k \in \mathbb{N}$, $l \in \{1,\ldots,n+1\}$ and $x_1,\ldots,x_n \in [0,1]$. ■

LEMMA 3. *There is a bijection $a\colon \mathbb{Z}^{n+1} \times \mathbb{N} \to 2\mathbb{Z}$ such that*

$$|s_1 - t_1| > 1 \;\Rightarrow\; |a(s_1,\ldots,s_{n+1},k) - a(t_1,\ldots,t_{n+1},l)| > 2 \tag{4}$$

for $(s_1,\ldots,s_{n+1}),(t_1,\ldots,t_{n+1}) \in \mathbb{Z}^{n+1}$ and $k,l \in \mathbb{N}$.

Proof. Let (see the figure) $\alpha\colon \mathbb{N}\cup\{0\} \to \mathbb{Z}\times(\mathbb{N}\cup\{0\})$ be a bijection such that $\alpha(0) = (0,0)$ and

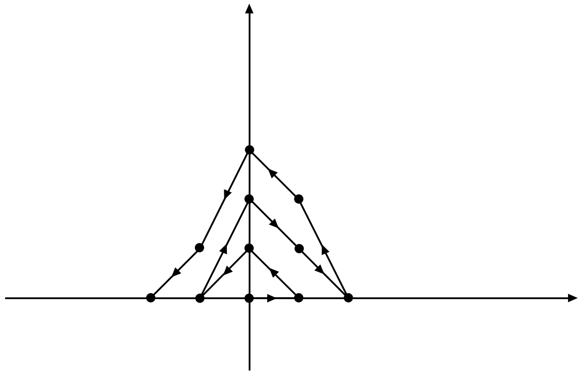
$$|k-l| \leq 1 \;\Rightarrow\; |\alpha_1(k) - \alpha_1(l)| \leq 1$$

for $k,l \in \mathbb{N}\cup\{0\}$, and $\beta\colon (-\mathbb{N}) \to \mathbb{Z}\times(-\mathbb{N})$ a bijection such that $\beta(-1) = (0,-1)$ and

$$|k-l| \leq 1 \;\Rightarrow\; |\beta_1(k) - \beta_1(l)| \leq 1$$

for $k,l \in -\mathbb{N}$. Putting $\gamma = \alpha\cup\beta$ we obtain a bijection $\gamma\colon \mathbb{Z} \to \mathbb{Z}^2$ such that

$$|k-l| \leq 1 \;\Rightarrow\; |\gamma_1(k) - \gamma_1(l)| \leq 1$$



for $k, l \in \mathbb{Z}$. Further, take an arbitrary bijection $\delta\colon \mathbb{Z} \to \mathbb{Z}^n \times \mathbb{N}$ and define $b\colon 2\mathbb{Z} \to \mathbb{Z}^{n+1} \times \mathbb{N}$ by

$$b(2k) = (\gamma_1(k), \delta(\gamma_2(k))) \quad \text{for } k \in \mathbb{Z}.$$

It is easy to see that b is a bijection and its inverse a has the desired properties. ∎

LEMMA 4. *If* $2^{\aleph_0} \le \aleph_n$, *then there exists a real constant* $\delta > 0$ *and functions* $f_1, \ldots, f_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ *with the following properties:*

(v) $|f_1(x) - f_1(y)| \le 3$ *for* $x \in \mathbb{R}^n$ *and* $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$;

(vi) *for every* $x \in \mathbb{R}^n$ *there exists an* $i \in \{1, \ldots, n+1\}$ *and a* $w \in \mathbb{Z}^n$ *such that*

$$(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n+1})(y) = y + w \quad \textit{for } y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1};$$

(vii) $(f_1, \ldots, f_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$.

Before the proof we propose to consider the special case $n = 1$ without (v).

LEMMA 4′. *If* $2^{\aleph_0} = \aleph_1$, *then there exist functions* $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ *and a* $\delta > 0$ *with the following properties:*

(vi′) *for every* $x \in \mathbb{R}$ *there is an integer* w *such that either*

$$f_1(y) = y + w \quad \textit{for } y \in (x - \delta, x + \delta)$$

or

$$f_2(y) = y + w \quad \textit{for } y \in (x - \delta, x + \delta);$$

(vii′) $(f_1, f_2)(\mathbb{R}) = \mathbb{R}^2$.

