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JOIN-SEMILATTICES WITH TWO-DIMENSIONAL CONGRUENCE AMALGAMATION

BY

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Abstract. We say that a $\langle\vee,0\rangle$-semilattice S is *conditionally co-Brouwerian* if (1) for all nonempty subsets X and Y of S such that $X \leq Y$ (i.e. $x \leq y$ for all $\langle x,y\rangle \in X \times Y$), there exists $z \in S$ such that $X \leq z \leq Y$, and (2) for every subset Z of S and all $a, b \in S$, if $a \leq b \vee z$ for all $z \in Z$, then there exists $c \in S$ such that $a \leq b \vee c$ and $c \leq Z$. By restricting this definition to subsets X, Y, and Z of less than κ elements, for an infinite cardinal κ, we obtain the definition of a *conditionally κ-co-Brouwerian* $\langle\vee,0\rangle$-semilattice.

We prove that for every conditionally co-Brouwerian lattice S and every partial lattice P, every $\langle\vee,0\rangle$-homomorphism $\varphi : \mathrm{Con_c}\, P \to S$ can be lifted to a lattice homomorphism $f : P \to L$ for some relatively complemented lattice L. Here, $\mathrm{Con_c}\, P$ denotes the $\langle\vee,0\rangle$-semilattice of compact congruences of P.

We also prove a two-dimensional version of this result, and we establish partial converses of our results and various of their consequences in terms of congruence lattice representation problems. Among these consequences, for every infinite regular cardinal κ and every conditionally κ-co-Brouwerian S of size κ, there exists a relatively complemented lattice L with zero such that $\mathrm{Con_c}\, L \cong S$.

1. Introduction. The present paper deals essentially with two categories of structures. The first one is the category **PL** of all *partial lattices* (see Definition 3.1) and their homomorphisms (see Definition 4.1), while the second one is the category **S** of all $\langle\vee,0\rangle$-semilattices and $\langle\vee,0\rangle$-homomorphisms. These categories are related by the functor $\mathrm{Con_c} : \mathbf{PL} \to \mathbf{S}$. For a partial lattice P, $\mathrm{Con_c}\, P$ is the $\langle\vee,0\rangle$-semilattice of compact congruences of P; see Section 3.

In the last few years some effort has been put on the investigation of the effect of the $\mathrm{Con_c}$ functor not only on the *objects* of **PL**, but also on the *diagrams* of **PL**, in fact, essentially on the diagrams of the full subcategory **L** of **PL** whose objects are all *lattices*. A complete account of the pre-1998 stages of this research is presented in [7]. Formally, a diagram of **PL** is a functor from a category **C** to **PL**. Most of the results of the last years on this topic can then be conveniently formulated *via* the following definition.

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Definition 1.1. Let $\mathcal{D}$ be a diagram of partial lattices. We denote the composition $\mathrm{Con_c} \circ \mathcal{D}$ by $\mathrm{Con_c}\, \mathcal{D}$. For a $\langle \vee, 0\rangle$-semilattice S and a partial lattice P, we say that a homomorphism $\varphi : \mathrm{Con_c}\, \mathcal{D} \to S$ can be

(i) *factored through* P if there are a homomorphism $f : \mathcal{D} \to P$ and a $\langle \vee, 0\rangle$-homomorphism $\psi : \mathrm{Con_c}\, P \to S$ such that $\varphi = \psi \circ \mathrm{Con_c}\, f$;

(ii) *lifted through* P if there are a homomorphism $f : \mathcal{D} \to P$ and an isomorphism $\psi : \mathrm{Con_c}\, P \to S$ such that $\varphi = \psi \circ \mathrm{Con_c}\, f$.

In (i) (resp., (ii)) above, we say that φ can be *factored to* (resp., *lifted to*) f.

Homomorphisms between diagrams have to be understood in the categorical sense, e.g., if $\mathcal{D} : \mathbf{C} \to \mathbf{PL}$ is a diagram of partial lattices and if P is a partial lattice, a homomorphism $f : \mathcal{D} \to P$ consists of a family $(f_X)_{X \in \mathrm{Ob}\, \mathbf{C}}$ of homomorphisms $f_X : \mathcal{D}(X) \to P$, for any object X of $\mathbf{C}$, such that if $u : X \to Y$ is a morphism in $\mathbf{C}$, then $f_X = f_Y \circ \mathcal{D}(u)$. Of particular interest to us will be the case where $\mathcal{D}$ consists exactly of one partial lattice, i.e., $\mathbf{C}$ is the trivial category with one object and one morphism, and the case where $\mathcal{D}$ is a *truncated square*, i.e., $\mathbf{C}$ consists of distinct objects 0, 1, and 2 together with nontrivial morphisms $e_1 : 0 \to 1$ and $e_2 : 0 \to 2$. In that case $\mathcal{D}$ can be described by partial lattices P_0, P_1, and P_2, together with homomorphisms $f_1 : P_0 \to P_1$ and $f_2 : P_0 \to P_2$. Moreover, if P is a partial lattice, a homomorphism from $\mathcal{D}$ to P can then be described by homomorphisms of partial lattices $g_i : P_i \to P$, for $i < 3$, such that $g_1 \circ f_1 = g_2 \circ f_2 = g_0$. The situation can be described by the following commutative diagrams:

$$\begin{tikzcd} P_1 & & P_2 \\ & P_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & P & \\ P_1 \arrow[ur, "g_1"] & & P_2 \arrow[ul, "g_2"'] \\ & P_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] \arrow[uu, "g_0"] & \end{tikzcd}$$

Illustrating $\mathcal{D}$ and a homomorphism from $\mathcal{D}$ to P

We shall call P_0 (resp., P_1 and P_2) the *bottom* (resp., the *sides*) of $\mathcal{D}$.

Then a typical lifting result for the $\mathrm{Con_c}$ functor is the following (see J. Tůma [10] and G. Grätzer, H. Lakser, and F. Wehrung [6]):

Theorem 1. *Let $\mathcal{D}$ be a truncated square of lattices and let S be a finite distributive $\langle \vee, 0\rangle$-semilattice. Then every homomorphism $\varphi : \mathrm{Con_c}\, \mathcal{D} \to S$ can be lifted through a relatively complemented lattice.*

For infinite S, completely different methods yield the following result (see Theorem C in F. Wehrung [15]).

THEOREM 2. *Let K be a lattice and let S be a distributive lattice with zero. Then every $\langle\vee,0\rangle$-homomorphism $\varphi : \mathrm{Con_c}\, K \to S$ can be lifted. Furthermore, a lift $f : K \to L$ can be found in such a way that*:

(i) *L is relatively complemented.*

(ii) *The range of f generates L as an ideal* (*resp.*, *a filter*).

(iii) *If the range of φ is cofinal in S, then the range of f generates L as a convex sublattice.*

We will express the fulfilment of conditions (i)–(iii) above by saying that φ *has a good lift*, although, strictly speaking, one would need to define good ideal lifts and good filter lifts. Moreover, this condition turns out to be somewhat looser than it appears, as, for example, it can be strengthened by many additional properties of L, such as the ones listed in the statement of Proposition 20.8 of [15]. For example, L has *definable principal congruences.*

One can then say that Theorem 1 is a two-dimensional lifting result for finite distributive $\langle\vee,0\rangle$-semilattices, while Theorem 2 is a one-dimensional lifting result for arbitrary distributive lattices with zero. In fact, the following stronger, "two-dimensional" result holds (see [15, Theorem D]):

THEOREM 3. *Let $\mathcal{D}$ be a truncated square of lattices with* finite *bottom*, *let S be a distributive lattice with zero*, *and let $\varphi : \mathrm{Con_c}\, \mathcal{D} \to S$ be a homomorphism. Then φ has a good lift.*

In the "good lift" statement, the range of $f : \mathcal{D} \to L$ has to be understood as the union of the ranges of the images under f of the individual objects in $\mathcal{D}$, while the range of φ is the $\langle\vee,0\rangle$-semilattice generated by the union of the ranges of the images under φ of the individual objects in $\mathrm{Con_c}\, \mathcal{D}$.

As we shall see in the present paper, the statement of Theorem 3 does *not* extend to the case where the bottom of $\mathcal{D}$ is an *infinite* lattice. However, we shall introduce a class of distributive lattices with zero, the so-called *conditionally co-Brouwerian* ones (see Definition 6.1), that includes all finite distributive lattices. Moreover, every complete sublattice of a complete Boolean lattice is conditionally co-Brouwerian. For those lattices, the stronger statement remains valid, and much more:

THEOREM 4. *Let P be a partial lattice and let S be a conditionally co-Brouwerian lattice. Then every homomorphism $\varphi : \mathrm{Con_c}\, P \to S$ has a good lift.*

Now the two-dimensional version of Theorem 4:

THEOREM 5. *Let $\mathcal{D}$ be a truncated square of partial lattices with bottom a lattice and let S be a conditionally co-Brouwerian lattice. Then every homomorphism $\varphi : \mathrm{Con_c}\, \mathcal{D} \to S$ has a good lift.*

As we shall prove in Sections 8 and 9, some of the assumptions on S are also necessary for the statements of Theorems 4 and 5 to hold.

All these results imply the following corollaries:

COROLLARY 6.4. *Let S be a distributive $\langle\vee,0\rangle$-semilattice that can be expressed as a $\langle\vee,0\rangle$-direct limit of at most $\aleph_1$ conditionally co-Brouwerian lattices. Then there exists a relatively complemented lattice L with zero such that* $\mathrm{Con_c}\, L \cong S$. *Furthermore*, *if S is bounded*, *then L can be taken bounded as well.*

This result extends a well known result of A. Huhn [8, 9] that states that every distributive $\langle\vee,0\rangle$-semilattice of size at most $\aleph_1$ is isomorphic to $\mathrm{Con_c}\, L$ for some lattice L.

Our next corollary also implies a positive solution for Problem 4 of [6].

COROLLARY 6.5. *Let K be a lattice that can be expressed as a direct union of* countably *many lattices whose congruence semilattices are conditionally co-Brouwerian. Then K embeds congruence-preservingly into some relatively complemented lattice L*, *which it generates as a convex sublattice.*

We also establish *relativizations* of the methods leading to Theorems 4 and 5. These statements involve a relativized version, for every infinite cardinal κ, of the notion of a conditionally co-Brouwerian lattice. We call the resulting objects *conditionally κ-co-Brouwerian $\langle\vee,0\rangle$-semilattices* (see Definition 7.1).

THEOREM 6. *Let κ be an infinite cardinal and let S be a conditionally κ-co-Brouwerian $\langle\vee,0\rangle$-semilattice of size κ. Then there exists a relatively complemented lattice L with zero such that* $\mathrm{Con_c}\, L \cong S$. *Furthermore*, *if S is bounded*, *then L can be taken bounded as well.*

2. Algebraic lattices. We recall that in a lattice L, an element a is *compact* if for any nonempty upward directed subset X of L, $a \leq \bigvee X$ implies that $a \leq x$ for some $x \in X$. We denote by $\mathcal{K}(L)$ the join-semilattice of compact elements of L. We say that L is *algebraic* if L is complete and every element of L is a join of compact elements. If L is an algebraic lattice, then $\mathcal{K}(L)$ is a $\langle\vee,0\rangle$-semilattice, while for every $\langle\vee,0\rangle$-semilattice S, the lattice $\mathrm{Id}\, S$ of all ideals of S is an algebraic lattice. These transformations can be extended to functors in a canonical way. The relevant definitions for the morphisms are the following. For $\langle\vee,0\rangle$-semilattices, they are the $\langle\vee,0\rangle$-homomorphisms, while for algebraic lattices, they are the compactness preserving complete join-homomorphisms; by definition, for complete lattices A and B, a map $f : A \to B$ is a *complete join-homomorphism* if $f(\bigvee X) = \bigvee f[X]$ for any subset X of A, while f is *compactness preserving* if $f[\mathcal{K}(A)] \subseteq \mathcal{K}(B)$. Then the aforementioned category equivalence can be stated in the following condensed form:

PROPOSITION 2.1. *The functors* $S \mapsto \operatorname{Id} S$ *and* $A \mapsto \mathcal{K}(A)$ *define a category equivalence between* $\langle \vee, 0\rangle$*-semilattices with* $\langle \vee, 0\rangle$*-homomorphisms and algebraic lattices with compactness preserving complete join-homomorphisms.*

3. Partial lattices. Our notations and definitions are the same as in [15]. If X is a subset of a *quasi-ordered* set P and if $a \in P$, let $a = \sup X$ (resp., $a = \inf X$) be the statement that a is a majorant (resp., minorant) of X and that every majorant (resp., minorant) x of X satisfies $a \le x$ (resp., $x \le a$). We observe that this statement determines a only up to equivalence.

