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CONVOLUTION OPERATORS WITH ANISOTROPICALLY HOMOGENEOUS MEASURES ON $\mathbb{R}^{2n}$ WITH n-DIMENSIONAL SUPPORT

BY

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Abstract. Let $\alpha_i, \beta_i > 0$, $1 \le i \le n$, and for $t > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $t \bullet x = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n)$, $t \circ x = (t^{\beta_1} x_1, \ldots, t^{\beta_n} x_n)$ and $\|x\| = \sum_{i=1}^n |x_i|^{1/\alpha_i}$. Let $\varphi_1, \ldots, \varphi_n$ be real functions in $C^\infty(\mathbb{R}^n - \{0\})$ such that $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies $\varphi(t \bullet x) = t \circ \varphi(x)$. Let $\gamma > 0$ and let μ be the Borel measure on $\mathbb{R}^{2n}$ given by

$$\mu(E) = \int_{\mathbb{R}^n} \chi_E(x, \varphi(x)) \|x\|^{\gamma - \alpha}\, dx,$$

where $\alpha = \sum_{i=1}^n \alpha_i$ and dx denotes the Lebesgue measure on $\mathbb{R}^n$. Let $T_\mu f = \mu * f$ and let $\|T_\mu\|_{p,q}$ be the operator norm of T_μ from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$, where the L^p spaces are taken with respect to the Lebesgue measure. The type set E_μ is defined by

$$E_\mu = \{(1/p, 1/q) : \|T_\mu\|_{p,q} < \infty,\ 1 \le p, q \le \infty\}.$$

In the case $\alpha_i \ne \beta_k$ for $1 \le i, k \le n$ we characterize the type set under certain additional hypotheses on φ.

1. Introduction. Let $\alpha_i, \beta_i > 0$, $1 \le i \le n$, and for $t > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$t \bullet x = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n), \qquad t \circ x = (t^{\beta_1} x_1, \ldots, t^{\beta_n} x_n)$$

and let $\|x\| = \sum_{i=1}^n |x_i|^{1/\alpha_i}$ be a homogeneous norm associated to the first group of dilations. Let $\varphi_1, \ldots, \varphi_n$ be real functions in $C^\infty(\mathbb{R}^n - \{0\})$ such that $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ is a homogeneous function with respect to these groups of dilations, i.e. $\varphi(t \bullet x) = t \circ \varphi(x)$. Let $\gamma > 0$ and let μ be the Borel measure on $\mathbb{R}^{2n}$ given by

$$\mu(E) = \int_{\mathbb{R}^n} \chi_E(x, \varphi(x)) \|x\|^{\gamma - \alpha}\, dx,$$

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where $\alpha = \sum_{i=1}^{n} \alpha_i$ and dx denotes the Lebesgue measure on $\mathbb{R}^n$. Let T_μ be the convolution operator defined, for $f \in S(\mathbb{R}^{2n})$, by $T_\mu f(x) = (\mu * f)(x)$ and let $\|T_\mu\|_{p,q}$ be the operator norm of T_μ from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$, where the L^p spaces are taken with respect to the Lebesgue measure. The *type set* E_μ is defined by

$$E_\mu = \{(1/p, 1/q) : \|T_\mu\|_{p,q} < \infty,\ 1 \le p, q \le \infty\}.$$

A very interesting survey of results concerning the type set for convolution operators with singular measures can be found in [R]. The type set associated with fractional measures on $\mathbb{R}^2$ supported on the graph of the parabola (t, t^2) has been characterized by M. Christ in [C], using a Littlewood–Paley decomposition of the operator. Also, convolution operators supported on surfaces of half the ambient dimension have been studied by S. W. Drury and K. Guo in [D-G], covering a wide amount of cases. As there, if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable function, we say that $x \in \mathbb{R}^n$ is an *elliptic point* for φ if there exists $\lambda = \lambda_x > 0$ such that $|\det(\varphi''(x)h)| \ge \lambda |h|^n$ for all $h \in \mathbb{R}^n$ ([D-G], p. 154).