Proof. We will construct the desired functions f_1, f_2 in two steps. First, applying Lemma 2, we define them on $\bigcup_{z \in \mathbb{Z}} [2z, 2z + 1]$ in such a manner that for every $z \in \mathbb{Z}$ there is an $i \in \{1, 2\}$ and a $c \in \mathbb{Z}$ such that $f_i(x) = x + c$ on $[2z, 2z + 1]$, and $(f_1, f_2)(\bigcup_{z \in \mathbb{Z}} [2z, 2z + 1]) = \mathbb{R}^2$. In the second step we extend them onto the whole $\mathbb{R}$ to functions satisfying (vi′).

Clearly $\mathbb{R}^2$ is the union of the squares $[s_1, s_1+1] \times [s_2, s_2+1]$ over all $s_1, s_2 \in \mathbb{Z}$. Furthermore, each such square is a countable union of graphs of functions and their inverses (as relations). Consequently, for every $n \in \mathbb{N}$ there exists a function $(h_{1,n}, h_{2,n})\colon [0,1] \to \mathbb{R}^2$ such that $\bigcup_{n\in\mathbb{N}}(h_{1,n}, h_{2,n})([0,1]) = \mathbb{R}^2$ and for every $n \in \mathbb{N}$ there exists $s \in \mathbb{Z}$ such that $h_{1,n} = \mathrm{id}_{[0,1]} + s$ or $h_{2,n} = \mathrm{id}_{[0,1]} + s$.

Take now any bijection $a\colon \mathbb{N} \to 2\mathbb{Z}$ and define $\tilde{f}\colon \bigcup_{z\in\mathbb{Z}}[2z, 2z+1] \to \mathbb{R}^2$ as follows:

$$(\tilde{f}_1, \tilde{f}_2)(x+2z) = (h_{1,a^{-1}(2z)}, h_{2,a^{-1}(2z)})(x) \quad \text{for } x \in [0,1] \text{ and } z \in \mathbb{Z}.$$

Consider any $\psi\colon \mathbb{Z} \to \{1,2\} \times \mathbb{Z}$ such that

$$\text{if} \quad \psi(z) = (i,c), \quad \text{then} \quad \tilde{f}_i|_{[2z,2z+1]} = \mathrm{id}_{[2z,2z+1]} + c.$$

As any extension (f_1, f_2) of $\tilde{f}$ onto $\mathbb{R}^2$ satisfies (vii′), the final step is to extend $\tilde{f}$ to a function f satisfying (vi′).

For each $z \in \mathbb{Z}$ we define f on $[2z-1/2, 2z+3/2)$ by putting $(i,c) = \psi(z)$ and

$$(f_1, f_2)(x) = \begin{cases} (\tilde{f}_1, \tilde{f}_2)(x) & \text{for } x \in [2z, 2z+1], \\ (x+c, x+c) & \text{for } x \in [2z-1/4, 2z+5/4] \setminus [2z, 2z+1], \\ (x+c, x) & \text{for } x \in [2z-1/2, 2z+3/2) \setminus [2z-1/4, 2z+5/4]. \end{cases}$$

If $z \in \mathbb{Z}$, then

$$\begin{aligned} &(2z-3/4, 2z-1/4) \\ &\subset ([2(z-1)-1/2, 2(z-1)+3/2) \setminus [2(z-1)-1/4, 2(z-1)+5/4]) \\ &\qquad \cup ([2z-1/2, 2z+3/2) \setminus [2z-1/4, 2z+5/4]) \end{aligned}$$

whence

$$f_2(x) = x \quad \text{for } x \in (2z-3/4, 2z-1/4);$$

moreover, if $\psi(z) = (i,c)$, then also

$$\begin{aligned} f_i(x) &= x+c \quad \text{for } x \in [2z-1/4, 2z+5/4], \\ f_1(x) &= x+c \quad \text{for } x \in [2z-1/2, 2z) \cup (2z+1, 2z+3/2], \end{aligned}$$

which yields (vi′) with $\delta = 1/8$. ∎

In the proof of Lemma 4 we will construct the desired functions similarly, but to have also (v) in the first step we rely on both Lemmas 3 and 2.

