CONVOLUTION OPERATORS WITH
ANISOTROPICALLY HOMOGENEOUS MEASURES ON $\mathbb{R}^{2n}$
WITH $n$-DIMENSIONAL SUPPORT

BY

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Abstract. Let $\alpha_i, \beta_i > 0$, $1 \leq i \leq n$, and for $t > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $t \cdot x = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n)$, $t \circ x = (t^{\beta_1} x_1, \ldots, t^{\beta_n} x_n)$ and $\|x\| = \sum_{i=1}^{n} |x_i|^{1/\alpha_i}$. Let $\varphi_1, \ldots, \varphi_n$ be real functions in $C^\infty(\mathbb{R}^n - \{0\})$ such that $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies $\varphi(t \cdot x) = t \circ \varphi(x)$. Let $\gamma > 0$ and let $\mu$ be the Borel measure on $\mathbb{R}^{2n}$ given by

$$\mu(E) = \int_{\mathbb{R}^n} \chi_E(x, \varphi(x)) \|x\|^{\gamma - \alpha} dx,$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$ and $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$. Let $T_\mu f = \mu * f$ and let $\|T_\mu\|_{p,q}$ be the operator norm of $T_\mu$ from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$, where the $L^p$ spaces are taken with respect to the Lebesgue measure. The type set $E_\mu$ is defined by

$$E_\mu = \{(1/p, 1/q) : \|T_\mu\|_{p,q} < \infty, 1 \leq p, q \leq \infty\}.$$

In the case $\alpha_i \neq \beta_k$ for $1 \leq i, k \leq n$ we characterize the type set under certain additional hypotheses on $\varphi$.

1. Introduction. Let $\alpha_i, \beta_i > 0$, $1 \leq i \leq n$, and for $t > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$t \cdot x = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n), \quad t \circ x = (t^{\beta_1} x_1, \ldots, t^{\beta_n} x_n)$$

and let $\|x\| = \sum_{i=1}^{n} |x_i|^{1/\alpha_i}$ be a homogeneous norm associated to the first group of dilations. Let $\varphi_1, \ldots, \varphi_n$ be real functions in $C^\infty(\mathbb{R}^n - \{0\})$ such that $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ is a homogeneous function with respect to these groups of dilations, i.e. $\varphi(t \cdot x) = t \circ \varphi(x)$. Let $\gamma > 0$ and let $\mu$ be the Borel measure on $\mathbb{R}^{2n}$ given by

$$\mu(E) = \int_{\mathbb{R}^n} \chi_E(x, \varphi(x)) \|x\|^{\gamma - \alpha} dx,$$
where $\alpha = \sum_{i=1}^{n} \alpha_i$ and $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$. Let $T_\mu$ be the convolution operator defined, for $f \in S(\mathbb{R}^{2n})$, by $T_\mu f(x) = (\mu * f)(x)$ and let $\|T_\mu\|_{p,q}$ be the operator norm of $T_\mu$ from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$, where the $L^p$ spaces are taken with respect to the Lebesgue measure. The type set $E_\mu$ is defined by

$$E_\mu = \{(1/p, 1/q) : \|T_\mu\|_{p,q} < \infty, 1 \leq p, q \leq \infty\}.$$  

A very interesting survey of results concerning the type set for convolution operators with singular measures can be found in [R]. The type set associated with fractional measures on $\mathbb{R}^2$ supported on the graph of the parabola $(t, t^2)$ has been characterized by M. Christ in [C], using a Littlewood–Paley decomposition of the operator. Also, convolution operators supported on surfaces of half the ambient dimension have been studied by S. W. Drury and K. Guo in [D-G], covering a wide amount of cases. As there, if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable function, we say that $x \in \mathbb{R}^n$ is an elliptic point for $\varphi$ if there exists $\lambda = \lambda_x > 0$ such that $|\det(\varphi''(x)h)| \geq \lambda|h|^n$ for all $h \in \mathbb{R}^n$ ([D-G], p. 154).