DEFINITION 3.1. A *partial prelattice* is a structure $\langle P, \le, \bigvee, \bigwedge\rangle$, where P is a nonempty set, $\le$ is a quasi-ordering on P, and $\bigvee$, $\bigwedge$ are partial functions from the set $[P]^{<\omega}_*$ of all nonempty finite subsets of P to P with the following properties:

(i) $a = \bigvee X$ implies that $a = \sup X$ for all $a \in P$ and all $X \in [P]^{<\omega}_*$.
(ii) $a = \bigwedge X$ implies that $a = \inf X$ for all $a \in P$ and all $X \in [P]^{<\omega}_*$.

We say that P is a *partial lattice* if $\le$ is antisymmetric.

A *congruence* of P is a quasi-ordering $\preceq$ of P containing $\le$ such that $\langle P, \preceq, \bigvee, \bigwedge\rangle$ is a partial prelattice.

For a partial lattice P, a congruence $\boldsymbol{c}$ of P, and elements x, y of P, we shall often write $x \le_{\boldsymbol{c}} y$ instead of $\langle x, y\rangle \in \boldsymbol{c}$, and $x \equiv_{\boldsymbol{c}} y$ instead of the conjunction of $x \le_{\boldsymbol{c}} y$ and $y \le_{\boldsymbol{c}} x$. The *quotient* $P/\boldsymbol{c}$ has underlying set $P/{\equiv_{\boldsymbol{c}}}$, and we endow it with the quotient quasi-ordering ${\le_{\boldsymbol{c}}}/{\equiv_{\boldsymbol{c}}}$ and the partial join defined by the rule

$$\boldsymbol{a} = \bigvee \boldsymbol{X} \quad \text{iff} \quad \begin{array}{l}\text{there are } X \in [P]^{<\omega}_* \text{ and } a \in P\\ \text{with } a = \bigvee X,\ \boldsymbol{a} = a_{/\boldsymbol{c}},\ \text{and } \boldsymbol{X} = X_{/\boldsymbol{c}},\end{array}$$

where $a_{/\boldsymbol{c}}$ denotes the equivalence class of a modulo $\boldsymbol{c}$ and we put $X_{/\boldsymbol{c}} = \{x_{/\boldsymbol{c}} \mid x \in X\}$. The partial meet on $P/\boldsymbol{c}$ is defined dually.

For a, $b \in P$, we denote by $\Theta^+_P(a, b)$ the least congruence $\boldsymbol{c}$ of P such that $a \le_{\boldsymbol{c}} b$, and we put $\Theta_P(a, b) = \Theta^+_P(a, b) \vee \Theta^+_P(b, a)$, the least congruence $\boldsymbol{c}$ of P such that $a \equiv_{\boldsymbol{c}} b$. Of course, the congruences of the form $\Theta^+_P(a, b)$ are generators of the join-semilattice $\operatorname{Con_c} P$.

We shall naturally identify *lattices* with partial lattices P such that $\bigvee$ and $\bigwedge$ are defined everywhere on $[P]^{<\omega}_*$.

PROPOSITION 3.2. *Let* P *be a partial prelattice. Then the set* $\operatorname{Con} P$ *of all congruences of* P *is a closure system in the powerset lattice of* $P \times P$, *closed under directed unions. In particular*, *it is an algebraic lattice.*

We denote by $\mathrm{Con_c}\, P$ the $\langle\vee,0\rangle$-semilattice of all *compact* congruences of P, by $\mathbf{0}_P$ the least congruence of P (that is, $\mathbf{0}_P$ is the quasi-ordering of P), and by $\mathbf{1}_P$ the largest (coarse) congruence of P.

If P is a lattice, then $\mathrm{Con}\, P$ is distributive, but this may not hold for a general partial lattice P.

Many $\langle\vee,0\rangle$-homomorphisms will be constructed by using the following notion of *measure*.

DEFINITION 3.3. Let P be a partial lattice and let S be a $\langle\vee,0\rangle$-semilattice. An *S-valued measure* on P is a map $\mu : P\times P \to S$ that has the following properties (we write $\mu(x,y)$ instead of $\mu(\langle x,y\rangle)$ from now on):

(i) $\mu(x,y)=0$ for all x, $y\in P$ such that $x\leq y$.

(ii) $\mu(x,z)\leq \mu(x,y)\vee\mu(y,z)$ for all $x,y,z\in P$.

(iii) $\mu(a,b)=\bigvee_{x\in X}\mu(x,b)$ for all $a,b\in P$ and all $X\in[P]^{<\omega}_*$ such that $a=\bigvee X$.

(iv) $\mu(a,b)=\bigvee_{y\in Y}\mu(a,y)$ for all $a,b\in P$ and all $Y\in[P]^{<\omega}_*$ such that $b=\bigwedge Y$.

We omit the easy proof of the following lemma (see also Proposition 13.1 in [15]). This lemma states that the notion of measure on P and the notion of $\langle\vee,0\rangle$-homomorphism from $\mathrm{Con_c}\, P$ are essentially equivalent.

LEMMA 3.4. *Let P be a partial lattice and let S be a $\langle\vee,0\rangle$-semilattice. Then*:

(i) *For every $\langle\vee,0\rangle$-homomorphism $\overline{\mu} : \mathrm{Con_c}\, P \to S$, the map $\mu : P\times P\to S$, $\langle x,y\rangle\mapsto\overline{\mu}\Theta_P^+(x,y)$, is an S-valued measure on P.*

(ii) *For any S-valued measure μ on P, there exists a unique $\langle\vee,0\rangle$-homomorphism $\overline{\mu} : \mathrm{Con_c}\, P\to S$ such that $\mu(x,y)=\overline{\mu}\Theta_P^+(x,y)$ for all $x,y\in P$.*

The homomorphism $\overline{\mu}$ (the "integral" with respect to μ) is of course defined by the formula

$$\overline{\mu}\Big(\bigvee_{i<n}\Theta_P^+(x_i,y_i)\Big)=\bigvee_{i<n}\mu(x_i,y_i)$$

for all $n<\omega$ and all $x_0,\dots,x_{n-1},y_0,\dots,y_{n-1}\in P$.

4. Homomorphisms of partial lattices

DEFINITION 4.1. If P and Q are partial prelattices, a *homomorphism of partial prelattices* from P to Q is an order preserving map $f : P\to Q$ such that $a=\bigvee X$ (resp., $a=\bigwedge X$) implies that $f(a)=\bigvee f[X]$ (resp., $f(a)=\bigwedge f[X]$) for all $a\in P$ and all $X\in[P]^{<\omega}_*$. We say that a homomorphism f is an *embedding* if $f(a)\leq f(b)$ implies that $a\leq b$ for all $a,b\in P$.

For a homomorphism $f : P \to Q$ of partial lattices, the *kernel* of f, denoted by $\ker f$, is defined as

$$\ker f = \{\langle x, y\rangle \in P \times P \mid f(x) \leq f(y)\}.$$

Moreover, we can define the following maps:

• The map $\operatorname{Con} f : \operatorname{Con} P \to \operatorname{Con} Q$, obtained by defining, for any congruence $\boldsymbol{a}$ of P, the congruence $(\operatorname{Con} f)(\boldsymbol{a})$ as the least congruence of Q that contains all the pairs $\langle f(x), f(y)\rangle$ for $\langle x, y\rangle \in \boldsymbol{a}$.

• The restriction $\operatorname{Con}_{\mathrm{c}} f$ of the map $\operatorname{Con} f$ from $\operatorname{Con}_{\mathrm{c}} P$ to $\operatorname{Con}_{\mathrm{c}} Q$.

• The map $\operatorname{Res} f : \operatorname{Con} Q \to \operatorname{Con} P$, obtained by defining, for any congruence $\boldsymbol{b}$ of Q, the congruence $(\operatorname{Res} f)(\boldsymbol{b})$ as the set of all $\langle x, y\rangle \in P \times P$ such that $\langle f(x), f(y)\rangle \in \boldsymbol{b}$. If, in particular, P is a partial sublattice of Q and $f : P \hookrightarrow Q$ is the inclusion map, then we shall write $\boldsymbol{b}\restriction_P$ instead of $(\operatorname{Res} f)(\boldsymbol{b})$.

This way the maps $P \mapsto \operatorname{Con} P$ and $P \mapsto \operatorname{Con}_{\mathrm{c}} P$ can be extended to functors from partial lattices and their homomorphisms to, respectively, complete lattices with compactness preserving join-complete homomorphisms, and $\langle\vee, 0\rangle$-semilattices with $\langle\vee, 0\rangle$-homomorphisms. On the other hand, $f \mapsto \operatorname{Res} f$ defines a contravariant functor from partial lattices to complete lattices with meet-complete homomorphisms that preserve nonempty directed joins.

The following lemma is a special case of a universal algebraic triviality:

Lemma 4.2. *Let $f : P \to Q$ be a homomorphism of partial lattices. Then the following are equivalent*:

(i) $\operatorname{Con} f$ *is one-to-one.*
(ii) $\operatorname{Con}_{\mathrm{c}} f$ *is one-to-one.*
(iii) $\boldsymbol{a} = (\operatorname{Res} f) \circ (\operatorname{Con} f)(\boldsymbol{a})$ *for all* $\boldsymbol{a} \in \operatorname{Con} P$.

If one of the items of Lemma 4.2 is satisfied, we say that f has the *congruence extension property.*

For a partial lattice P, we denote, as in [15], by $F_{\mathbf{L}}(P)$ the *free lattice over* P (see [2]). We denote by j_P the canonical embedding from P into $F_{\mathbf{L}}(P)$.

Proposition 4.3. *Let P be any partial lattice. Then j_P has the congruence extension property.*

Proof. For a congruence $\boldsymbol{a}$ of P, we denote by $p_{\boldsymbol{a}}$ the canonical projection from P onto $P/\boldsymbol{a}$. Since $k = j_{P/\boldsymbol{a}} \circ p_{\boldsymbol{a}}$ is a homomorphism of partial lattices from P to $F_{\mathbf{L}}(P/\boldsymbol{a})$, there exists, by the universal property of the map j_P, a unique lattice homomorphism $q_{\boldsymbol{a}} : F_{\mathbf{L}}(P) \twoheadrightarrow F_{\mathbf{L}}(P/\boldsymbol{a})$ such that $q_{\boldsymbol{a}} \circ j_P = k$, as in the following commutative diagram:

$$\begin{tikzcd} P \arrow[r, hook, "j_P"] \arrow[d, "p_{\boldsymbol{a}}"'] \arrow[dr, "k"] & F_{\mathbf{L}}(P) \arrow[d, "q_{\boldsymbol{a}}"] \\ P/\boldsymbol{a} \arrow[r, hook, "j_{P/\boldsymbol{a}}"'] & F_{\mathbf{L}}(P/\boldsymbol{a}) \end{tikzcd}$$

Put $\boldsymbol{b} = (\operatorname{Res} j_P) \circ (\operatorname{Con} j_P)(\boldsymbol{a})$, and let $x, y \in P$ be such that $x \leq_{\boldsymbol{b}} y$. This means that $j_P(x) \leq_{(\operatorname{Con} j_P)(\boldsymbol{a})} j_P(y)$, hence, by composing with $q_{\boldsymbol{a}}$, we obtain

$$k(x) \leq_{(\operatorname{Con} k)(\boldsymbol{a})} k(y). \tag{4.1}$$

However, $(\operatorname{Con} k)(\boldsymbol{a}) = \operatorname{Con}(j_{P/\boldsymbol{a}} \circ p_{\boldsymbol{a}})(\boldsymbol{a}) = (\operatorname{Con} j_{P/\boldsymbol{a}})(\mathbf{0}_{P/\boldsymbol{a}}) = \mathbf{0}_{F_{\mathbf{L}}(P/\boldsymbol{a})}$. Therefore, the relation (4.1) is equivalent to $k(x) \leq k(y)$, whence, since $j_{P/\boldsymbol{a}}$ is an embedding, $p_{\boldsymbol{a}}(x) \leq p_{\boldsymbol{a}}(y)$, that is, $x \leq_{\boldsymbol{a}} y$. Therefore, $\boldsymbol{b} \subseteq \boldsymbol{a}$. The converse inequality is trivial, hence $\boldsymbol{a} = \boldsymbol{b}$. The conclusion follows. ∎

Let us recall some further classical definitions, also used in [15]:

DEFINITION 4.4. Let P be a partial lattice.