When we deal with isotropic dilations, in [F-G-U] we have already obtained a complete description of E_μ in the case that every $x \neq 0$ is an elliptic point for φ. In this paper we obtain an explicit description of E_μ, for an anisotropically homogeneous and smooth φ, under the following assumptions:

(H1) The dilations satisfy $\alpha_i \neq \beta_k$ for $1 \le i, k \le n$.

(H2) The first differential $\varphi'(x)$ is invertible for all $x \in \mathbb{R}^n - \{0\}$.

(H3) Every $x \neq 0$ is an elliptic point for φ.

For some families of dilations, it is enough to require hypothesis (H3), since (H2) is its consequence. We will adapt M. Christ's arguments ([C]) to our setting, using some results obtained by S. W. Drury and K. Guo in [D-G]. Throughout the paper we will assume that all the hypotheses concerning φ and α_i, β_k, $1 \le i, k \le n$, stated in this introduction hold. Also c will denote a positive constant not necessarily the same at each occurrence.

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2. Preliminaries. The Riesz–Thorin theorem implies that E_μ is a convex set. On the other hand, it is well known that E_μ lies below the principal diagonal $1/q = 1/p$. Also, a result of Oberlin (see e.g. [O], Th. 1) says that

$$E_\mu \subset \{(1/p, 1/q) : 1/q \ge 2/p - 1\}. \tag{2.1}$$

Since the adjoint T_μ^* is a convolution operator with a measure of the same kind, we also have

$$E_\mu \subset \{(1/p, 1/q) : 1/q \geq 1/(2p)\}. \tag{2.2}$$

Let η be a function in $C_c^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\eta) \subset \{x \in \mathbb{R}^n : 1/4 \leq \|x\| \leq 2\},$$

$0 \leq \eta \leq 1$ and $\sum_{j\in\mathbb{Z}} \eta(2^j \bullet x) = 1$ if $x \neq 0$. For $j \in \mathbb{Z}$, let μ_j be the Borel measure on $\mathbb{R}^{2n}$ defined by

$$\mu_j(E) = \int\limits_{\mathbb{R}^n} \chi_E(x, \varphi(x))\eta(2^j \bullet x)\|x\|^{\gamma-\alpha}\, dx$$

and let T_{μ_j} be the associated convolution operator. So $T_\mu = \sum_{j\in\mathbb{Z}} T_{\mu_j}$. For $t > 0$ and $(x, y) \in \mathbb{R}^{2n}$ we set

$$t \diamond (x, y) = (t \bullet x, t \circ y)$$

and for $f : \mathbb{R}^{2n} \to \mathbb{C}$, we define $(t\diamond f)(x, y) = f(t\diamond(x, y))$. So $\|t\diamond f\|_\infty = \|f\|_\infty$ and $\|t \diamond f\|_q = t^{-(\alpha+\beta)/q}\|f\|_q$, $1 \leq q < \infty$, where $\beta = \sum_{k=1}^n \beta_k$. A standard homogeneity argument gives

Lemma 2.1. *Let $1 \leq p, q \leq \infty$. Then*

$$\|T_{\mu_j}\|_{p,q} = 2^{(-\gamma-(\alpha+\beta)/q+(\alpha+\beta)/p)j}\|T_{\mu_0}\|_{p,q}$$

for all $j \in \mathbb{Z}$. Moreover, if T_μ is bounded from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$ then $1/q = 1/p - \gamma/(\alpha + \beta)$.