Proof of Lemma 4. Let $a\colon \mathbb{Z}^{n+1} \times \mathbb{N} \to 2\mathbb{Z}$ be a bijection such that (4) holds, for $s = (s_1, \ldots, s_{n+1}) \in \mathbb{Z}^{n+1}$, $k \in \mathbb{N}$ put $a_k^s = a(s,k)$ and define $e_k^s\colon [a_k^s, a_k^s+1] \times [0,1]^{n-1} \to [0,1]^n$ and $l^s\colon [0,1]^{n+1} \to \prod_{i=1}^{n+1}[s_i, s_i+1]$ by

$$e_k^s(u) = u - (a_k^s, 0, \ldots, 0), \qquad l^s(v) = s + v.$$

Clearly e_k^s and l^s are bijections for all $s \in \mathbb{Z}^{n+1}$, $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ let $f_{1,k}, \ldots, f_{n+1,k} \colon [0,1]^n \to [0,1]$ satisfy (iii) and (iv), and put

$$(h_{1,k}^s, \ldots, h_{n+1,k}^s) = l^s \circ (f_{1,k}, \ldots, f_{n+1,k}) \circ e_k^s \quad \text{for } s \in \mathbb{Z}^{n+1}.$$

Then

$$\begin{aligned}(5)\quad & \bigcup_{k\in\mathbb{N}} (h_{1,k}^s, \ldots, h_{n+1,k}^s)([a_k^s, a_k^s+1] \times [0,1]^{n-1}) \\ &= \bigcup_{k\in\mathbb{N}} l^s((f_{1,k}, \ldots, f_{n+1,k})(e_k^s([a_k^s, a_k^s+1] \times [0,1]^{n-1}))) \\ &= l^s\Big(\bigcup_{k\in\mathbb{N}} (f_{1,k}, \ldots, f_{n+1,k})([0,1]^n)\Big) = l^s([0,1]^{n+1}) = [0,1]^{n+1} + s\end{aligned}$$

for $s \in \mathbb{Z}^{n+1}$. Moreover, if $k \in \mathbb{N}$ and $i \in \{1, \ldots, n+1\}$ is such that (3) holds, then for any $s \in \mathbb{Z}^{n+1}$ and $x \in [a_k^s, a_k^s+1] \times [0,1]^{n-1}$ we have

$$\begin{aligned}(6)\quad & (h_{1,k}^s, \ldots, h_{i-1,k}^s, h_{i+1,k}^s, \ldots, h_{n+1,k}^s)(x) \\ &= (f_{1,k}, \ldots, f_{i-1,k}, f_{i+1,k}, \ldots, f_{n+1,k}) \circ e_k^s(x) + (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n+1}) \\ &= e_k^s(x) + (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n+1}) \\ &= \begin{cases} x + (s_2 - a_k^s, s_3, \ldots, s_{n+1}) & \text{if } i = 1, \\ x + (s_1 - a_k^s, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n+1}) & \text{if } i > 1, \end{cases} \\ &= x + c\end{aligned}$$

with a suitable $c \in \mathbb{Z}^n$ (depending on s and k).

Since

$$[a_p^s, a_p^s+1] \cap [a_q^t, a_q^t+1] = \phi \quad \text{for } (p,s) \neq (q,t),$$

the formula

$$(\tilde{f}_1, \ldots, \tilde{f}_{n+1})(x) = (h_{1,k}^s, \ldots, h_{n+1,k}^s)(x)$$

for $x \in [a_k^s, a_k^s+1] \times [0,1]^{n-1}$, $s \in \mathbb{Z}^{n+1}$ and $k \in \mathbb{N}$ defines a function $(\tilde{f}_1, \ldots, \tilde{f}_{n+1}) \colon \bigcup_{z\in\mathbb{Z}} [2z, 2z+1] \times [0,1]^{n-1} \to \mathbb{R}^{n+1}$. According to (5) we have

$$\begin{aligned}(7)\quad & (\tilde{f}_1, \ldots, \tilde{f}_{n+1})\Big(\bigcup_{z\in\mathbb{Z}} [2z, 2z+1] \times [0,1]^{n-1}\Big) \\ &= \bigcup_{(s,k)\in\mathbb{Z}^{n+1}\times\mathbb{N}} (h_{1,k}^s, \ldots, h_{n+1,k}^s)([a_k^s, a_k^s+1] \times [0,1]^{n-1}) \\ &= \bigcup_{s\in\mathbb{Z}^{n+1}} ([0,1]^{n+1} + s) = \mathbb{R}^{n+1}.\end{aligned}$$