(i) A *partial sublattice* of P is a subset Q of P that is closed under $\bigvee$ and $\bigwedge$.

(ii) An *ideal* (resp., *filter*) of P is a lower (resp., upper) subset of P closed under $\bigvee$ (resp., $\bigwedge$).

We observe that both $\varnothing$ and P are simultaneously an ideal and a filter of P. For a subset X of P, we denote by $\mathcal{I}(X)$ (resp., $\mathcal{F}(X)$) the ideal (resp., filter) of P generated by X.

LEMMA 4.5. *Let $f : P \to Q$ be a homomorphism of partial lattices. If $\mathcal{I}(f[P]) = \mathcal{F}(f[P]) = Q$, then $\operatorname{Con_c} f$ is a cofinal map from $\operatorname{Con_c} P$ to $\operatorname{Con_c} Q$.*

Proof. Put $\boldsymbol{b} = (\operatorname{Con} f)(\mathbf{1}_P)$; it suffices to prove that $\boldsymbol{b} = \mathbf{1}_Q$.

Fix $x \in P$. Then $f(x) \leq_{\boldsymbol{b}} f(y)$ for all $y \in P$, thus the set $F_x = \{v \in Q \mid f(x) \leq_{\boldsymbol{b}} v\}$ contains $f[P]$. Since F_x is obviously a filter of Q, it follows from the assumptions that $F_x = Q$. Hence, we have established that

$$f(x) \leq_{\boldsymbol{b}} v \quad \text{for all } x \in P \text{ and all } v \in Q. \tag{4.2}$$

Now it follows from (4.2) that the set $I_v = \{u \in Q \mid u \leq_{\boldsymbol{b}} v\}$ contains $f[P]$ for all $v \in Q$. Since I_v is obviously an ideal of Q, it follows from the assumptions that $I_v = Q$. Therefore, $u \leq_{\boldsymbol{b}} v$ for all $u, v \in Q$, that is, $\boldsymbol{b} = \mathbf{1}_Q$. ∎

COROLLARY 4.6. *Let P be a partial lattice. Then the canonical map $\operatorname{Con_c} j_P : \operatorname{Con_c} P \to \operatorname{Con_c} F_{\mathbf{L}}(P)$ is a cofinal embedding.*

Proof. By Proposition 4.3, $\operatorname{Con_c} j_P$ is an embedding. Furthermore, P generates $F_{\mathbf{L}}(P)$ as a lattice, thus, *a fortiori*, P generates $F_{\mathbf{L}}(P)$ both as an ideal and as a filter. Therefore, by Lemma 4.5, $\operatorname{Con_c} j_P$ has cofinal range. ∎

5. Duality of complete lattices. The facts presented in this section are standard, although we do not know of any reference where they are recorded. Most of the proofs are straightforward, in which case we omit them. We shall mainly follow the presentation of [11].

In what follows, *complete meet-homomorphisms* are defined in a dual fashion as complete join-homomorphisms, and we denote by $\mathbf{C}_\vee$ (resp., $\mathbf{C}_\wedge$) the category of complete lattices with complete join-homomorphisms (resp., complete meet-homomorphisms).

DEFINITION 5.1. Let A and B be complete lattices. Two maps $f : A \to B$ and $g : B \to A$ are *dual* if the equivalence

$$f(a) \leq b \quad \text{if and only if} \quad a \leq g(b)$$

holds for all $\langle a, b\rangle \in A \times B$.

We recall some basic folklore facts stated in [11]. For complete lattices A and B, if $f : A \to B$ and $g : B \to A$ are dual, then f is a complete join-homomorphism and g is a complete meet-homomorphism. Also, for every complete join-homomorphism (resp., complete meet-homomorphism) $f : A \to B$ (resp., $g : B \to A$), there exists a unique $g : B \to A$ (resp., $f : A \to B$) such that f and g are dual, in symbols $g = f^*$ (resp., $f = g^\dagger$).

The basic categorical properties of the duality are recorded in the following lemma.

LEMMA 5.2. (i) *The correspondence $f \mapsto f^*$ defines a contravariant functor from $\mathbf{C}_\vee$ to $\mathbf{C}_\wedge$.*

(ii) *The correspondence $g \mapsto g^\dagger$ defines a contravariant functor from $\mathbf{C}_\wedge$ to $\mathbf{C}_\vee$.*

(iii) *If f is a complete join-homomorphism, then $(f^*)^\dagger = f$.*

(iv) *If g is a complete meet-homomorphism, then $(g^\dagger)^* = g$.*

Of particular importance is the effect of the duality on complete join-homomorphisms of the form $\operatorname{Con} f : \operatorname{Con} P \to \operatorname{Con} Q$, where $f : P \to Q$ is a homomorphism of partial lattices.

LEMMA 5.3. *Let P and Q be partial lattices, and let $f : P \to Q$ be a homomorphism of partial lattices. Then* $\operatorname{Con} f$ *and* $\operatorname{Res} f$ *are dual.*

LEMMA 5.4. *Let A and B be complete lattices, and let $g : B \to A$ be a complete meet-homomorphism. Then $g \circ g^\dagger \circ g = g$. In particular, if g is surjective, then $g^\dagger$ is an embedding.*

Let A and B be complete lattices. A map $f : A \to B$ is said to *preserve nonempty directed joins* if $f(\bigvee X) = \bigvee f[X]$ for any nonempty upward directed subset X of A.

LEMMA 5.5. *Let A and B be complete lattices.*

(i) *Let $g : B \to A$ be a complete meet-homomorphism. If g preserves nonempty directed joins, then the dual map $g^\dagger : A \to B$ preserves compactness.*

(ii) *Let $f : A \to B$ be a complete join-homomorphism. If A is algebraic and f preserves compactness, then the dual map $f^* : B \to A$ preserves nonempty directed joins.*

Proof. (i) Let $a \in \mathcal{K}(A)$; we prove that $b = g^\dagger(a)$ belongs to $\mathcal{K}(B)$. So let X be a nonempty upward directed subset of B such that $b \leq \bigvee X$. By the definition of $g^\dagger$, this means that $a \leq g(\bigvee X)$, which, by the assumption on g, can be written $a \leq \bigvee g[X]$. Therefore, since $a \in \mathcal{K}(A)$, there exists $x \in X$ such that $a \leq g(x)$, that is, $b \leq x$. Hence $b \in \mathcal{K}(B)$.

(ii) Let Y be an upward directed subset of B and put $b = \bigvee Y$. Let $a \in \mathcal{K}(A)$ be such that $a \leq f^*(b)$. This means that $f(a) \leq b$; but $f(a)$ is, by assumption on f, compact in B, thus $f(a) \leq y$ for some $y \in Y$, whence $a \leq \bigvee f^*[Y]$. Since A is algebraic, this proves that $f^*(b) \leq \bigvee f^*[Y]$. The converse inequality is trivial. ∎

As a corollary, we get the following well known fact (see, e.g., Lemma 1.3.3 in [4]):

LEMMA 5.6. *Let A be an algebraic lattice and let B be a* closure system *in A, i.e., a complete meet-subsemilattice of A that is closed under nonempty directed joins. Then B is an algebraic lattice.*

Proof. Let $g : B \hookrightarrow A$ be the inclusion map. By assumption and by Lemma 5.5(i), the dual map $f = g^\dagger$ preserves compactness. Let $b \in B$. For any $x \in \mathcal{K}(B)$ such that $x \leq b$, the inequalities $x \leq f(x) \leq f(b) = b$ hold, whence

$$b = \bigvee\{x \in \mathcal{K}(A) \mid x \leq b\} = \bigvee\{f(x) \mid x \in \mathcal{K}(A),\ x \leq b\}.$$

The conclusion follows from the fact that $f[\mathcal{K}(A)] \subseteq \mathcal{K}(B)$. ∎

6. Conditionally co-Brouwerian semilattices

DEFINITION 6.1. Let S be a $\langle \vee, 0\rangle$-semilattice. We say that S is

- *co-Brouwerian* if S is a complete lattice and it satisfies the infinite meet distributivity law (MID), that is, the infinitary identity

$$\text{(MID)} \qquad a \vee \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \vee x_i),$$

where a and the x_i's range over the elements of S;

- *conditionally co-Brouwerian* if every principal ideal of S is co-Brouwerian.

Equivalently, S is co-Brouwerian iff S is a dually relatively pseudo-complemented complete lattice (see [5] for the latter terminology).

We observe that every conditionally co-Brouwerian lattice is, of course, distributive.

The crucial point that we shall use about conditionally co-Brouwerian lattices is the following:

LEMMA 6.2. *Let S be a conditionally co-Brouwerian lattice and let A be a cofinal $\langle\vee,0\rangle$-subsemilattice of a $\langle\vee,0\rangle$-semilattice B. Then every $\langle\vee,0\rangle$-homomorphism from A to S extends to some $\langle\vee,0\rangle$-homomorphism from B to S.*

Proof. The conclusion follows immediately from Theorem 3.11 of [13]. However, it is worth observing that since we are dealing with semilattices, there is also a direct proof. Namely, if $f : A \to S$ is any $\langle\vee,0\rangle$-homomorphism, the completeness assumption on S and the fact that f has cofinal range make it possible to define a map $g : B \to S$ by the rule

$$g(b) = \bigwedge\{f(x) \mid x \in A \text{ and } b \leq x\}.$$

It then follows from (MID) that g is a join-homomorphism. It is obvious that g extends f. ∎

REMARK 6.3. By using some of the techniques of the proof of Theorem 3.11 of [13], it is not hard to prove that, in fact, Lemma 6.2 *characterizes* conditionally co-Brouwerian lattices.

Now we can provide the proofs of Theorems 4 and 5 stated in the Introduction.

Proof of Theorem 4. By Corollary 4.6 and Lemma 6.2, there exists a $\langle\vee,0\rangle$-homomorphism $\psi : \mathrm{Con_c}\, F_{\mathbf{L}}(P) \to S$ such that $\psi \circ \mathrm{Con_c}\, j_P = \varphi$. Then it suffices to apply Theorem 2 to ψ. ∎

Proof of Theorem 5. Let $\mathcal{D}$ be described by homomorphisms $f : K \to P$ and $g : K \to Q$ of partial lattices, with K a lattice, and let φ be described by $\langle\vee,0\rangle$-homomorphisms $\mu : \mathrm{Con_c}\, P \to S$ and $\nu : \mathrm{Con_c}\, Q \to S$ such that $\mu \circ \mathrm{Con_c}\, f = \nu \circ \mathrm{Con_c}\, g$. We shall construct a relatively complemented lattice L, homomorphisms $\overline{f} : P \to L$ and $\overline{g} : Q \to L$ of partial lattices, and an isomorphism $\varepsilon : \mathrm{Con_c}\, L \to S$ such that $\overline{f} \circ f = \overline{g} \circ g$, $\mu = \varepsilon \circ \mathrm{Con_c}\, \overline{f}$, $\nu = \varepsilon \circ \mathrm{Con_c}\, \overline{g}$, L is relatively complemented, $\overline{f}[P] \cup \overline{g}[Q]$ generates L as an ideal (resp., filter), and if S is generated as an ideal by $\operatorname{rng} \mu \cup \operatorname{rng} \nu$, then L is generated by $\overline{f}[P] \cup \overline{g}[Q]$ as a convex sublattice (where $\operatorname{rng} \mu$ stands for the range of μ).

We first reduce the problem to the case where both f and g are *embeddings*, as follows (see also the end of the proof of Proposition 18.5 of [15]).

