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L^2-DATA DIRICHLET PROBLEM FOR WEIGHTED FORM LAPLACIANS

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Abstract. We solve the L^2-data Dirichlet boundary problem for a weighted form Laplacian in the unit Euclidean ball. The solution is given explicitly as a sum of four series.

1. Introduction and preliminaries. We work with *weighted form Laplacians* $L = L_{a,b} = ad\delta + b\delta d$, $a, b > 0$, acting on the space of p-differential forms in $\mathbb{R}^n$. These operators give a subclass of so called *non-minimal* operators (cf. [2]). If $a = b = 1$, $L_{a,b}$ is just the Laplace–Beltrami operator $\Delta = d\delta + \delta d$. If $a = (n-1)/n$, $b = 1/2$ and $p = 1$, then $L_{a,b}$ corresponds to the Ahlfors–Laplace operator S^*S. The correspondence is given by the natural duality between the space of vector fields and one-forms. For more details see [9, 10].

Since $L = (\sqrt{a}\delta + \sqrt{b}d)^*(\sqrt{a}\delta + \sqrt{b}d)$, L is strongly elliptic, but in contrast to Δ, the principal symbol of L is not of metric type except when $a = b$. This causes the $L_{a,b}$ theory to be more complicated than the theory of Δ.

In [1] Ahlfors solved the Dirichlet boundary problem for S^*S in the hyperbolic ball. Reimann [11] solved the L^2-data Dirichlet problem for the Ahlfors–Laplace operator for the Euclidean ball and vector fields; the solution is given as a sum of three series. Next Lipowski [7] solved the equation $S^*S = 0$ for some boundary conditions of Neumann type.

In [5, 4] the author investigated the operator $L_{a,b}$ in the space of polynomial p-forms in $\mathbb{R}^n$ and solved the polynomial-data Dirichlet boundary problem for L and the Euclidean ball in [5]. Next in [6] A. Pierzchalski and the author solved the so called *elliptic boundary problems* in the sense of Gilkey and Smith for the operator L in the Euclidean ball for polynomial p-forms.

In the present paper we adopt Reimann's method and solve the L^2-data Dirichlet problem for the operator $L_{a,b}$ in the Euclidean unit ball and for

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differential forms of arbitrary degree. By analogy with [11], our solution is given as a sum of four series. In the special case of $p = 1$, $L = S^*S$, one of the series degenerates and our solution coincides with that from [11]. The main tool we use is an $\mathsf{SO}(n)$-invariant decomposition of $\ker L$.

1.1. *Spherical harmonics—basic facts.* We briefly review some basic properties of homogeneous polynomials and spherical harmonics. For more details we refer to [3] and [12, Ch. IV §2].

We work in $\mathbb{R}^n$, $n \geq 3$. Σ, $d\Sigma$ and $d\sigma$ denote the unit sphere in $\mathbb{R}^n$, the Lebesgue measure and the normalized Lebesgue measure on Σ, respectively. B and $\overline{B}$ denote the open and closed unit ball in $\mathbb{R}^n$, respectively. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index then $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$, $D^\alpha = (\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^n)^{\alpha_n}$. If $f = \sum_\alpha a_\alpha x^\alpha$ is a polynomial in $\mathbb{R}^n$ then $f(D) = \sum_\alpha a_\alpha D^\alpha$.

$\mathcal{P}_k$ denotes the space of all homogeneous polynomials in $\mathbb{R}^n$ of degree k. Obviously, $f(D)$ maps $\mathcal{P}_l$ into $\mathcal{P}_{l-k}$. Define an inner product $(\cdot,\cdot) = (\cdot,\cdot)_k$ in $\mathcal{P}_k$ as follows; $(f,g) = f(D)g$ for $f, g \in \mathcal{P}_k$. (Since f and g are both homogeneous polynomials of the same degree, $(f,g) = f(D)g$ is a constant function. We may and we will identify this function with its unique value.) Clearly, for any $f \in \mathcal{P}_k$, $g \in \mathcal{P}_l$ and $h \in \mathcal{P}_{k+l}$, $(gf,h)_{k+l} = (f, g(D)h)_k$.

$|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ denotes the Euclidean norm in $\mathbb{R}^n$. The polynomial $r^2(x) = |x|^2$ belongs to $\mathcal{P}_2$. The differential operator $-r^2(D)$ is the classical Laplace operator $\Delta = -\sum_{j=1}^n (\partial/\partial x^j)^2$.