Moreover, it follows from (6) that there exists a $\varphi \colon \mathbb{Z} \to \{1, \ldots, n+1\} \times \mathbb{Z}^{n+1}$ such that if $\varphi(z) = (i, c)$, then

$$(8)\quad (\tilde{f}_1, \ldots, \tilde{f}_{i-1}, \tilde{f}_{i+1}, \ldots, \tilde{f}_{n+1})|_{[2z,2z+1]\times[0,1]^{n-1}} = \mathrm{id}_{[2z,2z+1]\times[0,1]^{n-1}} + c.$$

For $z \in \mathbb{Z}$ we decompose the set $(2z-2/3, 2z+4/3] \times \mathbb{R}^{n-1}$ putting

$$B_z = [2z, 2z+1] \times [0,1]^{n-1}, \qquad C_z = (2z-2/3, 2z-1/3) \times \mathbb{R}^{n-1},$$

$$C_{z,j} = \left(\left[2z - \tfrac{n+1+j}{6n+3}, 2z - \tfrac{n+j}{6n+3}\right) \cup \left(2z+1+\tfrac{n+j}{6n+3}, 2z+1+\tfrac{n+j+1}{6n+3}\right]\right) \times \mathbb{R}^{n-1}$$

for $j \in \{1, \ldots, n\}$, and, with $(i,c) = \varphi(z)$,

$$B_{z,0} = \left[2z - \tfrac{i}{6n+3}, 2z+1+\tfrac{i}{6n+3}\right] \times \mathbb{R}^{n-1} \setminus B_z,$$

$$B_{z,j} = \left(\left[2z - \tfrac{i+j}{6n+3}, 2z - \tfrac{i+j-1}{6n+3}\right) \cup \left(2z+1+\tfrac{i+j-1}{6n+3}, 2z+1+\tfrac{i+j}{6n+3}\right]\right) \times \mathbb{R}^{n-1}$$

for $j \in \{1, \ldots, n+1-i\}$, and then we define $(f_1, \ldots, f_{n+1})$ on $(2z-2/3, 2z+4/3] \times \mathbb{R}^{n-1}$ by taking $s \in \mathbb{Z}^{n+1}$ and $k \in \mathbb{N}$ such that $a_k^s = 2z$ and putting

$$(f_1, \ldots, f_{n+1})(x) =$$
$$\begin{cases} (\tilde{f}_1, \ldots, \tilde{f}_{n+1})(x) & \text{for } x \in B_z, \\ (x_1+c_1, \ldots, x_{i-1}+c_{i-1}, s_1, x_i+c_i, \ldots, x_n+c_n) & \text{for } x \in B_{z,0}, \\ (x_1+s_1-2z, x_1+c_1, \ldots, x_n+c_n) & \text{for } x \in B_{z,1} \text{ if } i=1, \\ (x_1+c_1, \ldots, x_i+c_i, x_i+c_i, \ldots, x_n+c_n) & \text{for } x \in B_{z,1} \text{ if } i>1, \\ (x_1+s_1-2z, x_2+c_2, \ldots, x_{i-1+j}+c_{i-1+j}, & \text{for } x \in B_{z,j} \text{ and} \\ \qquad x_{i-1+j}+c_{i-1+j}, x_{i+j}+c_{i+j}, \ldots, x_n+c_n) & j \in \{2, \ldots, n+1-i\}, \\ (x_1+s_1-2z, x_2+c_2, \ldots, x_{n+1-j}+c_{n+1-j}, & \text{for } x \in C_{z,j} \text{ and} \\ \qquad x_{n+1-j}, \ldots, x_n) & j \in \{1, \ldots, n\}, \\ (s_1, x_1, \ldots, x_n) & \text{for } x \in C_z. \end{cases}$$