When we deal with isotropic dilations, in [F-G-U] we have already obtained a complete description of $E_\mu$ in the case that every $x \neq 0$ is an elliptic point for $\varphi$. In this paper we obtain an explicit description of $E_\mu$, for an anisotropically homogeneous and smooth $\varphi$, under the following assumptions:

(H1) The dilations satisfy $\alpha_i \neq \beta_k$ for $1 \leq i, k \leq n$.

(H2) The first differential $\varphi'(x)$ is invertible for all $x \in \mathbb{R}^n - \{0\}$.

(H3) Every $x \neq 0$ is an elliptic point for $\varphi$.

For some families of dilations, it is enough to require hypothesis (H3), since (H2) is its consequence. We will adapt M. Christ’s arguments ([C]) to our setting, using some results obtained by S. W. Drury and K. Guo in [D-G]. Throughout the paper we will assume that all the hypotheses concerning $\varphi$ and $\alpha_i, \beta_k$, $1 \leq i, k \leq n$, stated in this introduction hold. Also $c$ will denote a positive constant not necessarily the same at each occurrence.

Acknowledgments. We wish to express our thanks to Prof. Fulvio Ricci for his useful suggestions.

2. Preliminaries. The Riesz–Thorin theorem implies that $E_\mu$ is a convex set. On the other hand, it is well known that $E_\mu$ lies below the principal diagonal $1/q = 1/p$. Also, a result of Oberlin (see e.g. [O], Th. 1) says that

$$(2.1) \quad E_\mu \subset \{(1/p, 1/q) : 1/q \geq 2/p - 1\}.$$
Since the adjoint $T^*_\mu$ is a convolution operator with a measure of the same kind, we also have

\[(2.2) \quad E_\mu \subset \{(1/p, 1/q) : 1/q \geq 1/(2p)\}.
\]

Let $\eta$ be a function in $C^\infty_c(\mathbb{R}^n)$ such that

$$\text{supp}(\eta) \subset \{x \in \mathbb{R}^n : 1/4 \leq \|x\| \leq 2\},$$

and let $\mu_j$ be the Borel measure on $\mathbb{R}^{2n}$ defined by

$$\mu_j(E) = \int_{\mathbb{R}^n} \chi_E(x, \varphi(x))\eta(2^j \bullet x)\|x\|^{\gamma-\alpha} \, dx$$

and let $T_{\mu_j}$ be the associated convolution operator. So $T_\mu = \sum_{j \in \mathbb{Z}} T_{\mu_j}$. For $t > 0$ and $(x, y) \in \mathbb{R}^{2n}$ we set

$$t \circ (x, y) = (t \cdot x, t \circ y)$$

for $f : \mathbb{R}^{2n} \to \mathbb{C}$, we define $(t \circ f)(x, y) = f(t \circ (x, y))$. So $\|t \circ f\|_\infty = \|f\|_\infty$ and $\|t \circ f\|_q = t^{-(\alpha+\beta)/q} \|f\|_q$, $1 \leq q < \infty$, where $\beta = \sum_{k=1}^n \beta_k$. A standard homogeneity argument gives

**Lemma 2.1.** Let $1 \leq p, q \leq \infty$. Then

$$\|T_{\mu_j}\|_{p, q} = 2^{((-\gamma - (\alpha + \beta)/q + (\alpha + \beta)/p)j) \|T_{\mu_0}\|_{p, q}}$$

for all $j \in \mathbb{Z}$. Moreover, if $T_\mu$ is bounded from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$ then $1/q = 1/p - \gamma/(\alpha + \beta)$.

**Proof.** For $(x, y) \in \mathbb{R}^{2n}$ a change of variable gives

$$T_{\mu_0}(2^{-j} \circ f)(x, y) = \int_{\mathbb{R}^n} f(2^{-j} \bullet x - 2^{-j} \bullet w, 2^{-j} \circ y - \varphi(2^{-j} \bullet w))\eta(w)\|w\|^{\gamma-\alpha} \, dw$$

$$= 2^{j\alpha} \int_{\mathbb{R}^n} f(2^{-j} \bullet x - z, 2^{-j} \circ y - \varphi(z))\eta(2^{-j} \bullet z)\|2^{-j} \bullet z\|^{\gamma-\alpha} \, dz$$