Proof. For $(x, y) \in \mathbb{R}^{2n}$ a change of variable gives

$$\begin{aligned}
&T_{\mu_0}(2^{-j} \diamond f)(x, y)\\
&\quad= \int\limits_{\mathbb{R}^n} f(2^{-j} \bullet x - 2^{-j} \bullet w, 2^{-j} \circ y - \varphi(2^{-j} \bullet w))\eta(w)\|w\|^{\gamma-\alpha}\, dw\\
&\quad= 2^{j\alpha} \int\limits_{\mathbb{R}^n} f(2^{-j} \bullet x - z, 2^{-j} \circ y - \varphi(z))\eta(2^j \bullet z)\|2^j \bullet z\|^{\gamma-\alpha}\, dz\\
&\quad= 2^{j\gamma}(2^{-j} \diamond T_{\mu_j} f)(x, y).
\end{aligned}$$

So

$$\|T_{\mu_j}\|_{p,q} = 2^{(-\gamma-(\alpha+\beta)/q+(\alpha+\beta)/p)j}\|T_{\mu_0}\|_{p,q}$$

and the first assertion of the lemma follows. On the other hand, if T_μ is bounded then $\sup_{j\in\mathbb{Z}} \|T_{\mu_j}\|_{p,q} < \infty$ and so $-\gamma - (\alpha + \beta)/q + (\alpha + \beta)/p = 0$. ∎

Remark 2.2. Let D be the intersection, in the $(1/p, 1/q)$ plane, of the lines $1/q = 2/p - 1$, $1/q = 1/p - \gamma/(\alpha + \beta)$, and let D' be its reflection in the non-principal diagonal. So

$$D = \left(1 - \frac{\gamma}{\alpha+\beta}, 1 - \frac{2\gamma}{\alpha+\beta}\right) \quad\text{and}\quad D' = \left(\frac{2\gamma}{\alpha+\beta}, \frac{\gamma}{\alpha+\beta}\right).$$

Then (2.1), (2.2) and Lemma 2.1 imply that E_μ is the empty set for $\gamma > (\alpha+\beta)/3$, and, for $\gamma \le (\alpha+\beta)/3$, E_μ is contained in the closed segment with endpoints D and D'. Let ν_0 be the Borel measure given by $\nu_0(E) = \int \chi_E(w,\varphi(w))\eta(w)\,dw$. Then Theorem 3 of [D-G] and a compactness argument imply that $(2/3,1/3) \in E_{\nu_0}$. Now $T_{\mu_0} f \le cT_{\nu_0} f$ for $f \ge 0$, thus $(2/3,1/3) \in E_{\mu_0}$. Since $(1,1) \in E_{\mu_0}$, the Riesz–Thorin theorem implies that if $\gamma \le (\alpha+\beta)/3$ then D belongs to E_{μ_0}. Moreover, for these γ, if p_D, q_D are given by $D = (1/p_D, 1/q_D)$, Lemma 2.1 says that there exists c independent of j such that

$$\|T_{\mu_j}\|_{p_D,q_D} \le c \tag{2.3}$$

for all $j \in \mathbb{Z}$.

3. L^p-L^q estimates. We modify, to our present setting, Christ's arguments developed in [C], involving a Littlewood–Paley decomposition of the operator. A similar decomposition, in a different setting, can be found in [Se].

Consider the Fourier transform $\widehat{\mu}_0$. For $\xi = (\xi_1, \ldots, \xi_{2n}) \in \mathbb{R}^{2n}$ we put $\xi' = (\xi_1, \ldots, \xi_n)$, $\xi'' = (\xi_{n+1}, \ldots, \xi_{2n})$. Then

$$\widehat{\mu}_0(\xi) = \int\limits_{\mathbb{R}^n} e^{-i\langle \xi', w\rangle - i\langle \xi'', \varphi(w)\rangle} \eta(w)\|w\|^{\gamma-\alpha}\,dw.$$

For a fixed ξ, let $\Phi(w) = \langle \xi', w\rangle + \langle \xi'', \varphi(w)\rangle$, $w \in \mathbb{R}^n$. Suppose that Φ has a critical point w_0 belonging to the support of η. Then

$$\xi_j + \sum_{k=1}^{n} \xi_{n+k} \frac{\partial \varphi_k}{\partial w_j}(w_0) = 0 \quad \text{for } j = 1, \ldots, n.$$