Let $\mathcal{H}_k = \{h \in \mathcal{P}_k : \Delta h = 0\}$ be the space of all *harmonic* homogeneous polynomials of degree k. The *spherical harmonics* of degree k are the restrictions of the members of $\mathcal{H}_k$ to Σ. For the sake of the homogeneity, we may and will identify the space of spherical harmonics of degree k with $\mathcal{H}_k$.

Let $L^2(\Sigma)$ denote the Hilbert space of square integrable functions $\Sigma \to \mathbb{R}$ with the inner product $(f,g)_\sigma = \int_\Sigma fg\, d\sigma$ and the norm $\|f\|_\sigma = \sqrt{(f,f)_\sigma}$. The inner products in $L^2(\Sigma)$ and in $\mathcal{H}_k$ are related as follows ([3, p. 147]):

$$(f,g)_\sigma = (n-2) \prod_{j=0}^{k} \frac{1}{2j+n-2}\, (f,g)_k, \qquad f, g \in \mathcal{H}_k. \tag{1.1.1}$$

It is very well known ([12, Ch. IV, §2]) that if $k \neq l$ then $\mathcal{H}_k$ and $\mathcal{H}_l$ are mutually orthogonal (in $L^2(\Sigma)$) and

$$L^2(\Sigma) = \bigoplus_{k=0}^{\infty}{}^{\perp}\, \mathcal{H}_k, \tag{1.1.2}$$

i.e., if $h_{k,i}$, $i = 1, \ldots, d_k = \dim \mathcal{H}_k$, is an orthonormal (in $L^2(\Sigma)$) basis of $\mathcal{H}_k$, then $h_{k,i}$, $k = 0, 1, \ldots,$ $i = 1, \ldots, d_k$, is an orthonormal basis of $L^2(\Sigma)$.

Suppose $h \in \mathcal{H}_k$, $k > 0$. Fix $R, R' \in (0,1)$, $R' < R$, and put $C(R) = (1-R^2)(1-R)^{-n}$ and $\varepsilon = R'/R$. Clearly $\varepsilon < 1$. Then for any $|y| < R'$,

$$|h(y)| \le \varepsilon^k C(R)\|h\|_\sigma. \tag{1.1.3}$$

Proof of (1.1.3). If $y = 0$ then (1.1.3) is obvious. Let $y \neq 0$. By the Poisson formula

$$h(y) = \int_\Sigma P(x,y)h(x)\, d\sigma(x), \quad P(x,y) = \frac{1-|y|^2}{|y-x|^n}.$$

By the above, the homogeneity of h and the estimates

$$|y|R^{-1} < \varepsilon, \quad P(x, R|y|^{-1}y) \le C(R),$$

we obtain

$$\begin{aligned} |h(y)| &\le \varepsilon^k \Big| \int_\Sigma P(x, R|y|^{-1}y)h(x)\, d\sigma(x)\Big| \\ &\le \varepsilon^k \int_\Sigma |C(R)h(x)|\, d\sigma(x) \le \varepsilon^k C(R)\|h\|_\sigma. \end{aligned}$$

Let $(g_{k,j} : k \ge 0, 1 \le j \le l_k \le d_k)$ be any sequence of spherical harmonics such that $g_{k,j} \in \mathcal{H}_k$. Suppose that $(g_{k,j})$ is bounded in $L^2(\Sigma)$. Put

$$s = \sum_{k,j} g_{k,j}.$$

LEMMA 1.1.1. *Let $a_{k,j}$ be a sequence of reals such that $\sum_{k,j} a_{k,j}^2 < \infty$. Then for any sequence w_k of polynomial growth (i.e., $|w_k| < Mk^N$ for some positive integer N and $M > 0$) and $|x| < R'$ we have*

$$\sum_{k,j} |a_{k,j} w_k g_{k,j}(x)| \le C(R)\Big(\sum_{k,j} a_{k,j}^2\Big)^{1/2}\Big(\sum_{k,j} \varepsilon^{2k} w_k^2 \|g_{k,j}\|_\sigma^2\Big)^{1/2} < \infty.$$

LEMMA 1.1.2. *The series s converges absolutely and uniformly on compact sets in the unit ball to a harmonic function. Moreover, for any multi-index α, the series $\sum_{k,i} D^\alpha g_{k,j}$ converges absolutely and uniformly on compact sets to $D^\alpha s$.*

Lemma 1.1.1 follows from (1.1.3) and the fact that the series $\sum_k d_k k^N \varepsilon^k$ converges. Lemma 1.1.2 also follows from those two properties and the Weierstrass theorem. The details are left to the reader.

