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MINIMAL NON-INVERTIBLE TRANSFORMATIONS OF SOLENOIDS

BY

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Abstract. We construct a continuous non-invertible minimal transformation of an arbitrary solenoid. Since solenoids, as all other compact monothetic groups, also admit minimal homeomorphisms, our result allows one to classify solenoids among continua admitting both invertible and non-invertible continuous minimal maps.

1. Introduction. We consider discrete dynamical systems (X, f), where X is a compact metric space and $f\colon X \to X$ is a continuous map. A dynamical system is *minimal* if X has no non-empty proper closed f-invariant subsets ($M \subset X$ is *f-invariant* if $f(M) \subset M$; see e.g. [2]).

There is an interesting classification of compact metric spaces with respect to the existence of minimal homeomorphisms and non-invertible maps. Let us say that a space X is of *type* (ξ_1, ξ_2), where $\xi_1, \xi_2 \in \{0, 1\}$, according to the following rule: if there exists a minimal homeomorphism on X, then we set $\xi_1 = 1$, otherwise we set $\xi_1 = 0$; and if there exists a minimal non-invertible continuous map on X, then we set $\xi_2 = 1$, otherwise we set $\xi_2 = 0$.

There are well known easy examples of all four types of spaces. Moreover, all four types have representatives among connected spaces (continua). The unit interval $[0, 1]$ has the fixed point property, so it admits neither minimal homeomorphisms, nor minimal non-invertible maps (type $(0, 0)$). The 2-torus $\mathbb{T}$ admits both invertible and non-invertible minimal maps (type $(1, 1)$; see [11]). The pinched 2-torus (the 2-torus with two points identified) admits non-invertible minimal maps, while it has the fixed point property for homeomorphisms, so it does not admit minimal homeomorphisms (type $(0, 1)$; see [3]). The simple closed curve S^1 (i.e., the circle) admits minimal homeomorphisms but it does not admit minimal non-invertible maps (type $(1, 0)$; see [1]).

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There is no other known infinite continuum of type $(1,0)$ and it is a long standing open problem whether S^1 is indeed the unique continuum of this type. In search for other such continua it is natural to examine spaces as similar to the circle as possible. Let us mention that in [3] the authors conjecture that the pseudo-circle is of this type. It is known that there exists a minimal homeomorphism of the pseudo-circle (see [10]), but the existence of a non-invertible minimal map on this space is undecided. More information about this problem can be found in [3] and [11].

In this paper we focus on solenoids. Every solenoid, like the circle, has the structure of a compact monothetic group, hence it admits a minimal homeomorphism (rotation by a topological generator). Moreover, each solenoid is connected, which makes it even more similar to the circle. The same two properties (being a compact connected monothetic group) also hold for multidimensional tori, but solenoids and the circle are the only one-dimensional connected compact monothetic groups. Solenoids can be obtained from the circle via inverse limit procedures, a property they share with the pseudo-circle. All these facts make solenoids natural candidates to be tested for their type, and, to our knowledge, they have not yet been examined in this respect.

We will prove the following

Theorem 1.1. *Every solenoid admits a minimal non-invertible continuous map.*

That is, all solenoids are of type $(1,1)$. This result does not solve the general problem whether there exists a continuum of type $(1,0)$ different from a simple closed curve. But at least we decide how to classify solenoids, so we solve the restricted problem for monothetic groups of topological dimension one, and we eliminate a natural class of candidate spaces to behave (with respect to the existence of minimal maps) like the circle.

2. Construction. A solenoid (here denoted by Σ) was orginally defined as a continuum homeomorphic to the inverse limit $\varprojlim(S^1, f_n)$, where $f_n : S^1 \to S^1$ is a transformation of the form z^{q_n} and q_n is a sequence of natural numbers greater than 1 (see [12], [6] for more information on solenoids). As we will not use this definition, we refrain from providing the details of inverse limit constructions. Instead we will use a different characterization of solenoids relying on odometers (see [5] for more information on odometers). *Odometers* are characterized as infinite compact monothetic zero-dimensional groups. Since every odometer is homeomorphic to the Cantor set, we will denote odometers by $\mathfrak{C}$. By fixing a topological generator c_0 and defining $h : \mathfrak{C} \to \mathfrak{C}$ by $h(c) = c \oplus c_0$ (where $\oplus$ denotes addition in the odometer) we obtain a minimal equicontinuous dynamical system $(\mathfrak{C}, h)$. The homeomorphism h has the following properties: there exists a strictly increasing sequence $(p_k)_{k\geq 0}$ such that $p_0 = 1$ and $p_k \mid p_{k+1}$, and a decreasing sequence of closed-and-open sets V_k containing 0 such that $\{V_k, h(V_k), \ldots, h^{p_k-1}(V_k)\}$ is a partition of $\mathfrak{C}$ ($V_0 = \mathfrak{C}$). In fact, any subsequence of (p_k) starting with p_0 (and the corresponding subsequence of sets V_k) has the same properties. The simplest *dyadic* odometer is shown in Figure 1.