Now, the maps $w \mapsto \varphi'(w)$ and (from (H2)) $w \mapsto [\varphi'(w)]^{-1}$ are continuous on $\mathbb{R}^n - \{0\}$, hence there exist two positive constants c_1, c_2 (independent of ξ) such that ξ belongs to the cone

$$\Gamma_0 = \{\xi \in \mathbb{R}^{2n} : c_1|\xi''| < |\xi'| < c_2|\xi''|\}.$$

For $1 \le i,k \le n$, let

$$\Gamma_0^{i,k} = \{\xi \in \Gamma_0 : |\xi'| < 2\sqrt{n}\,|\xi_i| \text{ and } |\xi''| < 2\sqrt{n}\,|\xi_{n+k}|\},$$
$$C_0^{i,k} = \{\xi \in \Gamma_0 : |\xi'| < 4\sqrt{n}\,|\xi_i| \text{ and } |\xi''| < 4\sqrt{n}\,|\xi_{n+k}|\}.$$

LEMMA 3.1. *If $\alpha_i \ne \beta_k$ for $1 \le i,k \le n$, then*

$$\{j \in \mathbb{Z} : (2^j \diamond C_0^{i,k}) \cap C_0^{i,k} \ne \emptyset\}$$

is a finite set.

Proof. Let $j \in \mathbb{Z}$ and $\xi \in C_0^{i,k}$ be such that $2^j \diamond \xi \in C_0^{i,k}$. Then $|2^j \bullet \xi'| < 4\sqrt{n}\, 2^{j\alpha_i}|\xi_i|$ and $|2^j \circ \xi''| < 4\sqrt{n}\, 2^{j\beta_k}|\xi_{n+k}|$. Since $2^j \diamond \xi \in \Gamma_0$, we have $c_1|2^j \circ \xi''| < |2^j \bullet \xi'| < c_2|2^j \circ \xi''|$. Now,

$$\begin{aligned} c_1 2^{j\beta_k}|\xi_{n+k}| \le c_1|2^j \circ \xi''| < |2^j \bullet \xi'| < 4\sqrt{n}\, 2^{j\alpha_i}|\xi_i| \\ < 4\sqrt{n}\, 2^{j\alpha_i} c_2|\xi''| < 16n2^{j\alpha_i}c_2|\xi_{n+k}|. \end{aligned}$$

So $2^{j(\beta_k-\alpha_i)} < 16nc_2/c_1$. In a similar way we obtain $c_1/(16nc_2) < 2^{j(\beta_k-\alpha_i)}$. Since $\alpha_i \ne \beta_k$ for $1 \le i,k \le n$, the lemma follows. ∎

For $1 \le i,k \le n$, let $m_0^{i,k}$ be a function belonging to $C^\infty(\mathbb{R}^{2n} - \{0\})$ homogeneous of degree zero with respect to the Euclidean dilations on $\mathbb{R}^{2n}$ such that $m_0^{i,k} \equiv 1$ on $\Gamma_0^{i,k}$, $0 \le m_0^{i,k} \le 1$, $\operatorname{supp}(m_0^{i,k}) \subset C_0^{i,k}$, and let $m_j^{i,k}(y) = m_0^{i,k}(2^{-j} \diamond y)$. Let $Q_j^{i,k}$ be the operator with multiplier $m_j^{i,k}$. Let $h \in C_c^\infty(\mathbb{R}^{2n})$, $0 \le h \le 1$, be identically one in a neighborhood of the origin. Taking account of Proposition 4 in [S], p. 341, from the above observation about the critical points of Φ, we note that

$$\widehat{\mu}_0(1-h) \prod_{1\le i,k\le n} (1 - m_0^{i,k}) \in S(\mathbb{R}^{2n}). \tag{3.1}$$

Let $h_j(y) = h(2^{-j} \diamond y)$ and let P_j be the Fourier multiplier operator with symbol h_j.