We put $\lambda = \mu \circ \mathrm{Con_c}\, f = \nu \circ \mathrm{Con_c}\, g$, and we define congruences $\boldsymbol{d} \in \mathrm{Con}\, K$, $\boldsymbol{a} \in \mathrm{Con}\, P$, and $\boldsymbol{b} \in \mathrm{Con}\, Q$ as follows:

$$\begin{aligned}
\boldsymbol{d} &= \{\langle x, y\rangle \in K \times K \mid \lambda\Theta_K^+(x, y) = 0\},\\
\boldsymbol{a} &= \{\langle x, y\rangle \in P \times P \mid \mu\Theta_P^+(x, y) = 0\},\\
\boldsymbol{b} &= \{\langle x, y\rangle \in Q \times Q \mid \nu\Theta_Q^+(x, y) = 0\}.
\end{aligned}$$

We denote by $p_{\boldsymbol{d}} : K \twoheadrightarrow K/\boldsymbol{d}$, $p_{\boldsymbol{a}} : P \twoheadrightarrow P/\boldsymbol{a}$, $p_{\boldsymbol{b}} : Q \twoheadrightarrow Q/\boldsymbol{b}$ the canonical projections. Then there are unique homomorphisms of partial lattices $f' : K/\boldsymbol{d} \hookrightarrow P/\boldsymbol{a}$ and $g' : K/\boldsymbol{d} \hookrightarrow Q/\boldsymbol{b}$ such that $f' \circ p_{\boldsymbol{d}} = p_{\boldsymbol{a}} \circ f$ and $g' \circ p_{\boldsymbol{d}} = p_{\boldsymbol{b}} \circ g$, and both f' and g' are embeddings. Furthermore, we can define $\langle \vee, 0\rangle$-homomorphisms $\mu' : \mathrm{Con_c}(P/\boldsymbol{a}) \to S$ and $\nu' : \mathrm{Con_c}(Q/\boldsymbol{b}) \to S$ by the rules $\mu'(\boldsymbol{x} \vee \boldsymbol{a}/\boldsymbol{a}) = \mu(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathrm{Con_c}\, P$, and $\nu'(\boldsymbol{x} \vee \boldsymbol{b}/\boldsymbol{b}) = \nu(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathrm{Con_c}\, Q$.

Since $\mu' \circ \mathrm{Con_c}\, f' = \nu' \circ \mathrm{Con_c}\, g'$ and both f' and g' are embeddings, there are, by assumption, a relatively complemented lattice L, homomorphisms $\overline{f'} : P/\boldsymbol{a} \to L$ and $\overline{g'} : Q/\boldsymbol{b} \to L$ of partial lattices, and an isomorphism $\varepsilon : \mathrm{Con_c}\, L \to S$ such that $\overline{f'} \circ f' = \overline{g'} \circ g'$, $\mu' = \varepsilon \circ \mathrm{Con_c}\, \overline{f'}$, $\nu' = \varepsilon \circ \mathrm{Con_c}\, \overline{g'}$, L is relatively complemented, $\overline{f'}[P/\boldsymbol{a}] \cup \overline{g'}[Q/\boldsymbol{b}]$ generates L as an ideal (resp., filter), and if S is generated as an ideal by $\mathrm{rng}\, \mu' \cup \mathrm{rng}\, \nu'$, then L is generated by $\overline{f'}[P/\boldsymbol{a}] \cup \overline{g'}[Q/\boldsymbol{b}]$ as a convex sublattice. Then $f' = \overline{f'} \circ p_{\boldsymbol{a}}$ and $g' = \overline{g'} \circ p_{\boldsymbol{b}}$, together with ε and L, solve the amalgamation problem for f and g.

Hence we can reduce the problem to the case where both f and g are embeddings. Without loss of generality, f and g are the set-theoretical inclusions from K into P and Q respectively, and $K = P \cap Q$.

Then we define a partial lattice R as follows (this classical construction is also recalled in the statement of Proposition 3.4 in [15]). The underlying set of R is $P \cup Q$, and the partial ordering of R is defined as follows. For $x, y \in R$, the inequality $x \leq y$ holds iff one of the following cases holds:

(i) $x, y \in P$ and $x \leq y$ in P;

(ii) $x, y \in Q$ and $x \leq y$ in Q;

(iii) $x \in P$, $y \in Q$, and there exists $z \in K$ such that $x \leq z$ in P and $z \leq y$ in Q.

(iv) $x \in Q$, $y \in P$, and there exists $z \in K$ such that $x \leq z$ in Q and $z \leq y$ in P.

The partially ordered set R can be given a structure of partial lattice, as follows. For $a \in R$ and $X \in [R]_*^{<\omega}$, we have $a = \bigvee X$ in R if either $X \cup \{a\} \subseteq P$ and $a = \bigvee X$ in P, or $X \cup \{a\} \subseteq Q$ and $a = \bigvee X$ in Q. The meet operation on R is defined dually.

Let u (resp., v) be the inclusion map from P (resp., Q) into R. It is stated in [15], and very easy to prove, that $\langle R, u, v\rangle$ is a *pushout* of the diagram

$\langle K, P, Q, f, g\rangle$ in the category of partial lattices and their homomorphisms. We shall abuse the notation by stating this as $R = P \amalg_K Q$, the maps f and g then being understood.

Now we put $\boldsymbol{C} = \{\langle \boldsymbol{a}, \boldsymbol{b}\rangle \in \operatorname{Con} P \times \operatorname{Con} Q \mid \boldsymbol{a}{\restriction}_K = \boldsymbol{b}{\restriction}_K\}$. It is obvious that $\boldsymbol{C}$ is a complete meet-subsemilattice of $\operatorname{Con} P \times \operatorname{Con} Q$, closed under nonempty directed suprema. Hence, by Lemma 5.6, $\boldsymbol{C}$ is an algebraic lattice. Observe that $\langle \mathbf{0}_P, \mathbf{0}_Q\rangle \in \boldsymbol{C}$.

Let $\varphi : \operatorname{Con} R \to \boldsymbol{C}$, $\boldsymbol{c} \mapsto \langle \boldsymbol{c}{\restriction}_P, \boldsymbol{c}{\restriction}_Q\rangle$. Then φ is a complete meet-homomorphism, and it preserves nonempty directed joins. Hence, by Lemma 5.5(i), the dual map $\psi = \varphi^\dagger$ of φ is a compactness preserving complete join-homomorphism from $\boldsymbol{C}$ to $\operatorname{Con} R$.

Claim 1. *The map φ is surjective, while ψ is an embedding.*

Proof. By Lemma 5.4, it suffices to prove that φ is surjective. Let $\langle \boldsymbol{a}, \boldsymbol{b}\rangle \in \boldsymbol{C}$, put $\boldsymbol{d} = \boldsymbol{a}{\restriction}_K = \boldsymbol{b}{\restriction}_K$. Then the natural homomorphism from $K/\boldsymbol{d}$ into $P/\boldsymbol{a}$ (resp., $Q/\boldsymbol{b}$) is an embedding, therefore, by using the universal property of $R = P \amalg_K Q$, there exists a homomorphism $r : P \amalg_K Q \twoheadrightarrow (P/\boldsymbol{a}) \amalg_{K/\boldsymbol{d}} (Q/\boldsymbol{b})$ such that the following diagram commutes ($p_{\boldsymbol{a}}$ and $q_{\boldsymbol{b}}$ denote the canonical projections):

$$\begin{tikzcd}
 & & (P/\boldsymbol{a}) \amalg_{K/\boldsymbol{d}} (Q/\boldsymbol{b}) & & \\
P/\boldsymbol{a} \arrow[urr, hook] & & P \amalg_K Q \arrow[u, two heads, "r"] & & Q/\boldsymbol{b} \arrow[ull, hook'] \\
P \arrow[u, two heads, "p_{\boldsymbol{a}}"] \arrow[urr, hook, "u"] & & K/\boldsymbol{d} \arrow[ull, hook'] \arrow[urr, hook] & & Q \arrow[u, two heads, "q_{\boldsymbol{b}}"'] \arrow[ull, hook', "v"'] \\
 & & K \arrow[ull, hook', "f"] \arrow[u, two heads] \arrow[urr, hook, "g"'] & &
\end{tikzcd}$$

Put $\boldsymbol{c} = \ker r$. Then $\boldsymbol{c}$ is a congruence of R. Moreover, for any $x, y \in P$, $x \leq_{\boldsymbol{c}} y$ iff $r(x) \leq r(y)$, that is, $p_{\boldsymbol{a}}(x) \leq p_{\boldsymbol{a}}(y)$, or $x \leq_{\boldsymbol{a}} y$. Hence $\boldsymbol{c}{\restriction}_P = \boldsymbol{a}$. Similarly, $\boldsymbol{c}{\restriction}_Q = \boldsymbol{b}$, hence $\varphi(\boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{b}\rangle$. ■ Claim 1

Claim 2. *The map $\mathcal{K}(\psi)$ is cofinal from $\mathcal{K}(\boldsymbol{C})$ to* $\operatorname{Con_c} R$.

Proof. Put $\boldsymbol{c} = \psi(\langle \mathbf{1}_P, \mathbf{1}_Q\rangle)$. It follows from the definition of ψ that $\boldsymbol{c}{\restriction}_P = \mathbf{1}_P$ and $\boldsymbol{c}{\restriction}_Q = \mathbf{1}_Q$. Pick $z \in K$ (we have supposed that $K \neq \varnothing$). For any $x \in P$ and $y \in Q$, $x \leq_{\boldsymbol{c}} z$ (because $\boldsymbol{c}{\restriction}_P = \mathbf{1}_P$) and $z \leq_{\boldsymbol{c}} y$ (because $\boldsymbol{c}{\restriction}_Q = \mathbf{1}_Q$), hence $x \leq_{\boldsymbol{c}} y$. Similarly, $y \leq_{\boldsymbol{c}} x$. Therefore, $\psi(\langle \mathbf{1}_P, \mathbf{1}_Q\rangle) = \boldsymbol{c} = \mathbf{1}_R$. ■ Claim 2

Claim 3. *$\mathcal{K}(\boldsymbol{C})$ is a pushout of $\mathrm{Con_c}\, P$ and $\mathrm{Con_c}\, Q$ above $\mathrm{Con_c}\, f$ and $\mathrm{Con_c}\, g$ in the category of all $\langle\vee, 0\rangle$-semilattices.*

Proof. Let $\xi : \boldsymbol{C} \to \mathrm{Con}\, P$ and $\eta : \boldsymbol{C} \to \mathrm{Con}\, Q$ be the canonical projections. By the definition of $\boldsymbol{C}$, the diagram

$$\begin{tikzcd} & \boldsymbol{C} \arrow[dl, "\xi"'] \arrow[dr, "\eta"] & \\ \mathrm{Con}\, P \arrow[dr, "\mathrm{Res}\, f"'] & & \mathrm{Con}\, Q \arrow[dl, "\mathrm{Res}\, g"] \\ & \mathrm{Con}\, K & \end{tikzcd}$$

is a *pullback* in the category of all algebraic lattices with complete meet-homomorphisms that preserve nonempty directed joins. By dualizing this diagram (see Lemmas 5.2, 5.3, and 5.5), then by taking the image of the new diagram under the functor $\mathcal{K}$, and then by using Proposition 2.1, we obtain successively the two diagrams below; the left hand side is a pushout in the category of all algebraic lattices and compactness preserving complete join-homomorphisms, the right hand side is a pushout in the category of $\langle\vee, 0\rangle$-semilattices with $\langle\vee, 0\rangle$-homomorphisms:

$$\begin{tikzcd} & \boldsymbol{C} & \\ \mathrm{Con}\, P \arrow[ur, "\xi^\dagger"] & & \mathrm{Con}\, Q \arrow[ul, "\eta^\dagger"'] \\ & \mathrm{Con}\, K \arrow[ul, "\mathrm{Con}\, f"] \arrow[ur, "\mathrm{Con}\, g"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{K}(\boldsymbol{C}) & \\ \mathrm{Con_c}\, P \arrow[ur, "\alpha=\mathcal{K}(\xi^\dagger)"] & & \mathrm{Con_c}\, Q \arrow[ul, "\beta=\mathcal{K}(\eta^\dagger)"'] \\ & \mathrm{Con_c}\, K \arrow[ul, "\mathrm{Con_c}\, f"] \arrow[ur, "\mathrm{Con_c}\, g"'] & \end{tikzcd}$$