$$= 2^{j\gamma} (2^{-j} \circ T_{\mu_j} f)(x, y).$$

So

$$\|T_{\mu_j}\|_{p, q} = 2^{((-\gamma - (\alpha + \beta)/q + (\alpha + \beta)/p)j) \|T_{\mu_0}\|_{p, q}}$$

and the first assertion of the lemma follows. On the other hand, if $T_\mu$ is bounded then $\sup_{j \in \mathbb{Z}} \|T_{\mu_j}\|_{p, q} < \infty$ and so $-\gamma - (\alpha + \beta)/q + (\alpha + \beta)/p = 0$. □

**Remark 2.2.** Let $D$ be the intersection, in the $(1/p, 1/q)$ plane, of the lines $1/q = 2/p - 1$, $1/q = 1/p - \gamma/(\alpha + \beta)$, and let $D'$ be its reflection in the non-principal diagonal. So

$$D = \left(1 - \frac{\gamma}{\alpha + \beta}, 1 - \frac{2\gamma}{\alpha + \beta}\right) \quad \text{and} \quad D' = \left(\frac{2\gamma}{\alpha + \beta'}, \frac{\gamma}{\alpha + \beta'}\right).$$
Then (2.1), (2.2) and Lemma 2.1 imply that \( E_\mu \) is the empty set for \( \gamma > (\alpha + \beta)/3 \), and, for \( \gamma \leq (\alpha + \beta)/3 \), \( E_\mu \) is contained in the closed segment with endpoints \( D \) and \( D' \). Let \( \nu_0 \) be the Borel measure given by \( \nu_0(E) = \int E \chi_E(w, \varphi(w)) \eta(w) \, dw \). Then Theorem 3 of [D-G] and a compactness argument imply that \( (2/3, 1/3) \in E_{\nu_0} \). Now \( T_{\mu_0} f \leq c T_{\nu_0} f \) for \( f \geq 0 \), thus \( (2/3, 1/3) \in E_{\mu_0} \). Since \((1, 1) \in E_{\mu_0} \), the Riesz–Thorin theorem implies that if \( \gamma \leq (\alpha + \beta)/3 \) then \( D \) belongs to \( E_{\mu_0} \). Moreover, for these \( \gamma \), if \( p_D, q_D \) are given by \( D = (1/p_D, 1/q_D) \), Lemma 2.1 says that there exists \( c \) independent of \( j \) such that

\[
\|T_{\mu_j}\|_{p_D, q_D} \leq c
\]

for all \( j \in \mathbb{Z} \).

3. **\( L^p-L^q \) estimates.** We modify, to our present setting, Christ’s arguments developed in [C], involving a Littlewood–Paley decomposition of the operator. A similar decomposition, in a different setting, can be found in [Se].

Consider the Fourier transform \( \hat{\mu}_0 \). For \( \xi = (\xi_1, \ldots, \xi_{2n}) \in \mathbb{R}^{2n} \) we put \( \xi' = (\xi_1, \ldots, \xi_n) \), \( \xi'' = (\xi_{n+1}, \ldots, \xi_{2n}) \). Then

\[
\hat{\mu}_0(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi', w) - i(\xi'', \varphi(w))} \eta(w) \|w\|^{\gamma - \alpha} \, dw.
\]

For a fixed \( \xi \), let \( \Phi(w) = \langle \xi', w \rangle + \langle \xi'', \varphi(w) \rangle \), \( w \in \mathbb{R}^n \). Suppose that \( \Phi \) has a critical point \( w_0 \) belonging to the support of \( \eta \). Then

\[
\xi_j + \sum_{k=1}^{n} \xi_{n+k} \frac{\partial \varphi_k}{\partial w_j}(w_0) = 0 \quad \text{for } j = 1, \ldots, n.
\]

Now, the maps \( w \mapsto \varphi'(w) \) and (from (H2)) \( w \mapsto [\varphi'(w)]^{-1} \) are continuous on \( \mathbb{R}^n - \{0\} \), hence there exist two positive constants \( c_1, c_2 \) (independent of \( \xi \)) such that \( \xi \) belongs to the cone

\[
\Gamma_0 = \{ \xi \in \mathbb{R}^{2n} : c_1 |\xi''| < |\xi'| < c_2 |\xi''| \}.
\]