Remark 3.2. Lemma 3.1 implies that there exists $N > 0$ such that for all $y \in \mathbb{R}^{2n} - \{0\}$ the set $\{j \in \mathbb{Z} : 2^{-j} \diamond y \in C_0^{i,k}\}$ has at most N elements, so $m^{i,k}(y) := \sum_{j\in\mathbb{Z}} \varepsilon_j m_j^{i,k}(y)$, $\varepsilon_j = \pm 1$, is a well defined, $C^\infty(\mathbb{R}^{2n} - \{0\})$ and homogeneous function of degree zero. Moreover, for each $s = (s_1, \ldots, s_{2n})$, the function $\sum_{j\in\mathbb{Z}} |(\partial/\partial y)^s m_j^{i,k}(y)|$ is homogeneous of degree $-(s_1 + \ldots + s_{2n})$, so Theorem 3 in [St], p. 96, applies, showing that $m^{i,k}$ is an L^p multiplier, for $1 < p < \infty$, and that the norm of the associated operator has a bound independent of the choices of $\{\varepsilon_j\}_{j\in\mathbb{Z}}$. ∎

Theorem 3.3. *If $\gamma \le (\alpha+\beta)/3$ then E_μ is the closed segment with endpoints D and D'.*

Proof. By Remark 2.2, it is enough to prove that D and D' belong to E_μ. Let $\{Q_j^r\}_{1\le r\le n^2}$ be an arrangement of the set $\{Q_j^{i,k}\}_{1\le i,k\le n}$. For $J \in \mathbb{N}$, we write

$$\begin{aligned} \sum_{|j|<J} T_{\mu_j} &= \sum_{|j|<J} T_{\mu_j} P_j + \sum_{|j|<J} T_{\mu_j}(I - P_j) \\ &= \sum_{|j|<J} T_{\mu_j} P_j + \sum_{|j|<J} T_{\mu_j}(I - P_j) Q_j^1 + \sum_{|j|<J} T_{\mu_j}(I - P_j)(I - Q_j^1) \end{aligned}$$

$$= \sum_{|j|<J} T_{\mu_j} P_j + \sum_{|j|<J} T_{\mu_j}(I-P_j)Q_j^1 + \sum_{|j|<J} T_{\mu_j}(I-P_j)(I-Q_j^1)Q_j^2$$

$$+ \sum_{|j|<J} T_{\mu_j}(I-P_j)(I-Q_j^1)(I-Q_j^2) = \dots$$

$$= \sum_{|j|<J} T_{\mu_j} P_j + \sum_{|j|<J} T_{\mu_j}(I-P_j)Q_j^1$$

$$+ \sum_{1\le l\le n^2-1} \sum_{|j|<J} T_{\mu_j}(I-P_j) \prod_{1\le r\le l} (I-Q_j^r)Q_j^{l+1}$$

$$+ \sum_{|j|<J} T_{\mu_j}(I-P_j) \prod_{1\le r\le n^2} (I-Q_j^r).$$

The kernel K_J of the convolution operator

$$\sum_{|j|<J} T_{\mu_j}(I-P_j) \prod_{1\le r\le n^2} (I-Q_j^r)$$

satisfies

$$K_J(-\xi) = \sum_{|j|<J} 2^{j(\alpha+\beta-\gamma)} \widehat{g}(2^j \diamond \xi)$$

with $g = \widehat{\mu}_0(1-h)\prod_{1\le i,k\le n}(1-m_0^{i,k})$. So, by using (3.1), a standard homogeneity argument shows that, for all $J \in \mathbb{N}$,

$$|K_J(\xi)| \le \Big(\sum_{i=1}^{n} |\xi_i|^{1/\alpha_i} + \sum_{k=1}^{n} |\xi_{n+k}|^{1/\beta_k}\Big)^{-(\alpha+\beta-\gamma)}$$

and so they belong to the weak L^{p_D} space, with weak constant uniformly bounded in J. Also, a similar argument gives the same fact for the kernels of $\sum_{|j|<J} T_{\mu_j} P_j$. Then the weak Young inequality implies that there exists $c > 0$ independent of J such that

$$\Big\| \sum_{|j|<J} T_{\mu_j}(I-P_j) \prod_{1\le r\le n^2} (I-Q_j^r)\Big\|_{p_D,q_D} \le c$$