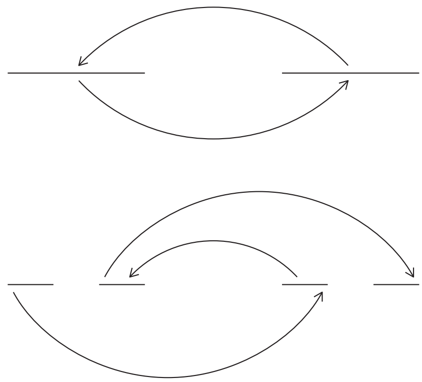

Fig. 1. The action on the dyadic odometer (first and second approximation)

It is known (see [7]) that every solenoid is homeomorphic to a quotient space of the product $[0,1] \times \mathfrak{C}$ of the interval with the Cantor set, with respect to the relation identifying the points $(1,c)$ and $(0,h(c))$, where h is as described above.

In this representation, we can equip the solenoid with the structure of an Abelian topological group, as follows:

$$(t,c) + (s,c') = \begin{cases} (t+s, c \oplus c') & \text{if } t+s < 1, \\ (t+s-1, h(c \oplus c')) & \text{if } t+s \geq 1. \end{cases}$$

Clearly, $(0,0)$ is the neutral element of Σ.

Being compact, metric, connected and Abelian, every solenoid Σ is necessarily a monothetic group (see [9]). It is not hard to see that any element of the form $(b,0)$ with b irrational is a topological generator (i.e., $\{n \cdot (b,0) : n \in \mathbb{N}\}$ is dense in Σ). Translation by any topological generator (in particular by $(b,0)$) is minimal (see e.g. [4]). From now on, we fix a solenoid Σ and an irrational number $b \in (1/5, 1/4)$, and we denote by α the translation by $(b,0)$. Clearly, α preserves arc lengths (in either representation of the solenoid).

Given a point p in a continuum X, the *composant* of X at p is the set

$$\kappa(p) = \{q \in X : \text{there exists a proper subcontinuum } A \text{ of } X \text{ such that } p, q \in A\}.$$

Solenoids are indecomposable continua (i.e., cannot be represented as a union of two proper subcontinua) and indecomposable continua have continuum many pairwise disjoint composants (see [12]).

We will construct a non-invertible, minimal map Ψ of the solenoid. We are going to define Ψ as a limit of homeomorphisms $\Psi_k\colon \Sigma \to \Sigma$ with $\Psi_k = \alpha \circ \beta_1 \circ \cdots \circ \beta_k$, where $\beta_i\colon \Sigma \to \Sigma$ will be referred to as the *ith perturbation.*

Before we start the construction we define two auxiliary families of functions on the interval $[0,1]$. For $k \in \{1,2,\ldots\}$ let F_k be the function defined by the following formula (see Figure 2):

$$F_k(x) = \begin{cases} \dfrac{x}{k+1}, & x \in \left[0, \dfrac{k+1}{2k+1}\right], \\ 2x-1, & x \in \left(\dfrac{k+1}{2k+1}, 1\right]. \end{cases}$$

For $k \in \{2,3,\ldots\}$ let us denote by f_k the function such that $F_{k-1} \circ f_{k-1} = F_k$ (see Figure 3). It is not hard to calculate that

$$f_{k-1}(x) = \begin{cases} \dfrac{k}{k+1}x, & x \in \left[0, \dfrac{k+1}{2k+1}\right], \\ 2kx - k, & x \in \left(\dfrac{k+1}{2k+1}, \dfrac{k}{2k-1}\right], \\ x, & x \in \left(\dfrac{k}{2k-1}, 1\right]. \end{cases}$$