For \( 1 \leq i, k \leq n \), let

\[
\Gamma_{0}^{i,k} = \{ \xi \in \Gamma_0 : |\xi'| < 2\sqrt{n} |\xi_i| \text{ and } |\xi''| < 2\sqrt{n} |\xi_{n+k}| \},
\]

\[
\Gamma_{0}^{i,k} = \{ \xi \in \Gamma_0 : |\xi'| < 4\sqrt{n} |\xi_i| \text{ and } |\xi''| < 4\sqrt{n} |\xi_{n+k}| \}.
\]

**Lemma 3.1.** If \( \alpha_i \neq \beta_k \) for \( 1 \leq i, k \leq n \), then

\[
\{ j \in \mathbb{Z} : (2^i \circ C_0^{i,k}) \cap C_0^{i,k} \neq \emptyset \}
\]

is a finite set.
Proof. Let \( j \in \mathbb{Z} \) and \( \xi \in C_{0}^{i,k} \) be such that \( 2^j \circ \xi \in C_{0}^{i,k} \). Then \( |2^j \cdot \xi'| < 4\sqrt{n} 2^{j\alpha_1} |\xi_i| \) and \( |2^j \circ \xi''| < 4\sqrt{n} 2^j |\xi_{n+k}| \). Since \( 2^j \circ \xi \in I_0 \), we have \( c_1 |2^j \circ \xi''| < |2^j \cdot \xi'| < c_2 |2^j \circ \xi''| \). Now,

\[
c_1 2^j |\xi_{n+k}| \leq c_1 |2^j \circ \xi''| < |2^j \cdot \xi'| < 4\sqrt{n} 2^j |\xi_{n+1}| \xi_i| < 4\sqrt{n} 2^j |\xi''| < 16n 2^j |\xi_{n+k}|. \]

So \( 2^j(\beta_k - \alpha_1) < 16nc_2/c_1 \). In a similar way we obtain \( c_1/(16nc_2) < 2^j(\beta_k - \alpha_1) \).

Since \( \alpha_i \neq \beta_k \) for \( 1 \leq i, k \leq n \), the lemma follows. \( \blacksquare \)

For \( 1 \leq i, k \leq n \), let \( m_{0}^{i,k} \) be a function belonging to \( C^{\infty}(\mathbb{R}^{2n} - \{0\}) \) homogeneous of degree zero with respect to the Euclidean dilations on \( \mathbb{R}^{2n} \) such that \( m_{0}^{i,k} \equiv 1 \) on \( I_{0}^{i,k} : 0 \leq m_{0}^{i,k} \leq 1 \), \( \text{supp}(m_{0}^{i,k}) \subset C_{0}^{i,k} \) and let \( m_{j}^{i,k}(y) = m_{0}^{i,k}(2^{-j} \circ y) \). Let \( Q_{j}^{i,k} \) be the operator with multiplier \( m_{j}^{i,k} \). Let \( h \in C_{c}^{\infty}(\mathbb{R}^{2n}) \), \( 0 \leq h \leq 1 \), be identically one in a neighborhood of the origin. Taking account of Proposition 4 in [S], p. 341, from the above observation about the critical points of \( \Phi \), we note that

\[
(3.1) \quad \hat{\mu}_0(1 - h) \prod_{1 \leq i, k \leq n} (1 - m_{0}^{i,k}) \in S(\mathbb{R}^{2n}).
\]

Let \( h_j(y) = h(2^{-j} \circ y) \) and let \( P_j \) be the Fourier multiplier operator with symbol \( h_j \).