and

$$\Big\| \sum_{|j|<J} T_{\mu_j} P_j\Big\|_{p_D,q_D} \le c.$$

Now Remark 3.2 allows us to use Littlewood–Paley inequalities. As in [C] we obtain, for $1 \le l \le n^2-1$,

$$\Big\| \sum_{|j|<J} T_{\mu_j}(I-P_j) \prod_{1\le r\le l} (I-Q_j^r)Q_j^{l+1} f\Big\|_q \le c\|\{T_{\mu_j}\}\|_{p,q,2}\|\{f_j\}\|_{L^p(l^2)}$$

where $f_j = \prod_{1\le r\le l}(I-Q_j^r)Q_j^{l+1}f$. Since the assertions of Remark 3.2 also hold when we replace $m_j^{i,k}$ by a finite product of the form $m_j^{r_1}\dots m_j^{r_s}$, $1\le r_1,\dots,r_s\le n^2$, we get $\|\{f_j\}\|_{L^p(l^2)}\le c\|f\|_p$.

A similar estimate holds for $\|\sum_{|j|<J}T_{\mu_j}(I-P_j)Q_j^1 f\|_q$.

Now, taking account of (2.3) we deduce (as in [C]) that there exist $0<\theta<1$ and $c>0$, independent of J, such that

$$\Big\|\sum_{|j|<J}T_{\mu_j}\Big\|_{p_D,q_D}<c\Big(1+\Big\|\sum_{|j|<J}T_{\mu_j}\Big\|_{p_D,q_D}^{\theta}\Big),$$

and so $\|\sum_{|j|<J}T_{\mu_j}\|_{p_D,q_D}$ is bounded independently of J. From Fatou's lemma, it follows that $D\in E_\mu$. Since T_μ^* is a convolution operator with a measure of the same kind, a duality argument shows that $D'\in E_\mu$. ∎

We now consider a local version of the problem, that is, we study the type set corresponding to the convolution operator T_σ with the Borel measure given by

$$\sigma(E)=\int\limits_{\|x\|\le 1}\chi_E(x,\varphi(x))\|x\|^{\gamma-\alpha}\,dx$$

with $\gamma>0$.

THEOREM 3.4. *If $\gamma>(\alpha+\beta)/3$, then E_σ is the closed triangular region with vertices $(2/3,1/3)$, $(0,0)$ and $(1,1)$. If $\gamma\le(\alpha+\beta)/3$ then E_σ is the closed polygonal region with vertices D, D', $(0,0)$ and $(1,1)$.*

Proof. We have $E_\mu\subset E_\sigma$. Since E_σ is a convex set and since σ is a finite measure, $(1,1)$ and $(0,0)$ belong to E_σ. On the other hand, the constraints (2.1) and (2.2) hold for E_σ. Moreover, Lemma 2.1 implies that if $(1/p,1/q)\in E_\sigma$, then $1/q\ge 1/p-\gamma/(\alpha+\beta)$. Thus the case $\gamma\le(\alpha+\beta)/3$ follows from Theorem 3.3.

If $\gamma>(\alpha+\beta)/3$, then $(2/3,1/3)$ lies above the line $1/q=1/p-\gamma/(\alpha+\beta)$ and we have noted in Remark 2.2 that $(2/3,1/3)$ belongs to E_{μ_0}, so Lemma 2.1 implies that $\|\sum_{j\ge0}T_{\mu_j}\|_{3/2,3}=c\|T_{\mu_0}\|_{3/2,3}$. Now, for $f\ge0$, $T_\sigma f\le\sum_{j\ge0}T_{\mu_j}f$ and the assertion follows. ∎

REMARK 3.5. If either $\alpha_1=\dots=\alpha_n$ or $\beta_1=\dots=\beta_n$, then (H3) implies (H2). Indeed, for $1\le i,k\le n$,

$$\frac{\partial\varphi_k}{\partial x_i}(t\bullet x)=t^{\beta_k-\alpha i}\,\frac{\partial\varphi_k}{\partial x_i}(x).$$