We will show that (v)–(vii) hold with

$$\delta = \frac{1}{12n+6}.$$

First, however, observe that if $z \in \mathbb{Z}$ and $a_k^s = 2z$, then for $i > 1$ according to the definition of c in (6), we have $s_1 = c_1 + 2z$ and so

$$\begin{aligned} f_1((2z-2/3, 2(z+1)-2/3] \times \mathbb{R}^{n-1}) \\ &\subset \tilde{f}_1(B_z) \cup ((2z-2/3, 2(z+1)-2/3] + c_1) \\ &\subset \tilde{f}_1(B_z) \cup (s_1-2/3, s_1+4/3]; \end{aligned}$$

and if $i = 1$, then

$$f_1((2z-2/3, 2(z+1)-2/3] \times \mathbb{R}^{n-1}) \subset \tilde{f}_1(B_z) \cup (s_1-2/3, s_1+4/3]$$

as well. Consequently, for any $z \in \mathbb{Z}$, if $a_k^s = 2z$, then

$$\begin{aligned} f_1((2z-2/3, 2(z+1)-2/3] \times \mathbb{R}^{n-1}) \\ &\subset h_{1,k}^s([2z, 2z+1] \times \mathbb{R}^{n-1}) \cup (s_1-2/3, s_1+4/3] \\ &\subset [s_1, s_1+1] \cup (s_1-2/3, s_1+4/3] = (s_1-2/3, s_1+4/3]. \end{aligned}$$

Let $x, y \in \mathbb{R}^n$ and $|x_1 - y_1| < 1$. Then there are integers z_1, z_2 such that

$$x_1 \in (2z_1 - 2/3, 2(z_1+1) - 2/3],\ y_1 \in (2z_2 - 2/3, 2(z_2+1) - 2/3]$$

and $|z_2 - z_1| \le 1$. If $a_k^s = 2z_1$ and $a_l^t = 2z_2$, then

$$|a(s_1, \dots, s_{n+1}, k) - a(t_1, \dots, t_{n+1}, l)| = |2z_1 - 2z_2| \le 2$$

and

$$f_1(x) \in (s_1 - 2/3, s_1 + 4/3],\ f_1(y) \in (t_1 - 2/3, t_1 + 4/3],$$

which together with (4) shows that $|s_1 - t_1| \le 1$ and $|f_1(x) - f_1(y)| \le 3$ and proves (v).

To prove (vi) fix an $x \in \mathbb{R}^n$, let

$$x \in (2z - 2/3 - \delta, 2(z+1) - 2/3 - \delta] \times \mathbb{R}^{n-1}$$

for a $z \in \mathbb{Z}$ and put $(i, c) = \varphi(z)$. We will distinguish five cases depending on the first coordinate x_1 of x.

1. If $x_1 \in \left[2z - \frac{i}{6n+3} + \delta, 2z + 1 + \frac{i}{6n+3} - \delta\right]$, then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset B_z \cup B_{z,0},$$

and taking into account (8) for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n+1})(y) = y + c.$$

2. If

$$x_1 \in \left[2z - \frac{i+j}{6n+3} + \delta, 2z - \frac{i+j-2}{6n+3} - \delta\right) \cup \left(2z + 1 + \frac{i+j-2}{6n+3} + \delta, 2z + 1 + \frac{i+j}{6n+3} - \delta\right]$$

for some $j \in \{1, \dots, n+1-i\}$, then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset B_{z,j} \cup B_{z,j-1},$$

and for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have (note that if $i > 1$, then $c_1 = s_1 - 2z$)

$$(f_1, \dots, f_{i+j-2}, f_{i+j}, \dots, f_{n+1})(y) = \begin{cases} y + c & \text{if } j = 1, \\ y + (s_1 - 2z, c_2, \dots, c_n) & \text{if } j > 1. \end{cases}$$

3. If

$$x_1 \in \left[2z - \frac{n+2}{6n+3} + \delta, 2z - \frac{n}{6n+3} - \delta\right) \cup \left(2z + 1 + \frac{n}{6n+3} + \delta, 2z + 1 + \frac{n+2}{6n+3} - \delta\right],$$

then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset B_{z,n+1-i} \cup C_{z,1}$$

and for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \dots, f_n)(y) = y + (s_1 - 2z, c_2, \dots, c_n).$$