This completes the proof of Claim 3. $\blacksquare_{\text{Claim 3}}$

By applying the sequence of two functors used in the proof of Claim 3 to the commutative diagram

$$\begin{tikzcd} & \mathrm{Con}\, R \arrow[d, "\varphi"'] \arrow[ddl, bend right, "\mathrm{Res}\, u"'] \arrow[ddr, bend left, "\mathrm{Res}\, v"] & \\ & \boldsymbol{C} \arrow[dl, "\xi"'] \arrow[dr, "\eta"] & \\ \mathrm{Con}\, P \arrow[dr, "\mathrm{Res}\, f"'] & & \mathrm{Con}\, Q \arrow[dl, "\mathrm{Res}\, g"] \\ & \mathrm{Con}\, K & \end{tikzcd}$$

we obtain, successively, the following two commutative diagrams:

$$
\begin{tikzcd}
& \operatorname{Con} R & \\
& \boldsymbol{C} \arrow[u, hook, "\psi"] & \\
\operatorname{Con} P \arrow[uur, bend left, "\operatorname{Con} u"] \arrow[ur, "\xi^{\dagger}"] & & \operatorname{Con} Q \arrow[uul, bend right, "\operatorname{Con} v"'] \arrow[ul, "\eta^{\dagger}"'] \\
& \operatorname{Con} K \arrow[ul, "\operatorname{Con} f"] \arrow[ur, "\operatorname{Con} g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \operatorname{Con_c} R & \\
& \mathcal{K}(\boldsymbol{C}) \arrow[u, hook, "\mathcal{K}(\psi)"] & \\
\operatorname{Con_c} P \arrow[uur, bend left, "\operatorname{Con_c} u"] \arrow[ur, "\alpha"] & & \operatorname{Con_c} Q \arrow[uul, bend right, "\operatorname{Con_c} v"'] \arrow[ul, "\beta"'] \\
& \operatorname{Con_c} K \arrow[ul, "\operatorname{Con_c} f"] \arrow[ur, "\operatorname{Con_c} g"'] &
\end{tikzcd}
$$

Since $\mu \circ \operatorname{Con_c} f = \nu \circ \operatorname{Con_c} g$ and by Claim 3, there exists a $\langle \vee, 0 \rangle$-homomorphism $\gamma : \mathcal{K}(\boldsymbol{C}) \to S$ such that the diagram

$$
\begin{tikzcd}
& S & \\
& \mathcal{K}(\boldsymbol{C}) \arrow[u, "\gamma"] & \\
\operatorname{Con_c} P \arrow[uur, bend left, "\mu"] \arrow[ur, "\alpha"] & & \operatorname{Con_c} Q \arrow[uul, bend right, "\nu"'] \arrow[ul, "\beta"'] \\
& \operatorname{Con_c} K \arrow[ul, "\operatorname{Con_c} f"] \arrow[ur, "\operatorname{Con_c} g"'] &
\end{tikzcd}
$$

is commutative. Furthermore, by Claims 1 and 2, $\mathcal{K}(\psi)$ is a cofinal embedding from $\mathcal{K}(\boldsymbol{C})$ into $\operatorname{Con_c} R$, while, by Corollary 4.6, $\operatorname{Con_c} j_R$ is a cofinal embedding from $\operatorname{Con_c} R$ into $\operatorname{Con_c} F_{\mathbf{L}}(R)$. Therefore, the map $(\operatorname{Con_c} j_R) \circ \mathcal{K}(\psi)$ is a cofinal embedding from $\mathcal{K}(\boldsymbol{C})$ into $\operatorname{Con_c} F_{\mathbf{L}}(R)$. By Lemma 6.2, there exists a $\langle \vee, 0 \rangle$-homomorphism $\pi : \operatorname{Con_c} F_{\mathbf{L}}(R) \to S$ such that $\pi \circ (\operatorname{Con_c} j_R) \circ \mathcal{K}(\psi) = \gamma$. By Theorem 2, there are a relatively complemented lattice L, a lattice homomorphism $h : F_{\mathbf{L}}(R) \to L$, and an isomorphism $\varepsilon : \operatorname{Con_c} L \to S$ such that $\pi = \varepsilon \circ \operatorname{Con_c} h$, the range of h generates L as an ideal (resp., filter), and, if the range of π is cofinal in S, then the range of h generates L as a convex sublattice. The latter condition is certainly satisfied if $\operatorname{rng} \mu \cup \operatorname{rng} \nu$ is cofinal in S (because $\operatorname{rng} \gamma$ contains $\operatorname{rng} \mu \cup \operatorname{rng} \nu$). Some of this information is summarized in the following commutative diagram:

$$
\begin{tikzcd}
S & & & \\
\mathcal{K}(\boldsymbol{C}) \arrow[u, "\gamma"] \arrow[r, hook, "\mathcal{K}(\psi)"'] & \operatorname{Con_c} R \arrow[r, hook, "\operatorname{Con_c} j_R"'] & \operatorname{Con_c} F_{\mathbf{L}}(R) \arrow[ull, "\pi"'] \arrow[r, "\operatorname{Con_c} h"'] & \operatorname{Con_c} L \arrow[ulll, bend right, "\varepsilon"']
\end{tikzcd}
$$

Now we consider the commutative diagram

$$\begin{tikzcd} & F_{\mathbf{L}}(R) & \\ P \arrow[ur, "f'=j_R\circ u"] & & Q \arrow[ul, "g'=j_R\circ v"'] \\ & K \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

We further compute:

$$\pi \circ \mathrm{Con_c}\, f' = \pi \circ \mathrm{Con_c}\, j_R \circ \mathrm{Con_c}\, u = \pi \circ \mathrm{Con_c}\, j_R \circ \mathcal{K}(\psi) \circ \alpha = \gamma \circ \alpha = \mu.$$

A similar argument proves the equality $\pi \circ \mathrm{Con_c}\, g' = \nu$. The fact that $F_{\mathbf{L}}(R)$ is generated, as a lattice, by $f'[P] \cup g'[Q]$, trivially follows from $R = P \cup Q$. Therefore, the maps $\overline{f} = h \circ f'$ and $\overline{g} = h \circ g'$, together with the isomorphism ε, satisfy the required conditions. ∎

The following corollary generalizes Theorem 2 of [6]:

COROLLARY 6.4. *Let S be a distributive $\langle\vee, 0\rangle$-semilattice that can be expressed as a $\langle\vee, 0\rangle$-direct limit of at most $\aleph_1$ conditionally co-Brouwerian lattices. Then there exists a relatively complemented lattice L with zero such that* $\mathrm{Con_c}\, L \cong S$. *Furthermore, if S has a largest element, then L can be taken bounded.*

Proof. Write $S = \varinjlim (S_i)_{i\in I}$ with transition $\langle\vee, 0\rangle$-homomorphisms $f_{i,j} : S_i \to S_j$ and limiting maps $f_i : S_i \to S$, where I is an upward directed partially ordered set of size at most $\aleph_1$ and all the S_i's are conditionally co-Brouwerian lattices. As at the beginning of the proof of Theorem 2 of [6], we may assume without loss of generality that I is a 2-*ladder*, that is, a lattice with zero in which every principal ideal is finite and every element has at most two immediate predecessors. The rest of the proof goes as the proof of Theorem 2 of [6], by using Theorem 5 for the amalgamation step. ∎

The following corollary generalizes Theorem 3 of [6]. Its proof is similar, again by using Theorem 5.

COROLLARY 6.5. *Let K be a lattice that can be expressed as a direct union of* countably *many lattices whose congruence semilattices are conditionally co-Brouwerian. Then K embeds congruence-preservingly into some relatively complemented lattice L, which it generates as a convex sublattice.*

7. Conditionally κ-co-Brouwerian semilattices

DEFINITION 7.1. Let S be a $\langle\vee, 0\rangle$-semilattice and let κ be an infinite cardinal. We say that S is *conditionally κ-co-Brouwerian* if it satisfies the following conditions:

(i) $<\kappa$-*interpolation property*: for all nonempty $X, Y \subseteq S$ such that $|X|, |Y| < \kappa$ and $X \leq Y$ (that is, $x \leq y$ for all $\langle x, y\rangle \in X \times Y$), there exists $z \in S$ such that $X \leq z \leq Y$.

(ii) $<\kappa$-*interval axiom*: for all $X \subseteq S$ such that $|X| < \kappa$ and all $a, b \in S$ such that $a \leq b \vee x$ for all $x \in X$, there exists $c \in S$ such that $a \leq b \vee c$ and $c \leq X$.

Observe that every conditionally κ-co-Brouwerian $\langle\vee, 0\rangle$-semilattice is obviously distributive (take X a pair in (ii)).

Now we prove the following analogue of Lemma 6.2:

LEMMA 7.2. *Let κ be an infinite cardinal, let S be a conditionally κ-co-Brouwerian $\langle\vee, 0\rangle$-semilattice, and let A be a cofinal $\langle\vee, 0\rangle$-subsemilattice of a $\langle\vee, 0\rangle$-semilattice B such that $|B| < \kappa$. Then every $\langle\vee, 0\rangle$-homomorphism from A to S extends to some $\langle\vee, 0\rangle$-homomorphism from B to S.*

Proof. It suffices to consider the case where B is a monogenic extension of A, i.e., $B = A[b] = A \cup \{x \vee b \mid x \in A\}$, where b is an element of B. Let $f : A \to S$ be a $\langle\vee, 0\rangle$-homomorphism. Let $\{\langle x_i, y_i\rangle \mid i \in I\}$ enumerate all elements $\langle x, y\rangle$ of $A \times A$ such that $x \leq y \vee b$, and let $\{z_j \mid j \in J\}$ enumerate all elements z of A such that $b \leq z$, with $|I|, |J| < \kappa$. Observe that $I \neq \varnothing$, and, since A is cofinal in B, $J \neq \varnothing$. For all $\langle i, j\rangle \in I \times J$, the inequality $x_i \leq y_i \vee z_j$ holds, thus $f(x_i) \leq f(y_i) \vee f(z_j)$. By the $<\kappa$-interval axiom, for all $i \in I$, there exists $\boldsymbol{b}_i \in S$ such that $f(x_i) \leq f(y_i) \vee \boldsymbol{b}_i$ and $\boldsymbol{b}_i \leq f(z_j)$ for all $j \in J$. By the $<\kappa$-interpolation property, there exists $\boldsymbol{b} \in S$ such that $\boldsymbol{b}_i \leq \boldsymbol{b} \leq f(z_j)$ for all $\langle i, j\rangle \in I \times J$. Hence, $f(x_i) \leq f(y_i) \vee \boldsymbol{b}$ for all $i \in I$, and $\boldsymbol{b} \leq f(z_j)$ for all $j \in J$, so that there exists a unique $\langle\vee, 0\rangle$-homomorphism $g : B \to S$ extending f such that $g(b) = \boldsymbol{b}$. ∎

We shall now outline a proof of the following analogue of Proposition 18.5 of [15].

LEMMA 7.3. *Let κ be an uncountable cardinal, let $\mathcal{D}$ be a truncated square of partial lattices, with lattice bottom, of size less than κ, and let S be a conditionally κ-co-Brouwerian $\langle\vee, 0\rangle$-semilattice. Then every homomorphism $\varphi : \mathrm{Con_c}\,\mathcal{D} \to S$ has a factor of the form $f : \mathcal{D} \to L$, where L is a lattice generated by the range of f (thus, $|L| < \kappa$).*

Proof. We use the same notation as in the proof of Theorem 5 in Section 6. In particular, $|K|, |P|, |Q| < \kappa$. Then the proof of Theorem 5 applies *mutatis mutandis*, by using Lemma 7.2 instead of Lemma 6.2, to establish that the canonical pushout homomorphism $f : \mathcal{D} \to F_{\mathbf{L}}(R)$, with $R = P \amalg_K Q$, is a factor of φ: all semilattices that need to be of size less than κ are indeed of size less than κ, moreover, the last extension step from $F_{\mathbf{L}}(R)$ to L made in the proof of Theorem 5 is no longer necessary since we require only "factor" instead of "lift". Observe that since κ is uncountable, $L = F_{\mathbf{L}}(R)$ still has size less than κ. ∎

Our next definitions are borrowed from [15]:

Definition 7.4. Let S be a $\langle\vee, 0\rangle$-semilattice. An *S-measured partial lattice* is a pair $\langle P, \mu\rangle$, where P is a partial lattice and $\mu : \mathrm{Con_c}\, P \to S$ is a $\langle\vee, 0\rangle$-homomorphism. If, in addition, P is a lattice, we say that $\langle P, \mu\rangle$ is an *S-measured lattice*.