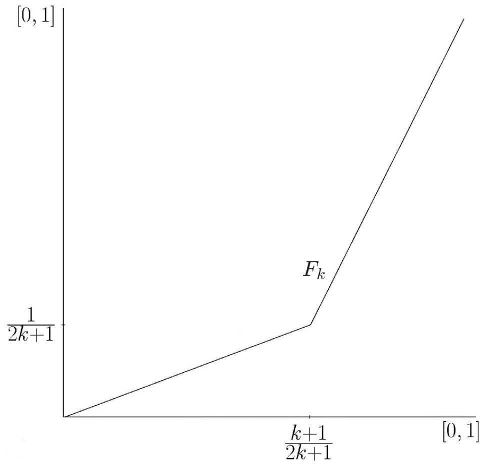

Fig. 2. Graph of the function F_k

Since $b < 1$ and we have $\alpha((0,0)) = (b,0)$, it is obvious that $\alpha((0,0)) \in [0,1] \times \{0\}$. In that case α preserves the composant of $(0,0)$ and thus it preserves all composants.

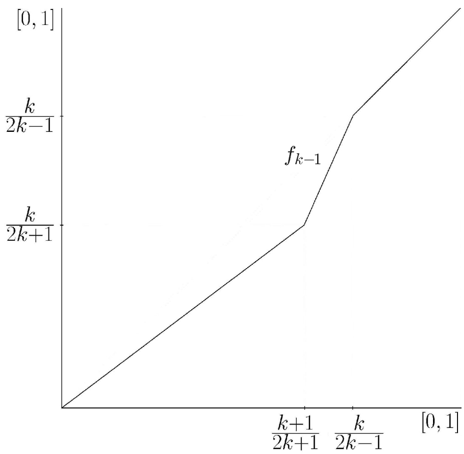

Fig. 3. Graph of the function f_{k-1}

Let $U_k = [0, b) \times V_k$. Let us observe that for each k there exists a number m_k such that the images of U_k under the first m_k iterates of α, i.e., $U_k, \alpha(U_k), \ldots, \alpha^{m_k-1}(U_k)$, are pairwise disjoint, and together with $\alpha^{m_k}(U_k)$ they cover Σ (in fact, $m_0 = 4$, $4p_k \le m_k < 5p_k$, see Figures 4 and 5).

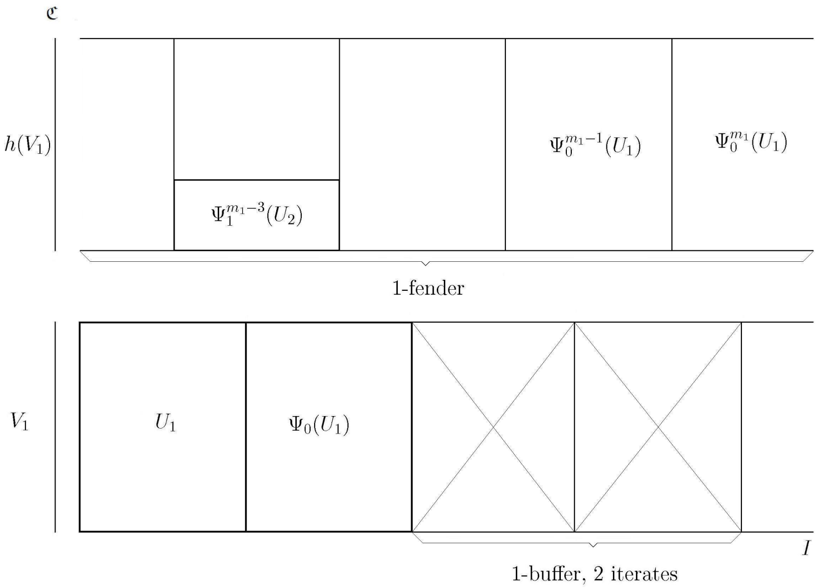

Fig. 4. The first m_1+1 (counting from 0 to m_1) iterates of U_1 under the transformation Ψ_0. The picture shows the case with $p_1 = 2$, and $m_1 = 8$. Note that the eighth image of U_k completes the cover of Σ (and is not disjoint from U_k). The 1-fender is covered by the last five of the nine images.

The sets $[0,1]\times h^{-1}(V_k)$ will be called the *k-fenders*. Note that they form a decreasing sequence of sets.

Let $\Psi_0 = \alpha$ and set $N_0 = 0$.