Taking the derivative with respect to t, at $t=1$, we obtain

$$(\beta_k-\alpha_i)\,\frac{\partial\varphi_k}{\partial x_i}(x)=\sum_{l=1}^{n}x_l\alpha_l\,\frac{\partial^2\varphi_k}{\partial x_i\partial x_l}(x).$$

Thus, the matrix, with respect to the canonical basis of $\mathbb{R}^n$, of the linear operator $\varphi''(x)(\alpha_1 x_1, \ldots, \alpha_n x_n)$ is given by

$$\begin{bmatrix} (\beta_1-\alpha_1)\frac{\partial\varphi_1}{\partial x_1}(x) & \ldots & (\beta_n-\alpha_1)\frac{\partial\varphi_n}{\partial x_1}(x) \\ \vdots & & \vdots \\ (\beta_1-\alpha_n)\frac{\partial\varphi_1}{\partial x_n}(x) & \ldots & (\beta_n-\alpha_n)\frac{\partial\varphi_n}{\partial x_n}(x) \end{bmatrix} \tag{3.2}$$

So if either $\alpha_1 = \ldots = \alpha_n$ or $\beta_1 = \ldots = \beta_n$ then

$$c\det(\varphi'(x)) = \det(\varphi''(x)(\alpha_1 x_1, \ldots, \alpha_n x_n)).$$

Hence (H3) implies (H2).

On the other hand, if either $\alpha_1 = \ldots = \alpha_n$ or $\beta_1 = \ldots = \beta_n$ and (H1) fails, then there does not exist a homogeneous function φ that satisfies (H3). Indeed, in this case, (3.2) implies that

$$\det(\varphi''(x)(\alpha_1 x_1, \ldots, \alpha_n x_n)) = 0$$

for every $x \in \mathbb{R}^n$. Then no $x \in \mathbb{R}^n$ is an elliptic point for φ.

EXAMPLES. Let us show two examples of functions φ that satisfy the hypothesis of Theorems 3.4 and 3.5.

1) Let

$$\varphi(x_1, x_2) = (x_1^4 - 6x_1^2x_2^2 + x_2^4, (4x_1^3x_2 - 4x_1x_2^3)\sqrt{x_1^2+x_2^2}).$$

In this case $\varphi(t \bullet x) = t \circ \varphi(x)$ with $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = 4$, $\beta_2 = 5$. Taking account of Remark 3.5, we only need to check (H3). An explicit computation shows that the discriminant of the quadratic form

$$(h_1, h_2) \mapsto \det(\varphi''(x_1, x_2)(h_1, h_2))$$

is negative for $(x_1, x_2) \neq (0, 0)$, so (H3) holds.

2) Let

$$\varphi(x_1, x_2) = (x_1x_2, x_2^2 - x_1^2\sqrt{x_2^2 + x_1^4}).$$

In this case $\varphi(t \bullet x) = t \circ \varphi(x)$ with $\alpha_1 = 1$, $\alpha_2 = 2$ and $\beta_1 = 3$, $\beta_2 = 4$. A computation shows that

$$\det(\varphi'(x_1, x_2)) = \frac{2\sqrt{x_2^2+x_1^4}\,x_2^2 + x_1^2x_2^2 + 4x_1^6}{\sqrt{x_2^2+x_1^4}} \neq 0, \quad (x_1, x_2) \neq (0,0)$$

and so (H2) holds. On the other hand, the discriminant of the quadratic form

$$(h_1, h_2) \mapsto \det(\varphi''(x_1, x_2)(h_1, h_2))$$

is

$$\frac{-8\sqrt{x_2^2+x_1^4}\,x_2^2 - 8\sqrt{x_2^2+x_1^4}\,x_1^4 + 4x_1^6}{(x_2^2+x_1^4)^3} < 0, \quad (x_1, x_2) \neq (0,0),$$

so that (H3) holds.

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