**Remark 3.2.** Lemma 3.1 implies that there exists \( N > 0 \) such that for all \( y \in \mathbb{R}^{2n} - \{0\} \) the set \( \{ j \in \mathbb{Z} : 2^{-j} \circ y \in C_{0}^{i,k} \} \) has at most \( N \) elements, so \( m_{j}^{i,k}(y) := \sum_{j \in \mathbb{Z}} \varepsilon_j m_{j}^{i,k}(y) \), \( \varepsilon_j = \pm 1 \), is a well defined, \( C^{\infty}(\mathbb{R}^{2n} - \{0\}) \) and homogeneous function of degree zero. Moreover, for each \( s = (s_1, \ldots, s_{2n}) \), the function \( \sum_{j \in \mathbb{Z}} \left( \partial/\partial y \right)^s m_{j}^{i,k}(y) \) is homogeneous of degree \(- (s_1 + \ldots + s_{2n})\), so Theorem 3 in [St], p. 96, applies, showing that \( m_{j}^{i,k} \) is an \( L^p \) multiplier, for \( 1 < p < \infty \), and that the norm of the associated operator has a bound independent of the choices of \( \{\varepsilon_j\}_{j \in \mathbb{Z}} \). \( \blacksquare \)

**Theorem 3.3.** If \( \gamma \leq (\alpha + \beta)/3 \) then \( E_\mu \) is the closed segment with endpoints \( D \) and \( D' \).

**Proof.** By Remark 2.2, it is enough to prove that \( D \) and \( D' \) belong to \( E_\mu \). Let \( \{Q_{j}^{i,k}\}_{1 \leq r \leq n^2} \) be an arrangement of the set \( \{Q_{j}^{i,k}\}_{1 \leq i, k \leq n} \). For \( J \in \mathbb{N} \), we write

\[
\sum_{|j| < J} T_{\mu_j} = \sum_{|j| < J} T_{\mu_j} P_j + \sum_{|j| < J} T_{\mu_j} (I - P_j)
\]

\[
= \sum_{|j| < J} T_{\mu_j} P_j + \sum_{|j| < J} T_{\mu_j} (I - P_j) Q_{j}^{1} + \sum_{|j| < J} T_{\mu_j} (I - P_j)(I - Q_{j}^{1}).
\]
\[
= \sum_{|j| < J} T_{\mu_j} P_j + \sum_{|j| < J} T_{\mu_j} (I - P_j) Q^1_j + \sum_{|j| < J} T_{\mu_j} (I - P_j) (I - Q^1_j) Q^2_j \\
+ \sum_{|j| < J} T_{\mu_j} (I - P_j) (I - Q^1_j) (I - Q^2_j) = \ldots \\
= \sum_{|j| < J} T_{\mu_j} P_j + \sum_{|j| < J} T_{\mu_j} (I - P_j) Q^1_j \\
+ \sum_{1 \leq l \leq n^2 - 1} \sum_{|j| < J} T_{\mu_j} (I - P_j) \prod_{1 \leq r \leq l} (I - Q^r_j) Q^{l+1}_j \\
+ \sum_{|j| < J} T_{\mu_j} (I - P_j) \prod_{1 \leq r \leq n^2} (I - Q^r_j).
\]