An S-measured partial lattice $\langle P, \mu\rangle$ is *proper* if μ *isolates zero*, that is, $\mu^{-1}\{0\} = \{\mathbf{0}_P\}$.

Definition 7.5. Let S be a $\langle\vee, 0\rangle$-semilattice, and let $\langle P, \mu\rangle$ and $\langle Q, \nu\rangle$ be S-measured partial lattices. A *homomorphism* from $\langle P, \mu\rangle$ to $\langle Q, \nu\rangle$ is a homomorphism $f : P \to Q$ of partial lattices such that $\nu \circ \mathrm{Con_c}\, f = \mu$. If, in addition, f is an embedding of partial lattices, we say that f is an *embedding* of S-measured partial lattices.

Definition 7.6. Let S be a $\langle\vee, 0\rangle$-semilattice, and let $\langle P, \mu\rangle$ and $\langle L, \varphi\rangle$ be S-measured partial lattices, with L a lattice. We say that an embedding $f : \langle P, \mu\rangle \hookrightarrow \langle L, \varphi\rangle$ is a *lower embedding* (resp., *upper embedding*, *internal embedding*) if the filter (resp., ideal, convex sublattice) of L generated by P equals L.

Definition 7.7. Let S be a $\langle\vee, 0\rangle$-semilattice and let X be a subset of S. A proper S-measured lattice $\langle L, \varphi\rangle$ is *X-saturated* (resp., *lower X-saturated*, *upper X-saturated*, *internally X-saturated*) if for every embedding (resp., lower embedding, upper embedding, internal embedding) $e : \langle K, \lambda\rangle \hookrightarrow \langle P, \mu\rangle$ of finite proper S-measured partial lattices such that $\operatorname{rng}\mu \subseteq X \cup \operatorname{rng}\varphi$, with K a *lattice*, and every homomorphism $f : \langle K, \lambda\rangle \to \langle L, \varphi\rangle$, there exists a homomorphism $g : \langle P, \mu\rangle \to \langle L, \varphi\rangle$ such that $g \circ e = f$.

Now a standard increasing chain argument makes it possible to prove the following result.

Proposition 7.8. *Let κ be an uncountable cardinal, let S be a conditionally κ-co-Brouwerian $\langle\vee, 0\rangle$-semilattice, and let $X \subseteq S$ be such that $|X| < \kappa$. Every proper S-measured partial lattice $\langle P, \varphi\rangle$ such that $|P| < \kappa$ admits an embedding* (*resp., a lower embedding, an upper embedding, an internal embedding*) *into an X-saturated* (*resp., lower X-saturated, upper X-saturated, internally X-saturated*) *S-measured lattice $\langle L, \psi\rangle$ such that $|L| = |P| + |X| + \aleph_0$.*

Proof. We proceed as in the proof of Proposition 19.3 of [15]. We first use Corollary 4.6 and Lemma 7.2 to extend $\langle P, \varphi\rangle$ by $\langle F_{\mathbf{L}}(P), \psi\rangle$ for some ψ. Then the S-measured partial lattice $\langle F_{\mathbf{L}}(P), \psi\rangle$ may not be proper, so we need to replace it by its quotient under the congruence of $F_{\mathbf{L}}(P)$ that consists of all pairs $\langle x, y\rangle$ such that $\psi\Theta^{+}(x, y) = 0$ (called the *kernel projection* in [15]).

This way, we deduce that P may be assumed to be a lattice from the start. Furthermore, there are at most $|P| + |X| + \aleph_0$ pairs of the form $\langle e, f\rangle$ where $e : \langle K, \lambda\rangle \to \langle Q, \nu\rangle$ and $f : \langle K, \lambda\rangle \to \langle P, \varphi\rangle$ are homomorphisms of S-measured partial lattices with K a lattice, both K and Q finite, e an embedding, and $\operatorname{rng}\nu$ contained in $X \cup \operatorname{rng}\varphi$. We increase $\langle P, \varphi\rangle$ by a transfinite sequence of length $|P| + |X| + \aleph_0$ of S-measured lattices. At each stage $\langle L, \psi\rangle$ of the construction, we pick the corresponding pair $\langle e, f\rangle$ of homomorphisms. The amalgamation result of Lemma 7.3 makes it possible to find an S-measured lattice $\langle L', \psi'\rangle$, together with homomorphisms e' and f', such that the following diagram commutes:

$$\begin{tikzcd}
\langle Q, \nu\rangle \arrow[r, "f'"] & \langle L', \psi'\rangle \\
\langle K, \lambda\rangle \arrow[u, "e"] \arrow[r, "f"'] & \langle L, \psi\rangle \arrow[u, "e'"']
\end{tikzcd}$$

Again, by replacing $\langle L', \psi'\rangle$ by its quotient under its kernel projection, we may assume that $\langle L', \psi'\rangle$ is proper; let, then, $\langle L', \psi'\rangle$ be the next step of the construction.

We denote by $\langle P, \varphi\rangle^{*}$ the direct limit of that construction. Iterating ω times the operation $\langle P, \varphi\rangle \hookrightarrow \langle P, \varphi\rangle^{*}$ and taking again the direct limit yields the desired result. ∎

Now, by using Proposition 7.8, we argue as in Section 20 of [15] to obtain the following analogue of Proposition 20.8 of [15]. Observe that the proof is, in fact, much simpler than that of Proposition 20.8 of [15]. The reason for this is that we no longer need to check that the corresponding S-measured partial lattices are “balanced”, which removes lots of technical complexity.

PROPOSITION 7.9. *Let κ be an uncountable cardinal, let S be a conditionally κ-co-Brouwerian $\langle\vee, 0\rangle$-semilattice, and let $\langle L, \varphi\rangle$ be an internally X-saturated S-measured partial lattice. Then*:

(i) *L is relatively complemented.*
(ii) *The map φ is an embedding from* $\operatorname{Con_c} L$ *into S, and $X \cap {\downarrow}\operatorname{rng}\varphi \subseteq$* $\operatorname{rng}\varphi$. (*For a subset Y of S, ${\downarrow}Y$ denotes the lower subset of S generated by Y.*)
(iii) *For $o, a, b, i \in L$ such that $o \leq \{a, b\} \leq i$, we have $\Theta_L(o, a) = \Theta_L(o, b)$ iff there are $a_0, a_1, b_0, b_1 \in [o, i]$ such that*:

(a) *$a = a_0 \vee a_1$, $b = b_0 \vee b_1$, and $a_0 \wedge a_1 = b_0 \wedge b_1 = o$.*

(b) *a_0 and b_0 (resp., a_1 and b_1) are perspective in $[o,i]$, i.e., for all $l < 2$, there exists $x \in [o,i]$ such that $a_l \wedge x = b_l \wedge x = o$ and $a_l \vee x = b_l \vee x = i$.*

(iv) *If, in addition, $\langle L, \varphi\rangle$ is either lower X-saturated or upper X-saturated, then $X \subseteq \operatorname{rng}\varphi$.*

Outline of proof. We imitate the proof of Proposition 20.8 of [15]. We first show, for example, that L is relatively complemented. For $a < b < c$ in L, let $K = \{a, b, c\}$ be the three-element chain, let $f : K \hookrightarrow L$ be the natural embedding, and put $\lambda = \varphi \circ \operatorname{Con_c} f$. Then $\langle K, \lambda\rangle$ is a finite, proper S-measured lattice and f is an embedding from $\langle K, \lambda\rangle$ into $\langle L, \varphi\rangle$. Next, we put $P = \{a, b, c, t\}$, the Boolean lattice with bottom a, top c, and atoms b and t, endowed with the homomorphism $\mu : \operatorname{Con_c} P \to S$ defined by

$$\mu\Theta_P(a,b) = \mu\Theta_P(t,c) = \varphi\Theta_L(a,b),$$
$$\mu\Theta_P(a,t) = \mu\Theta_P(b,c) = \varphi\Theta_L(b,c).$$

Then $\langle P, \mu\rangle$ is a proper S-measured lattice, with $\operatorname{rng}\mu \subseteq \operatorname{rng}\varphi \subseteq X \cup \operatorname{rng}\varphi$, and the inclusion map $j : K \hookrightarrow P$ is an embedding from $\langle K, \lambda\rangle$ into $\langle P, \mu\rangle$. By assumption on $\langle L, \varphi\rangle$, there exists a homomorphism $g : \langle P, \mu\rangle \to \langle L, \varphi\rangle$ such that $g \circ j = f$. Put $x = g(t)$. Then $a = b \wedge x$ and $c = b \vee x$.

The proofs of (ii)–(iv) proceed in the same way, as shown in 20.2–20.7 in [15]. For proving the containment $X \cap {\downarrow}\operatorname{rng}\varphi \subseteq \operatorname{rng}\varphi$, we need to imitate the second part of the proof of Lemma 20.7 in [15]. More specifically, let $\alpha \in X$, let $o < i$ in L be such that $0 < \alpha < \varphi\Theta_L(o,i)$, put $K = \{o, i\}$, let $f : K \hookrightarrow L$ be the inclusion map, and let $\lambda = \varphi \circ \operatorname{Con_c} f$. Furthermore, let $P = \{o, x, i\}$ be the three-element chain, with $o < x < i$, and let $j : K \hookrightarrow P$ be the inclusion map. Endow P with the $\langle\vee, 0\rangle$-homomorphism $\mu : \operatorname{Con_c} P \to S$ defined by $\mu\Theta_P(o,x) = \alpha$ and $\mu\Theta_P(x,i) = \varphi\Theta_L(o,i)$. Observe that the range of μ is contained in $X \cup \operatorname{rng}\varphi$. By assumption on $\langle L, \varphi\rangle$, there exists a homomorphism $g : \langle P, \mu\rangle \to \langle L, \varphi\rangle$ such that $g \circ j = f$. Hence the element $\alpha = \mu\Theta_P(o,x) = (\varphi \circ \operatorname{Con_c} g)\Theta_P(o,x)$ belongs to the range of φ.

The proof of (iii) goes along similar lines, although the relevant lattice K and partial lattice P are much more complicated (see 20.2–20.6 in [15] for details). ■

Now we come to the main result (stated in the Introduction) of Section 7:

Proof of Theorem 6. We first deal separately with the case where $\kappa = \aleph_0$, i.e., S is countable. Then, by Bergman's Theorem [1, 3] and Corollary 7.5 in [3], there exists a relatively complemented *modular* lattice L with zero such that $\operatorname{Con_c} L \cong S$; moreover, if S is bounded, then L is bounded.

Suppose now that $\kappa > \aleph_0$. We can decompose S as $S = \bigcup_{\xi<\kappa} S_\xi$, for an increasing family $(S_\xi)_{\xi<\kappa}$ of infinite $\langle\vee, 0\rangle$-subsemilattices of S such that $|\xi| \leq |S_\xi| < \kappa$ for all $\xi < \kappa$. Furthermore, if S is bounded, then we may assume that $1 \in S_\xi$ for all $\xi < \kappa$.

Now we construct S-measured lattices $\langle L_\xi, \varphi_\xi\rangle$, for $\xi < \kappa$, as follows. For $\xi < \kappa$, suppose that $\langle L_\eta, \varphi_\eta\rangle$ has been constructed for all $\eta < \xi$, such that $\langle L_\zeta, \varphi_\zeta\rangle$ is an extension of $\langle L_\eta, \varphi_\eta\rangle$, $|L_\eta| \leq |S_\eta|$, and $\langle L_\eta, \varphi_\eta\rangle$ is lower S_η-saturated, for $\eta \leq \zeta < \xi$. Put $L'_\xi = \bigcup_{\eta<\xi} L_\eta$ and $\varphi'_\xi = \bigcup_{\eta<\xi} \varphi_\eta$. Observe that $|L'_\xi| \leq |S_\xi|$. Hence, by Proposition 7.8 applied to $\langle L'_\xi, \varphi'_\xi\rangle$, there exists a lower S_ξ-saturated $\langle L_\xi, \varphi_\xi\rangle$ with $|L_\xi| \leq |S_\xi|$ such that $\langle L'_\xi, \varphi'_\xi\rangle$ admits a 0-lattice embedding into $\langle L_\xi, \varphi_\xi\rangle$ (the embedding condition is vacuously satisfied for $\xi = 0$). In particular, L_ξ is a lattice with zero. Furthermore, if S is bounded, then this embedding may be taken internal, with L_0 bounded and $1 \in \operatorname{rng}\varphi_0$.