1.2. *Spaces Λ_k^p and $L^{2,p}(\Sigma)$.* Consider any p-form ω defined in a subset $A \subset \mathbb{R}^n$. If $p = 0$ we identify ω with a function on A. Assume that any p-form, $p < 0$, is the zero form. If $p \ge 1$ then ω has the unique expression

$$\omega = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n} \omega_{i_1,\dots,i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p},$$

where the coefficients $\omega_{i_1,\dots,i_p}$ are skew-symmetric with respect to their indices. Let A be any subset of $\mathbb{R}^n$. We say that ω is a *differential* p-form on A, and we write $\omega \in \Lambda^p(A)$, if ω is defined and smooth, i.e., C^∞, on some open set containing A. If α and β are p-forms defined in A, their pointwise inner product is simply the function $\alpha\beta : A \to \mathbb{R}$ given by

$$\alpha\beta = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n} \alpha_{i_1,\dots,i_p}\beta_{i_1,\dots,i_p}. \tag{1.2.1}$$

Let d and δ denote the (exterior) differential and co-differential, respectively. Put $\nu^\star = (1/2)dr^2$, i.e., $\nu^\star_x = x^1dx^1 + \dots + x^ndx^n$.

Let $\varepsilon_\nu = \nu^\star\wedge$ and $\iota_\nu = \nu^\star\vee$ denote the operators of exterior product and contraction with $\nu^\star$. Recall that ι_ν is adjoint to ε_ν with respect to the pointwise inner product defined above, i.e., $(\varepsilon_\nu\alpha)\beta = \alpha(\iota_\nu\beta)$. We have

$$r^2\omega = (\iota_\nu\varepsilon_\nu + \varepsilon_\nu\iota_\nu)\omega. \tag{1.2.2}$$

Relations between d, δ, ι_ν and ε_ν will play an important role in our considerations. For example, Theorem 1.2.1 shows that with respect to the inner product $(\cdot|\cdot)$ (introduced below) the adjoint operators to d and δ are ι_ν and $-\varepsilon_\nu$. Combining Theorem 1.2.1 with (1.1.1) we get in a simple way some orthogonality relations in the space $L^{2,p}(\Sigma)$ (defined below). In fact, the kernel of a weighted form Laplacian (in the space of polynomial forms) can be reconstructed from the space $\ker\Delta \cap \ker\delta \cap \ker\iota_\nu$.

As a consequence of the above and the Green formula we find that for any smooth $\omega, \eta \in \Lambda^p(\overline{B})$,

$$\int_B (d\omega)\eta\,dx = \int_B \omega(\delta\eta)\,dx + \int_\Sigma \omega(\iota_\nu\eta)\,d\Sigma, \tag{1.2.3}$$

where dx denotes Lebesgue measure in $\mathbb{R}^n$.

Remark. (1.2.3) is a very special case of a much more general formula for manifolds with boundary (cf. [8, Ch. IV]).

Let $L^{2,p}(\Sigma)$ be the space of p-forms (defined on Σ) with all coefficients in $L^2(\Sigma)$. Equipped with the inner product $(\cdot,\cdot)_\sigma = (\cdot,\cdot)_{\sigma,p}$, where

$$(\alpha,\beta)_\sigma = \int_\Sigma \alpha\beta\,d\sigma, \qquad \alpha,\beta \in L^{2,p}(\Sigma),$$

$L^{2,p}(\Sigma)$ is a Hilbert space. Here $\alpha\beta$ denotes the pointwise inner product (1.2.1).