4. If

$$x_1 \in \left[2z - \frac{n+1+j}{6n+3} + \delta, 2z - \frac{n-1+j}{6n+3} - \delta\right) \cup \left(2z + 1 + \frac{n-1+j}{6n+3} + \delta, 2z + 1 + \frac{n+1+j}{6n+3} - \delta\right]$$

for some $j \in \{2, \ldots, n\}$, then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset C_{z,j} \cup C_{z,j-1},$$

and for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have

$$\begin{aligned}(f_1, \ldots, f_{n+1-j}, f_{n+3-j}, \ldots, f_{n+1})(y) \\ &= (y_1 + c_1, \ldots, y_{n+1-j} + c_{n+1-j}, y_{n+2-j}, \ldots, y_n) \\ &= y + (s_1 - 2z, c_2, \ldots, c_{n+1-j}, 0, \ldots, 0).\end{aligned}$$

5. If $x_1 \in (2z - 2/3 - \delta, 2z - 1/3 + \delta)$, then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset C_z \cup C_{z,n} \cup C_{z-1,n}$$

and for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_2, \ldots, f_{n+1})(y) = y.$$

To get (vii) it is enough to observe that according to (7) we have

$$\begin{aligned}(f_1, \ldots, f_{n+1})(\mathbb{R}^n) \supset \bigcup_{z \in \mathbb{Z}} (f_1, \ldots, f_{n+1})(B_z) = \bigcup_{z \in \mathbb{Z}} (\tilde{f}_1, \ldots, \tilde{f}_{n+1})(B_z) \\ = \mathbb{R}^{n+1}. \ \blacksquare\end{aligned}$$

LEMMA 5. *If $2^{\aleph_0} \le \aleph_n$, then for every $d \in (0, \infty)$ there exists an $M \in (0, \infty)$ and functions $h_1, \ldots, h_{n+1} : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

(v″) $|h_1(x) - h_1(y)| \le M$ *for* $x \in \mathbb{R}^n$ *and* $y \in x + (-d, d) \times \mathbb{R}^{n-1}$;

(vi″) *for every $x \in \mathbb{R}^n$ there exists an $l \in \{1, \ldots, n+1\}$ and a $w \in \mathbb{Z}^n$ such that*

(9) $(h_1, \ldots, h_{l-1}, h_{l+1}, \ldots, h_{n+1})(y) = y + w$ *for* $y \in x + (-d, d) \times \mathbb{R}^{n-1}$;

(vii″) $(h_1, \ldots, h_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$.

Proof. Fix a $d \in (0, \infty)$ and making use of Lemma 4 choose a positive real constant δ and functions $f_1, \ldots, f_{n+1} : \mathbb{R}^n \to \mathbb{R}$ with properties (v)–(vii).

Let m be a natural number such that $d < m\delta$. Defining $h_1, \ldots, h_{n+1} : \mathbb{R}^n \to \mathbb{R}$ by

$$h_j(y) = m f_j\left(\frac{1}{m} y\right) \quad \text{for } j \in \{1, \ldots, n+1\}$$

we easily see that (v″)–(vii″) hold with $M = 3m$. ∎

Proof of the Theorem. The proof goes by induction on m and Lemma 5 provides its first step. Let m be a positive integer and fix $d \in (0, \infty)$. Applying Lemma 5 we obtain an $M \in (0, \infty)$ and functions $h_1, \dots, h_{n+1} \colon \mathbb{R}^n \to \mathbb{R}$ with properties (v″)–(vii″).

Making use of the induction hypothesis consider functions $f_1, \dots, f_{n+m} : \mathbb{R}^n \to \mathbb{R}$ satisfying (i) with $c = \max\{d, M\}$ and (ii). It follows from (ii) that $(\tilde{f}_1, \dots, \tilde{f}_{n+m+1}) \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+m+1}$ given by

$$(\tilde{f}_1, \dots, \tilde{f}_{n+m+1})(x_1, \dots, x_{n+1}) = ((f_1, \dots, f_{n+m})(x_1, \dots, x_n), x_{n+1})$$

is a surjection and so is the function $(g_1, \dots, g_{n+m+1}) \colon \mathbb{R}^n \to \mathbb{R}^{n+m+1}$ defined by

$$(g_1, \dots, g_{n+m+1}) = (\tilde{f}_1, \dots, \tilde{f}_{n+m+1}) \circ (h_1, \dots, h_{n+1}).$$