Take $L = \bigcup_{\xi<\kappa} L_\xi$, a lattice with zero. Then $\varphi = \bigcup_{\xi<\kappa} \varphi_\xi$ is, by Proposition 7.9, an isomorphism from $\operatorname{Con_c} L$ onto S. If S is bounded, then so is L. Furthermore, by Proposition 7.9, L is relatively complemented. ∎

We observe that the lattice L constructed in the proof of Theorem 6 has many other properties besides being relatively complemented, such as item (iii) in the statement of Proposition 7.9.

8. The spaces $P^*_{\kappa,\lambda}$, $P_{\kappa,\lambda}$, $A_{\kappa,\lambda}$, $U_{\kappa,\lambda}$, $V_{\kappa,\lambda}$. For a partially ordered set P, we denote by $\operatorname{Int} P$ the Boolean subalgebra of the powerset algebra of P generated by all lower subsets of P. For a limit ordinal λ, we define a subset x of λ to be *bounded* if $x \subseteq \alpha$ for some $\alpha < \lambda$, and then we define a map $\chi_\lambda : \operatorname{Int}\lambda \to \mathbf{2}$ by the rule

$$\chi_\lambda(x) = \begin{cases} 0 & \text{if } x \text{ is bounded,} \\ 1 & \text{otherwise.} \end{cases}$$

We leave to the reader the easy proof of the following lemma:

LEMMA 8.1. *The map χ_λ is a $\langle\vee, \wedge, 0, 1\rangle$-homomorphism from* $\operatorname{Int}\lambda$ *onto* $\mathbf{2}$, *for any limit ordinal λ.*

For the remainder of this section, we fix infinite cardinals κ and λ. Then we put

$$A_{\kappa,\lambda} = \operatorname{Int}\kappa \times \operatorname{Int}\lambda \times \operatorname{Int}\lambda,$$
$$U_{\kappa,\lambda} = \{\langle x_0, x_1, x_2\rangle \in A_{\kappa,\lambda} \mid \chi_\kappa(x_0) = \chi_\lambda(x_1) = \chi_\lambda(x_2)\},$$
$$V_{\kappa,\lambda} = \{\langle x_0, x_1, x_2\rangle \in A_{\kappa,\lambda} \mid \chi_\kappa(x_0) = \chi_\lambda(x_1)\},$$
$$P^*_{\kappa,\lambda} = \{\langle x_0, x_1, x_2\rangle \in A_{\kappa,\lambda} \mid \text{either } \chi_\kappa(x_0) = 0 \text{ or } x_1 \cup x_2 \neq \varnothing\},$$
$$P_{\kappa,\lambda} = \{\langle x_0, x_1, x_2\rangle \in A_{\kappa,\lambda} \mid \chi_\kappa(x_0) = \chi_\lambda(x_1) \vee \chi_\lambda(x_2)\}.$$

We endow each of the sets $A_{\kappa,\lambda}$, $U_{\kappa,\lambda}$, $V_{\kappa,\lambda}$, $P^*_{\kappa,\lambda}$, $P_{\kappa,\lambda}$ with the structure of partial lattice inherited from the (Boolean) lattice structure of $A_{\kappa,\lambda}$, i.e., for a nonempty finite subset X of $P_{\kappa,\lambda}$ and $a \in P_{\kappa,\lambda}$, $a = \bigvee X$ if a is the join of X in $A_{\kappa,\lambda}$, and similarly for the meet.

The following easy lemma summarizes the elementary properties of these objects:

LEMMA 8.2. (i) *$A_{\kappa,\lambda}$, $U_{\kappa,\lambda}$, and $V_{\kappa,\lambda}$ are Boolean algebras such that $U_{\kappa,\lambda} \subset V_{\kappa,\lambda} \subset A_{\kappa,\lambda}$.*

(ii) *$P^*_{\kappa,\lambda}$ is a $\langle\vee, 0, 1\rangle$-subsemilattice of $A_{\kappa,\lambda}$ and contains $V_{\kappa,\lambda}$.*

(iii) *For all $x, y \in P_{\kappa,\lambda}$, $x \smallsetminus y$ belongs to $P^*_{\kappa,\lambda}$.*

Of course, $x \smallsetminus y$ is an abbreviation for $x \wedge \neg y$.

Now let S be a $\langle\vee, 0\rangle$-semilattice, and let $\vec{\boldsymbol{a}} = (\boldsymbol{a}_\xi)_{\xi<\kappa}$ (resp., $\vec{\boldsymbol{b}} = (\boldsymbol{b}_\eta)_{\eta<\lambda}$) be an increasing (resp., decreasing) κ-sequence (resp., λ-sequence) of elements of S such that $\vec{\boldsymbol{a}} \le \vec{\boldsymbol{b}}$, i.e., $\boldsymbol{a}_\xi \le \boldsymbol{b}_\eta$ for all $\xi < \kappa$ and all $\eta < \lambda$. We suppose, in addition, that $\boldsymbol{a}_0 = 0$.

We define a map $\sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}} : P^*_{\kappa,\lambda} \to S$ by the rule

$$\sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(\langle x_0, x_1, x_2\rangle) = \begin{cases} \boldsymbol{a}_{\sup x_0} & \text{if } x_1 \cup x_2 = \varnothing, \\ \boldsymbol{b}_{\min(x_1\cup x_2)} & \text{otherwise.} \end{cases}$$

LEMMA 8.3. *The map $\sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}$ is a $\langle\vee, 0\rangle$-homomorphism from $P^*_{\kappa,\lambda}$ to S.*

Now we define a map $\mu_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}} : P_{\kappa,\lambda} \times P_{\kappa,\lambda} \to S$ by the rule

$$\mu_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(x, y) = \sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(x \smallsetminus y) \quad \text{for all } x, y \in P_{\kappa,\lambda}.$$

This definition is consistent, by Lemma 8.2(iii).

LEMMA 8.4. *The map $\mu_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}$ is a measure (see Definition 3.3) on $P_{\kappa,\lambda}$.*

Proof. This follows immediately from Lemma 8.3. ∎

By Lemma 3.4, there exists a unique $\langle\vee, 0\rangle$-homomorphism $\varphi_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}$: $\mathrm{Con_c}\, P_{\kappa,\lambda} \to S$ such that $\varphi_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}\Theta^+_{P_{\kappa,\lambda}}(x, y) = \mu_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(x, y)$ for all $x, y \in P_{\kappa,\lambda}$.

Now we come to the main result of this section.

PROPOSITION 8.5. *Suppose that the map $\varphi_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}} : \mathrm{Con_c}\, P_{\kappa,\lambda} \to S$ can be factored through a lattice. Then there exists $\boldsymbol{c} \in S$ such that $\boldsymbol{a}_\xi \le \boldsymbol{c} \le \boldsymbol{b}_\eta$ for all $\xi < \kappa$ and all $\eta < \lambda$.*

Proof. Suppose that there are a lattice L, a homomorphism $f : P_{\kappa,\lambda} \to L$ of partial lattices, and a $\langle\vee, 0\rangle$-homomorphism $\psi : \mathrm{Con_c}\, L \to S$ such that $\varphi_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}} = \psi \circ \mathrm{Con_c}\, f$. We put

$$\boldsymbol{c} = \psi\Theta_L(f(\langle\varnothing, \varnothing, \varnothing\rangle), f(\langle\kappa, \lambda, \varnothing\rangle) \wedge f(\langle\kappa, \varnothing, \lambda\rangle)).$$

We prove that $\boldsymbol{c}$ satisfies the required inequalities.

Let $\xi < \kappa$. From the inequality

$$f(\langle\kappa, \lambda, \varnothing\rangle) \wedge f(\langle\kappa, \varnothing, \lambda\rangle) \ge f(\langle\xi + 1, \varnothing, \varnothing\rangle)$$

it follows that

$$\begin{aligned}\boldsymbol{c} &\geq \psi\Theta_L(f(\langle\varnothing,\varnothing,\varnothing\rangle), f(\langle\xi+1,\varnothing,\varnothing\rangle))\\ &= \varphi_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}\Theta_P(\langle\varnothing,\varnothing,\varnothing\rangle,\langle\xi+1,\varnothing,\varnothing\rangle)\\ &= \mu_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(\langle\xi+1,\varnothing,\varnothing\rangle,\langle\varnothing,\varnothing,\varnothing\rangle) = \sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(\langle\xi+1,\varnothing,\varnothing\rangle) = \boldsymbol{a}_\xi.\end{aligned}$$

Now let $\eta < \lambda$. We first observe that $f(\langle\kappa,\varnothing,\lambda\rangle) \leq f(\langle\kappa,\lambda\smallsetminus\eta,\lambda\rangle)$ and that $\langle\kappa,\lambda,\varnothing\rangle \wedge \langle\kappa,\lambda\smallsetminus\eta,\lambda\rangle$ is defined in $P_{\kappa,\lambda}$, with value $\langle\kappa,\lambda\smallsetminus\eta,\varnothing\rangle$. It follows that

$$\begin{aligned}\boldsymbol{c} &\leq \psi\Theta_L(f(\langle\varnothing,\varnothing,\varnothing\rangle), f(\langle\kappa,\lambda\smallsetminus\eta,\varnothing\rangle))\\ &= \sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}(\langle\kappa,\lambda\smallsetminus\eta,\varnothing\rangle) = \boldsymbol{b}_{\min(\lambda\smallsetminus\eta)} = \boldsymbol{b}_\eta. \ \blacksquare\end{aligned}$$

DEFINITION 8.6. Let P be a partially ordered set, and let κ and λ be infinite cardinals. We say that P has the *$\langle\kappa,\lambda\rangle$-interpolation property* if for every increasing κ-chain $\vec{\boldsymbol{a}}$ and every decreasing λ-chain $\vec{\boldsymbol{b}}$ of P such that $\vec{\boldsymbol{a}} \leq \vec{\boldsymbol{b}}$, there exists $\boldsymbol{c} \in P$ such that $\vec{\boldsymbol{a}} \leq \boldsymbol{c} \leq \vec{\boldsymbol{b}}$.

Observe that if $\kappa = \lambda = \aleph_0$, then $P_{\kappa,\lambda} = P_{\omega,\omega}$ is countable, and we obtain the following result:

PROPOSITION 8.7. *Let S be a $\langle\vee,0\rangle$-semilattice that does not have the $\langle\omega,\omega\rangle$-interpolation property. Then there exists a $\langle\vee,0\rangle$-homomorphism $\varphi : \mathrm{Con_c}\, P_{\omega,\omega} \to S$ that cannot be factored through a lattice.*

9. Necessity of the conditional completeness

DEFINITION 9.1. Let P be a partially ordered set. We say that P is *conditionally complete* if every nonempty majorized subset of P has a least upper bound.

We recall the following elementary fact about conditional completeness:

LEMMA 9.2. *For any lattice S, if S has the $\langle\kappa,\lambda\rangle$-interpolation property for all infinite cardinals κ and λ, then S is conditionally complete.*

Then we immediately get the following result:

PROPOSITION 9.3. *Let S be a $\langle\vee,0\rangle$-semilattice such that for every partial lattice P, every $\langle\vee,0\rangle$-homomorphism $\varphi : \mathrm{Con_c}\, P \to S$ can be lifted. Then S is a conditionally complete lattice.*

Proof. By [12], the condition above, even restricted to Boolean lattices P, is sufficient to imply that S is a lattice. The conclusion then follows from Lemma 9.2 and Proposition 8.5. $\blacksquare$

In order to be able to formulate the forthcoming Proposition 9.4, we introduce some additional notation. For infinite cardinals κ and λ, let $e_{\kappa,\lambda} : U_{\kappa,\lambda} \hookrightarrow V_{\kappa,\lambda}$ be the inclusion map, let $s_{\kappa,\lambda} : U_{\kappa,\lambda} \to U_{\kappa,\lambda}$, $\langle x_0,x_1,x_2\rangle \mapsto \langle x_0,x_2,x_1\rangle$, be the natural symmetry, and put $e'_{\kappa,\lambda} = e_{\kappa,\lambda} \circ s_{\kappa,\lambda}$.