We turn to the construction of the first map Ψ_1. We define a selfhomeomorphism γ_1 of U_1 by the formula

$$\gamma_1(x,y) = (\hat{F}_1(x), y),$$

where $\hat{F}_1 : [0,b] \to [0,b]$ is rescaled F_1. Next, we let $\beta_1 = \gamma_1$ on U_1, and on $\Psi_0(U_1)$ we define

$$\beta_1 = \alpha\gamma_1^{-1}\alpha^{-1}.$$

On the rest of the space we let β_1 be the identity.

Since the set V_1 is closed and open, and the function $\hat{F}_1$ matches the identity at 0 and b, the perturbation β_1 is easily seen to be continuous. Now we set

$$\Psi_1 = \Psi_0 \circ \beta_1.$$

Note that for any point $p \in U_1$, $\Psi_1^2(p) = \alpha^2(p)$. It is crucial that the action of Ψ_1 is topologically conjugate to that of α. Indeed, the reader will easily verify that $\Psi_1 = \phi_1^{-1} \circ \alpha \circ \phi_1$, where ϕ_1 equals the identity except on $\Psi_1(U_1)$ where it matches β_1.

Note that the perturbation β_1 is the identity on the 1-fender. This is where the future "inverse modifications" will take place.

Assume that for some $k \geq 1$ we have $4p_k \geq m_{k-1} + N_{k-1} + 4$ (this was satisfied for $k = 1$, because $p_1 \geq 2$, $m_0 = 4$ and $N_0 = 0$), and that we have defined the map Ψ_k as the composition $\Psi_k = \alpha \circ \beta_1 \circ \cdots \circ \beta_k$ which is topologically conjugate to α via a map ϕ_k which is the identity on U_k. Further, we assume that the perturbation β_k differs from the identity only on U_k and on the set $\Psi_{k-1}^{m_{k-1}-3}(U_k)$ (this was also satisfied for $k = 1$ because $m_0 = 4$ and the modifications affected U_1 and $\Psi_0^1(U_1)$), and that for any $p \in U_k$, $\Psi_k^{m_{k-1}-2}(p) = \alpha^{m_{k-1}-2}(p)$ (which also holds for $k = 1$).

At this point we note some properties implied by the inductive assumptions. The perturbation introduced in $\Psi_{k-1}^{m_{k-1}-3}(U_k)$ "reverses" the perturbation made in U_k, so that the orbit under Ψ_k of any given point $p \in U_1$ returns to its original orbit under α. Now we look at the orbit of p through the iterates numbered $m_{k-1} - 2, \ldots, m_k$. The orbit visits (many times) the set U_1, where it is subject to a perturbation (depending on how "deep" inside U_1 it falls, i.e., to which set U_j $(1 \leq j \leq k)$), but then it always visits the set $\Psi_{j-1}^{m_{j-1}-3}(U_j)$, where this perturbation is "undone". Eventually, before reaching (for the first time) the k-fender, the orbit returns to its original bed of the orbit of α. Within the k-fender the orbit under Ψ_k matches that under α. This implies, among other things, that the k-fender is covered by the "last five" of the observed images $\Psi_k^{m_k-4}(U_k),\dots,\Psi_k^{m_k}(U_k)$ (because they coincide with $\alpha^{m_k-4}(U_k),\dots,\alpha^{m_k}(U_k)$).

On the other hand, any future perturbations affect only (subsets of) U_k and the k-fender. Thus, the set (which we will call the *k-buffer*)

$$\Psi_k^{m_{k-1}-2}(U_k)\cup\dots\cup\Psi_k^{m_k-5}(U_k)$$

is not affected by any perturbation in the future, which means that on this set we have $\Psi_l=\Psi_{k-1}$ for all $l\ge k$. Note that, since $m_k\ge 4p_k\ge m_{k-1}+N_{k-1}+4$, there are at least $N_{k-1}+2$ items in the union defining the k-buffer. All these objects (the sets U_k, the k-fender, the k-buffer, the places affected by the perturbations, etc.) are shown for $k=1$ and 2 in Figures 4 and 5.