The kernel \( K_J \) of the convolution operator
\[
\sum_{|j| < J} T_{\mu_j} (I - P_j) \prod_{1 \leq r \leq n^2} (I - Q^r_j)
\]
satisfies
\[
K_J(-\xi) = \sum_{|j| < J} 2^{j(\alpha + \beta - \gamma)} \hat{g}(2^j \circ \xi)
\]
with \( g = \hat{\mu}_0(1 - h) \prod_{1 \leq i, k \leq n} (1 - m_{0,i,k}^i) \). So, by using (3.1), a standard homogeneity argument shows that, for all \( J \in \mathbb{N} \),
\[
|K_J(\xi)| \leq \left( \sum_{i=1}^{n} |\xi_i|^{1/\alpha_i} + \sum_{k=1}^{n} |\xi_{n+k}|^{1/\beta_k} \right)^{-(\alpha + \beta - \gamma)}
\]
and so they belong to the weak \( L^{p,D} \) space, with weak constant uniformly bounded in \( J \). Also, a similar argument gives the same fact for the kernels of \( \sum_{|j| < J} T_{\mu_j} P_j \). Then the weak Young inequality implies that there exists \( c > 0 \) independent of \( J \) such that
\[
\left\| \sum_{|j| < J} T_{\mu_j} (I - P_j) \prod_{1 \leq r \leq n^2} (I - Q^r_j) \right\|_{p,D,q,D} \leq c
\]
and
\[
\left\| \sum_{|j| < J} T_{\mu_j} P_j \right\|_{p,D,q,D} \leq c.
\]
Now Remark 3.2 allows us to use Littlewood–Paley inequalities. As in [C] we obtain, for \( 1 \leq l \leq n^2 - 1 \),
\[
\left\| \sum_{|j| < J} T_{\mu_j} (I - P_j) \prod_{1 \leq r \leq l} (I - Q^r_j) Q^{l+1}_j f \right\|_q \leq c \left\{ T_{\mu_j} \right\}_{p,q,2} \left\{ f_j \right\}_{L^p(I^2)}
\]
where \( f_j = \prod_{1 \leq r \leq l} (I - Q_j^r)Q_j^{r+1}f \). Since the assertions of Remark 3.2 also hold when we replace \( m_{j}^{i,k} \) by a finite product of the form \( m_{j}^{r_1} \ldots m_{j}^{r_s} \), \( 1 \leq r_1, \ldots, r_s \leq n^2 \), we get \( \| \{ f_j \} \|_{L_p(I^2)} \leq c \| f \|_p \).

A similar estimate holds for \( \| \sum_{|j| < J} T_{\mu_j} (I - P_j)Q_j^f \|_q \).

Now, taking account of (2.3) we deduce (as in [C]) that there exist \( 0 < \theta < 1 \) and \( c > 0 \), independent of \( J \), such that

\[
\left\| \sum_{|j| < J} T_{\mu_j} \right\|_{p_D, q_D} < c \left( 1 + \left\| \sum_{|j| < J} T_{\mu_j} \right\|_{p_D, q_D}^{\theta} \right),
\]

and so \( \| \sum_{|j| < J} T_{\mu_j} \|_{p_D, q_D} \) is bounded independently of \( J \). From Fatou’s lemma, it follows that \( D \in E_\mu \). Since \( T_{\mu}^* \) is a convolution operator with a measure of the same kind, a duality argument shows that \( D^t \in E_\mu \).

We now consider a local version of the problem, that is, we study the type set corresponding to the convolution operator \( T_\sigma \) with the Borel measure given by

\[
\sigma(E) = \int_{\| x \| \leq 1} \chi_E(x, \varphi(x)) \| x \|^{\gamma - \alpha} dx
\]

with \( \gamma > 0 \).

**Theorem 3.4.** If \( \gamma > (\alpha + \beta)/3 \), then \( E_\sigma \) is the closed triangular region with vertices \( (2/3, 1/3) \), \( (0, 0) \) and \( (1, 1) \). If \( \gamma \leq (\alpha + \beta)/3 \) then \( E_\sigma \) is the closed polygonal region with vertices \( D, D', (0, 0) \) and \( (1, 1) \).

**Proof.** We have \( E_\mu \subset E_\sigma \). Since \( E_\sigma \) is a convex set and since \( \sigma \) is a finite measure, \( (1, 1) \) and \( (0, 0) \) belong to \( E_\sigma \). On the other hand, the constraints (2.1) and (2.2) hold for \( E_\sigma \). Moreover, Lemma 2.1 implies that if \( (1/p, 1/q) \in E_\sigma \), then \( 1/q \geq 1/p - \gamma/(\alpha + \beta) \). Thus the case \( \gamma \leq (\alpha + \beta)/3 \) follows from Theorem 3.3.

If \( \gamma > (\alpha + \beta)/3 \), then \( (2/3, 1/3) \) lies above the line \( 1/q = 1/p - \gamma/(\alpha + \beta) \) and we have noted in Remark 2.2 that \( (2/3, 1/3) \) belongs to \( E_{\mu_0} \), so Lemma 2.1 implies that \( \| \sum_{j \geq 0} T_{\mu_j} \|_{3/2, 3} = c \| T_{\mu_0} \|_{3/2, 3} \). Now, for \( f \geq 0 \), \( T_{\sigma} f \leq \sum_{j \geq 0} T_{\mu_j} f \) and the assertion follows.