PROPOSITION 9.4. *Let S be a $\langle \vee, 0\rangle$-semilattice, let κ and λ be infinite cardinal numbers, and let $\vec{\boldsymbol{a}}$ (resp., $\vec{\boldsymbol{b}}$) be an increasing (resp., decreasing) κ-sequence (resp., λ-sequence) of elements of S such that $\vec{\boldsymbol{a}} \leq \vec{\boldsymbol{b}}$. Denote by μ (resp., ν) the restriction of $\sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}$ to $U_{\kappa,\lambda}$ (resp., $V_{\kappa,\lambda}$). Suppose that there are a meet-semilattice L, meet-homomorphisms $f, f' : V_{\kappa,\lambda} \to L$, and an order-preserving map $\varrho : L \to S$ such that $f \circ e_{\kappa,\lambda} = f' \circ e'_{\kappa,\lambda}$ and $\varrho \circ f = \varrho \circ f' = \nu$. Then there exists $\boldsymbol{c} \in S$ such that $\boldsymbol{a}_\xi \leq \boldsymbol{c} \leq \boldsymbol{b}_\eta$ for all $\xi < \kappa$ and all $\eta < \lambda$.*

The statement of Proposition 9.4 means that if the amalgamation problem described by the diagram

$$\begin{tikzcd}
& \langle L, \varrho\rangle & \\
\langle V_{\kappa,\lambda}, \nu\rangle \arrow[ur, dashed, "f"] & & \langle V_{\kappa,\lambda}, \nu\rangle \arrow[ul, dashed, "f'"'] \\
& \langle U_{\kappa,\lambda}, \mu\rangle \arrow[ul, hook, "e_{\kappa,\lambda}"] \arrow[ur, hook', "e'_{\kappa,\lambda}"'] &
\end{tikzcd}$$

can be solved for a meet-semilattice L, meet-homomorphisms $f, f' : V \to L$, and an order preserving $\varrho : L \to S$, then there exists $\boldsymbol{c} \in S$ such that $\vec{\boldsymbol{a}} \leq \boldsymbol{c} \leq \vec{\boldsymbol{b}}$.

Proof. Suppose that L, ϱ, f, and f' are as required. We define $\boldsymbol{c} \in S$ by

$$\boldsymbol{c} = \varrho(f(\langle \kappa, \lambda, \varnothing\rangle) \wedge f'(\langle \kappa, \lambda, \varnothing\rangle)).$$

Now we prove that $\boldsymbol{a}_\xi \leq \boldsymbol{c}$ for all $\xi < \kappa$. Indeed, from the inequalities

$$\begin{aligned}
f(\langle \kappa, \lambda, \varnothing\rangle) &\geq f(\langle \xi+1, \varnothing, \varnothing\rangle),\\
f'(\langle \kappa, \lambda, \varnothing\rangle) &\geq f'(\langle \xi+1, \varnothing, \varnothing\rangle) = f(\langle \xi+1, \varnothing, \varnothing\rangle)
\end{aligned}$$

it follows that

$$\boldsymbol{c} \geq \varrho(f(\langle \xi+1, \varnothing, \varnothing\rangle)) = \nu(\langle \xi+1, \varnothing, \varnothing\rangle) = \boldsymbol{a}_\xi.$$

Next, we prove that $\boldsymbol{c} \leq \boldsymbol{b}_\eta$ for all $\eta < \lambda$. Indeed, put $g = f \circ e = f' \circ e'$. We compute:

$$\begin{aligned}
f'(\langle \kappa, \lambda, \varnothing\rangle) &\leq f'(\langle \kappa, \lambda, \lambda \smallsetminus \eta\rangle) \qquad (\text{observe that } \langle \kappa, \lambda, \lambda \smallsetminus \eta\rangle \in U_{\kappa,\lambda})\\
&= f' \circ s_{\kappa,\lambda}(\langle \kappa, \lambda \smallsetminus \eta, \lambda\rangle) = f(\langle \kappa, \lambda \smallsetminus \eta, \lambda\rangle),
\end{aligned}$$

thus, since f is a meet-homomorphism,

$$f(\langle \kappa, \lambda, \varnothing\rangle) \wedge f'(\langle \kappa, \lambda, \varnothing\rangle) \leq f(\langle \kappa, \lambda, \varnothing\rangle \wedge \langle \kappa, \lambda \smallsetminus \eta, \lambda\rangle) = f(\langle \kappa, \lambda \smallsetminus \eta, \varnothing\rangle).$$

Therefore,

$$\boldsymbol{c} \leq \varrho(f(\langle \kappa, \lambda \smallsetminus \eta, \varnothing\rangle)) = \nu(\langle \kappa, \lambda \smallsetminus \eta, \varnothing\rangle) = \boldsymbol{b}_\eta. \blacksquare$$

As a corollary, even a weak version of Theorem 5 requires the assumption that S is a conditionally complete distributive lattice:

COROLLARY 9.5. *Let S be a $\langle\vee,0\rangle$-semilattice such that for every truncated square $\mathcal{D}$ of Boolean lattices and homomorphisms of Boolean lattices, every homomorphism $\varphi : \operatorname{Con_c} \mathcal{D} \to S$ can be lifted through a lattice. Then S is a conditionally complete distributive lattice.*

Proof. The one-dimensional version of Theorem 5 states that every $\langle\vee,0\rangle$-homomorphism from $\operatorname{Con_c} K$ to S can be lifted, for any lattice K. By [12], even the restriction of this result to the case where K is Boolean is already sufficient to imply that S is a distributive lattice. Of course, the two-dimensional amalgamation property above is stronger (take $B_0 = B_1 = B_2$ and $e_1 = e_2 = \mathrm{id}_{B_0}$).

Now, if S is not conditionally complete, then, by Lemma 9.2, there are infinite cardinals κ and λ, an increasing κ-chain $\vec{\boldsymbol{a}}$ of S, and a decreasing λ-chain $\vec{\boldsymbol{b}}$ of S such that $\vec{\boldsymbol{a}} \le \vec{\boldsymbol{b}}$ but there exists no $\boldsymbol{c} \in S$ such that $\vec{\boldsymbol{a}} \le \boldsymbol{c} \le \vec{\boldsymbol{b}}$. Let μ and ν be the restrictions of $\sigma_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}}$ to $U_{\kappa,\lambda}$ and $V_{\kappa,\lambda}$, respectively, let φ denote the canonical isomorphism from $\operatorname{Con_c} V_{\kappa,\lambda}$ onto $V_{\kappa,\lambda}$, and put $\overline{\nu} = \nu\circ\varphi$. Hence $\overline{\nu}$ is a $\langle\vee,0\rangle$-homomorphism from $\operatorname{Con_c} V_{\kappa,\lambda}$ to S. By assumption on S, there are a lattice L, lattice homomorphisms $f, f' : V_{\kappa,\lambda} \to L$, and an isomorphism $\varepsilon : \operatorname{Con_c} L \to S$ such that $f \circ e_{\kappa,\lambda} = f' \circ e'_{\kappa,\lambda}$ (denote this map by h) and $\overline{\nu} = \varepsilon \circ \operatorname{Con_c} f = \varepsilon \circ \operatorname{Con_c} f'$. Define a map $\varrho : L \to S$ by the rule

$$\varrho(x) = \varepsilon\Theta_L(h(0_{U_{\kappa,\lambda}}), x) \quad \text{for all } x \in L.$$

Then ϱ is order preserving and $\varrho \circ f = \varrho \circ f' = \nu$, a contradiction. ∎

10. Lifting truncated cubes. The question whether the results of this paper can be extended from truncated squares to truncated *cubes* of lattices has a trivial, negative answer. Indeed, consider the following diagram $\mathcal{D}$ of lattices and 0-preserving lattice embeddings:

$$\begin{tikzcd}
\mathbf{2} & & M_3 & & \mathbf{2} \\
\mathbf{2} \arrow[u, twoheadrightarrow] \arrow[urr, hook, "f"] & & \mathbf{2} \arrow[ull, hook] \arrow[urr, hook] & & \mathbf{2} \arrow[u, twoheadrightarrow] \arrow[ull, hook', "g"'] \\
& & \mathbf{1} \arrow[ull, hook] \arrow[u, hook] \arrow[urr, hook] & &
\end{tikzcd}$$

where $\mathbf{1} = \{0\}$, $M_3 = \{0, a, b, c, 1\}$ is the five-element modular nondistributive lattice (with atoms a, b, c), the unlabelled arrows are uniquely determined, $f(1) = a$, and $g(1) = c$. Then the image of $\mathcal{D}$ under the $\operatorname{Con_c}$ functor is obtained by truncating the top $\mathbf{2}$ from the following commutative diagram of $\langle\vee,0\rangle$-semilattices and $\langle\vee,0\rangle$-embeddings:

$$
\begin{tikzcd}
& \mathbf{2} & \\
\mathbf{2} \arrow[ur] & \mathbf{2} \arrow[u] & \mathbf{2} \arrow[ul] \\
\mathbf{2} \arrow[u] \arrow[ur] & \mathbf{2} \arrow[ul] \arrow[ur] & \mathbf{2} \arrow[u] \arrow[ul] \\
& \mathbf{1} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

that defines a homomorphism $\varphi : \mathrm{Con_c}\, \mathcal{D} \to \mathbf{2}$. Suppose that φ can be lifted to a homomorphism from $\mathcal{D}$ to some partial lattice P, in particular, P is *simple*. Let $u : \mathbf{2} \to P$, $w : M_3 \to P$, and $v : \mathbf{2} \to P$ be the homomorphisms of partial lattices that correspond to the top part of such a lifting. Travelling through the diagram $\mathcal{D}$, we obtain

$$w(a) = wf(1) = u(1) = v(1) = wg(1) = w(c),$$

but $\mathrm{Con_c}\, w$ isolates zero, i.e., w is an embedding, hence $a = c$, a contradiction.

Therefore, even the simplest nontrivial lattice $\mathbf{2}$ does not satisfy what could be called the “three-dimensional amalgamation property”.

11. Open problems. The first two open problems ask whether the sufficient conditions underlying Theorems 4 and 5 are also necessary (we conjecture that yes). Possible formulations are the following:

PROBLEM 1. *Let S be a distributive $\langle\vee, 0\rangle$-semilattice. If, for every partial lattice P, every $\langle\vee, 0\rangle$-homomorphism from* $\mathrm{Con_c}\, P$ *to S can be factored through a lattice, is S conditionally co-Brouwerian?*

By Proposition 8.5, S has the $\langle\kappa, \lambda\rangle$-interpolation property for all infinite cardinals κ and λ.

PROBLEM 2. *Let S be a distributive $\langle\vee, 0\rangle$-semilattice. If, for every truncated square $\mathcal{D}$ of lattices, every homomorphism from* $\mathrm{Con_c}\, \mathcal{D}$ *to S can be factored through a partial lattice, is S conditionally co-Brouwerian?*

The proof of Corollary 9.5 shows that if, for every truncated square $\mathcal{D}$ of lattices, every homomorphism from $\mathrm{Con_c}\, \mathcal{D}$ to S can be factored through a *lattice*, then S has the $\langle\kappa, \lambda\rangle$-interpolation property for all infinite cardinals κ and λ.

On the positive side, we formulate the following question, related to Theorem 6:

PROBLEM 3. *Let κ be an infinite cardinal and let S be a conditionally κ-co-Brouwerian $\langle\vee, 0\rangle$-semilattice. Does there exist a relatively complemented modular lattice L with zero such that* $\mathrm{Con_c}\, L \cong S$*?*

By Bergman's Theorem and the main result of [14], the answer to Problem 3 is known to be affirmative for $\kappa = \aleph_0$ and for $\kappa = \aleph_1$.

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