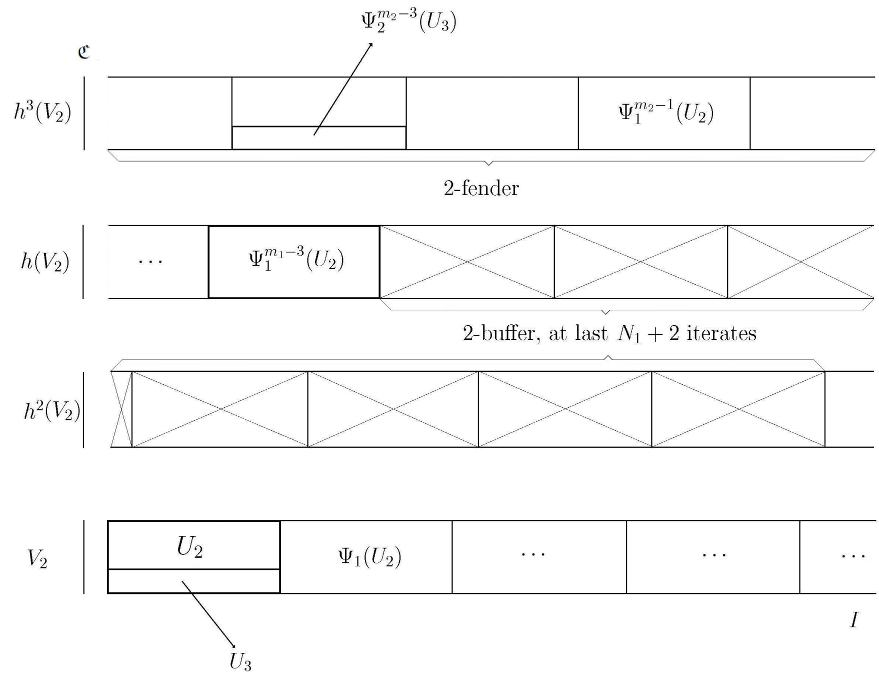

Fig. 5. The first m_2 images of U_2 under the transformation Ψ_1. (In this picture $p_2=4$ and $m_2=17$. Also, we have ignored the perturbation β_1 which slightly shifts some of the vertical division lines.) The perturbation β_2 affects U_2 and $\Psi_1^5(U_2)$. The 2-buffer consists of $\Psi_1^6(U_2),\dots,\Psi_1^{12}(U_2)$, while $\Psi_1^{13}(U_2),\dots,\Psi_1^{17}(U_2)$ cover the 2-fender. We have also marked the set $\Psi_2^{14}(U_3)$ where the perturbation β_3 will take place (to show that this set falls within the 2-fender).

We are in a position to construct Ψ_{k+1}. We cover the space by finitely many open $1/k$-balls. By minimality of Ψ_k, there exists a number N_k such that every orbit under Ψ_k visits every selected ball before N_k iterates.

By passing to a subsequence of (p_k) we can assume that $4p_{k+1} \geq m_k + N_k + 4$. Since Ψ_k is conjugate to α via a map ϕ_k which is the identity on U_k (and hence also on U_{k+1}), the iterates $\Psi_k^n(U_{k+1})$ for $0 \leq n \leq m_{k+1} - 1$ are pairwise disjoint, and together with $\Psi_k^{m_{k+1}}(U_{k+1})$ they cover Σ.

We define a selfhomeomorphism γ_{k+1} of U_{k+1} by the formula

$$\gamma_{k+1}(x, y) = (\hat{f}_k(x), y),$$

where $\hat{f}_k : [0, b] \to [0, b]$ is rescaled f_k. Next, we let $\beta_{k+1} = \gamma_{k+1}$ on U_{k+1}. Note that $\Psi_k^{m_k-3}(U_{k+1}) \subset \Psi_k^{m_k-3}(U_k)$ is contained in the k-fender. On $\Psi_k^{m_k-3}(U_{k+1})$, we define

$$\beta_{k+1} = \alpha^{m_k-3}\gamma_{k+1}^{-1}\alpha^{-m_k+3},$$

and we let β_{k+1} be the identity on the remainder of the space.

At this point we set

$$\Psi_{k+1} = \Psi_k \circ \beta_{k+1} = \alpha \circ \beta_1 \circ \beta_2 \circ \cdots \circ \beta_{k+1}.$$

The verification of the condition that $\Psi_{k+1}^{m_k-2}(p) = \alpha^{m_{k-1}-2}(p)$ for any $p \in U_{k+1}$ is straightforward, by multiple substitution.

It is fairly easy to check that Ψ_{k+1} is topologically conjugate to Ψ_k (and hence to α), where the conjugating map ϕ_{k+1} is the identity on U_{k+1} (ϕ_{k+1} differs from the identity only on $\Psi_k^n(U_{k+1})$ with $n = 1, \ldots, m_k - 3$; we skip the details).