Fix now an $x \in \mathbb{R}^n$ and let $l \in \{1, \dots, n+1\}$ and $w \in \mathbb{Z}^n$ be such that (9) holds. Then

$$\text{(10)} \qquad \text{if } l > 1, \text{ then } h_1(y) = y_1 + w_1 \quad \text{for } y \in x + (-d, d) \times \mathbb{R}^{n-1}.$$

It follows from (i) that there exist a strictly increasing sequence $(i_1, \dots, i_n) \in \{1, \dots, n+m\}^n$ and a $u \in \mathbb{Z}^n$ such that

$$\text{(11)} \qquad (f_{i_1}, \dots, f_{i_n})(y) = y + u \quad \text{for } y \in (h_1(x) + (-c, c)) \times \mathbb{R}^{n-1}.$$

If $l \le n$, then put

$$i_{n+1} = n + m + 1, \qquad j_k = \begin{cases} i_k & \text{for } k < l, \\ i_{k+1} & \text{for } l \le k \le n, \end{cases}$$

$$v = (w_1 + u_1, \dots, w_{l-1} + u_{l-1}, w_l + u_{l+1}, \dots, w_{n-1} + u_n, w_n).$$

Clearly, $1 \le j_1 < \dots < j_n \le n+m+1$ and $v \in \mathbb{Z}^n$. Fix $y \in x + (-d, d) \times \mathbb{R}^{n-1}$. According to (9) we have

$$\begin{aligned}(g_{j_1}, \dots, g_{j_n})(y) &= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_{l-1}}, \tilde{f}_{i_{l+1}}, \dots, \tilde{f}_{i_n}, \tilde{f}_{n+m+1})(h_1, \dots, h_{n+1})(y) \\ &= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_{l-1}}, \tilde{f}_{i_{l+1}}, \dots, \tilde{f}_{i_n}, \tilde{f}_{n+m+1})(y_1 + w_1, \dots, \\ &\qquad\qquad y_{l-1} + w_{l-1}, h_l(y), y_l + w_l, \dots, y_n + w_n) \\ &= ((f_{i_1}, \dots, f_{i_{l-1}}, f_{i_{l+1}}, \dots, f_{i_n})(y_1 + w_1, \dots, y_{l-1} + w_{l-1}, \\ &\qquad\qquad h_l(y), y_l + w_l, \dots, y_{n-1} + w_{n-1})), y_n + w_n).\end{aligned}$$

Moreover, as follows from (v″),

$$(h_1(y), y_1 + w_1, \dots, y_{n-1} + w_{n-1}) \in (h_1(x) + (-c, c)) \times \mathbb{R}^{n-1}$$

and if $l > 1$, then (10) shows that the point

$$(y_1 + w_1, \dots, y_{l-1} + w_{l-1}, h_l(y), y_l + w_l, \dots, y_{n-1} + w_{n-1})$$

belongs to $(h_1(x) + (-c, c)) \times \mathbb{R}^{n-1}$. Consequently, taking also (11) into account,

$$(g_{j_1}, \dots, g_{j_n})(y) = (y_1 + w_1 + u_1, \dots, y_{l-1} + w_{l-1} + u_{l-1},$$
$$y_l + w_l + u_{l+1}, \dots, y_{n-1} + w_{n-1} + u_n, y_n + w_n)$$
$$= y + v.$$

If $l = n + 1$, then taking (10), (v″) and (11) into account we see that

$$\begin{aligned}(g_{i_1}, \dots, g_{i_n})(y) &= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_n})((h_1, \dots, h_{n+1})(y)) \\ &= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_n})(y_1 + w_1, \dots, y_n + w_n, h_{n+1}(y)) \\ &= (f_{i_1}, \dots, f_{i_n})(y + w) = y + w + u. \ \blacksquare\end{aligned}$$

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Tomasz Słonka
Uniwersytet Śląski
40-007 Katowice, Poland
E-mail: t.wodzu@gmail.com

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