**Remark 3.5.** If either \( \alpha_1 = \ldots = \alpha_n \) or \( \beta_1 = \ldots = \beta_n \), then (H3) implies (H2). Indeed, for \( 1 \leq i, k \leq n \),

\[
\frac{\partial \varphi_k}{\partial x_i} (t \cdot x) = i^{\beta_k - \alpha_i} \frac{\partial \varphi_k}{\partial x_i} (x).
\]

Taking the derivative with respect to \( t \), at \( t = 1 \), we obtain

\[
(\beta_k - \alpha_i) \frac{\partial \varphi_k}{\partial x_i} (x) = \sum_{l=1}^{n} x_l \alpha_l \frac{\partial^2 \varphi_k}{\partial x_i \partial x_l} (x).
\]
Thus, the matrix, with respect to the canonical basis of $\mathbb{R}^n$, of the linear operator $\varphi''(x)(\alpha_1 x_1, \ldots, \alpha_n x_n)$ is given by
\[
\begin{pmatrix}
(\beta_1 - \alpha_1) \frac{\partial \varphi_1}{\partial x_1}(x) & \cdots & (\beta_1 - \alpha_1) \frac{\partial \varphi_n}{\partial x_1}(x) \\
\vdots & \ddots & \vdots \\
(\beta_n - \alpha_n) \frac{\partial \varphi_1}{\partial x_n}(x) & \cdots & (\beta_n - \alpha_n) \frac{\partial \varphi_n}{\partial x_n}(x)
\end{pmatrix}
\]
(3.2)

So if either $\alpha_1 = \ldots = \alpha_n$ or $\beta_1 = \ldots = \beta_n$ then
\[
\det(\varphi''(x)(\alpha_1 x_1, \ldots, \alpha_n x_n)) = 0
\]
for every $x \in \mathbb{R}^n$. Then no $x \in \mathbb{R}^n$ is an elliptic point for $\varphi$.

**Examples.** Let us show two examples of functions $\varphi$ that satisfy the hypothesis of Theorems 3.4 and 3.5.

1) Let
\[
\varphi(x_1, x_2) = (x_1^4 - 6x_1^2 x_2 + x_2^4, (4x_1^3 x_2 - 4x_1 x_2^3)\sqrt{x_1^2 + x_2^2}).
\]
In this case $\varphi(t \cdot x) = t \circ \varphi(x)$ with $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = 4$, $\beta_2 = 5$. Taking account of Remark 3.5, we only need to check (H3). An explicit computation shows that the discriminant of the quadratic form
\[
(h_1, h_2) \mapsto \det(\varphi''(x_1, x_2)(h_1, h_2))
\]
is negative for $(x_1, x_2) \neq (0, 0)$, so (H3) holds.

2) Let
\[
\varphi(x_1, x_2) = (x_1 x_2, x_2^2 - x_1^2 \sqrt{x_2^2 + x_1^2}).
\]
In this case $\varphi(t \cdot x) = t \circ \varphi(x)$ with $\alpha_1 = 1$, $\alpha_2 = 2$ and $\beta_1 = 3$, $\beta_2 = 4$. A computation shows that
\[
\det(\varphi'(x_1, x_2)) = \frac{2\sqrt{x_2^2 + x_1^2} x_2^3 + x_1^7}{\sqrt{x_2^2 + x_1^2}} \neq 0, \quad (x_1, x_2) \neq (0, 0)
\]
and so (H2) holds. On the other hand, the discriminant of the quadratic form
\[
(h_1, h_2) \mapsto \det(\varphi''(x_1, x_2)(h_1, h_2))
\]
is
\[
-8\sqrt{x_2^2 + x_1^2} x_2^2 - 8\sqrt{x_2^2 + x_1^2} x_2^4 + 4x_1^6
\]
\[
(x_2^2 + x_1^4)^3 < 0, \quad (x_1, x_2) \neq (0, 0),
\]
so that (H3) holds.
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Received 12 July 2001;
revised 28 February 2002 (4089)