3. Properties of the limit transformation Ψ

LEMMA 3.1. *The transformations Ψ_k converge uniformly.*

Proof. It suffices to show that the sequence Ψ_k is Cauchy in the supremum metric for continuous maps. For given two natural numbers k and l note that the transformations Ψ_k and Ψ_{k+l} differ on the disjoint sets U_{k+1} and $\Psi_k^{m_k-3}(U_{k+1}), \ldots, \Psi_k^{m_{k+l-1}-3}(U_{k+l})$.

On U_{k+1}, Ψ_k is the composition of α with $\hat{F}_k$, while Ψ_{k+1} is piecewise the composition of α with $\hat{F}_{k+j}$ for $j = 1, \ldots, l$. Since $\hat{F}_k$ converges uniformly, it is obvious that, on this set, Ψ_k is uniformly close to Ψ_{k+l} for large k (we also use the fact that α is uniformly continuous). On each of the sets $\Psi_k^{m_{k+j-1}}(U_{k+j})$ $(j = 1, \ldots, l)$ the map Ψ_k is just α while Ψ_{k+l} equals the composition of α with $\hat{f}_{k+j}^{-1}$. Since the functions $\hat{f}_{k+j}^{-1}$ converge uniformly to the identity map, the uniform Cauchy condition follows on these sets as well. ∎

We denote by Ψ the resulting limit transformation, $\Psi = \lim_{k\to\infty} \Psi_k$. Let us observe that Ψ is not invertible since for any point $x \in [0, b/2]$ we have $\lim_{k\to\infty} \hat{F}_k(x) = 0$, hence $\Psi((x, 0)) = \alpha((0, 0)) = (b, 0)$.

It remains to show that the transformation Ψ is minimal. Before we proceed to the next theorem we recall one fact concerning the composants of solenoids. Given two points p and q belonging to the same composant, the *ray starting at p and containing q* is by definition the union of all arcs containing q and having an endpoint at p. Every composant of a solenoid is the union of two rays starting at the same point; moreover, every ray is a dense subset of the solenoid (see [8]).

THEOREM 3.2. *The transformation Ψ is minimal.*

Proof. Using the same arguments as in Lemma 3.1 it is not hard to show that $d(\alpha(p), \Psi_k(p)) < b/2$ for any $p \in \Sigma$ and k (where d is the distance on Σ measured along the composant). Every orbit under α "walks forward" along a ray with steps of length b, hence every orbit under Ψ_k "walks forward" along a ray with steps of length strictly between $b/2$ and $3b/2$, while every orbit under Ψ has steps between $b/2$ and $3b/2$ (inclusive). The intersection of the set $D_k = \Psi_k^{m_{k-1}-2}(U_k) \cup \Psi_k^{m_{k-1}-1}(U_k) \cup \Psi_k^{m_{k-1}}(U_k)$ with any ray is an arc bounded by some point p and $\Psi_k^3(p)$, hence has length strictly larger than $3b/2$. This implies that every orbit under Ψ must visit the set D_k.

Note that the k-buffer consists of D_k together with at least $N_k - 1$ further images (under the iterates of Ψ_k) of U_k, and that on the k-buffer we have $\Psi_k = \Psi$. This implies that every orbit under Ψ visits the k-buffer at least N_k consecutive times, during which it visits every ball in the selected cover by $1/k$-balls. This ends the proof of minimality. ∎

Questions:

- It seems that the simplified construction with only the perturbations defined on U_k (without inverses and fenders) also leads to a minimal limit transformation. Is it indeed so?

We repeat the main long standing open question:

- Is it true that the simple closed curve is the only infinite continuum of type $(1, 0)$ (i.e., such that it admits a minimal homeomorphism but does not admit non-invertible minimal maps)?

One can ask more specific versions of the above question:

- If we drop connectedness, then the only known infinite compact metric spaces of type $(1, 0)$ are disjoint unions of finitely many circles. Every such space can be equipped with the structure of a compact monothetic group (the circle times a finite cyclic group). Is it true that among infinite compact monothetic groups these are the only ones of type $(1, 0)$? (We have just eliminated solenoids.)
- Is it true that the pseudo-circle admits non-invertible minimal maps? (See [12] for general information concerning the pseudo-circle.)

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