The proofs of the properties stated below can be found in [5, §2.2]. A p-form ω is called a *polynomial p-form* if $A = \mathbb{R}^n$ and the $\omega_{i_1,\dots,i_p}$'s are polynomials. Denote by Λ^p the vector space of all polynomial p-forms in $\mathbb{R}^n$. A polynomial p-form ω is called *homogeneous* if all coefficients are from $\mathcal{P}_k$, for some k. Such a form will also be called a (p/k)-*form*. Λ^p_k denotes the vector space of all (p/k)-forms. We have $\Lambda_k^0 = \mathcal{P}^k$, and Λ_k^n is isomorphic to $\mathcal{P}^k$ in a natural way. Moreover, it is convenient to put $\Lambda_k^p = \{0\}$ if either $p < 0$ or $k < 0$. We extend the inner product $(\cdot,\cdot)_k$ to Λ_k^p by setting

$$(\omega|\eta)_{p,k} = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n} (\omega_{i_1,\dots,i_p}, \eta_{i_1,\dots,i_p})_k, \tag{1.2.4}$$

where $\omega, \eta \in \Lambda_k^p$ while $\omega_{i_1,\dots,i_p}$'s and $\eta_{i_1,\dots,i_p}$'s denote their coefficients. Notice that $(\cdot|\cdot)_{0,k}$ and $(\cdot,\cdot)_k$ coincide. We will frequently write $(\cdot|\cdot)$ instead of $(\cdot|\cdot)_{p,k}$ if the values of p and k are evident. We have

$$d\varepsilon_\nu = -\varepsilon_\nu d \quad \text{and} \quad \delta\iota_\nu = -\iota_\nu\delta. \tag{1.2.5}$$

PROPOSITION 1.2.1. *Suppose ω is a (p/k)-form. Then*

$$\delta\varepsilon_\nu\omega = -\varepsilon_\nu\delta\omega - (n-p+k)\omega, \qquad d\iota_\nu\omega = -\iota_\nu d\omega + (p+k)\omega.$$

PROPOSITION 1.2.2. *For any polynomial form ω we have*

$$d(r^2\omega) = r^2 d\omega + 2\varepsilon_\nu\omega, \qquad \delta(r^2\omega) = r^2\delta\omega - 2\iota_\nu\omega.$$

THEOREM 1.2.1. *Consider d and δ as operators $d : \Lambda_k^p \to \Lambda_{k-1}^{p+1}$ and $\delta : \Lambda_k^p \to \Lambda_{k-1}^{p-1}$. Let $d^\star$ and $\delta^\star$ denote their respective adjoints (with respect to the inner product $(\cdot,\cdot)$). Then, for any (p/k)-form ω, $\delta^\star\omega = -\varepsilon_\nu\omega$ and $d^\star\omega = \iota_\nu\omega$.*

1.3. *Weighted form Laplacians.* We briefly review the relevant facts of $L_{a,b}$ theory. Demonstrations of the assertions listed without proofs in this section can be found in [5, §3]. Consider a *weighted form Laplacian* $L = L_{a,b} = ad\delta + b\delta d$, $a, b > 0$. For $a = b = 1$, $L_{1,1}$ is just the Laplace–Beltrami operator $\Delta = d\delta + \delta d$. Notice that in the case of differential 0-forms, i.e. smooth functions, the Laplace–Beltrami operator $L_{1,1}$ and the classical Laplace operator coincide.

For any differential form ω in $\mathbb{R}^n$, $(\Delta\omega)_{i_1,\dots,i_p} = \Delta\omega_{i_1,\dots,i_p}$; thus ω is harmonic ($\Delta\omega = 0$) iff its coefficients are harmonic functions. In particular, a (p/k)-form ω is harmonic iff every $\omega_{i_1,\dots,i_p} \in \mathcal{H}_k$.

Denote by $\mathfrak{H}_k^p$ the space of all harmonic (p/k)-forms, i.e., $\mathfrak{H}_k^p = \ker\Delta \cap \Lambda_k^p$. Consider $L = L_{a,b}$ as an operator $L : \Lambda_k^p \to \Lambda_{k-2}^p$, and let $\mathfrak{L}_k^p$ be its kernel. If $k = 0, 1$ then $\Lambda_k^p = \mathfrak{L}_k^p = \mathfrak{H}_k^p$. Moreover, $\mathfrak{L}_k^0 = \mathcal{H}_k$ and $\mathfrak{L}_k^n$ is isomorphic to $\mathcal{H}_k$ in a natural way.

For any $0 \le p \le n$ and $k \ge 0$ put

$$\chi_{p,k}^0 = \mathfrak{H}_k^p \cap \ker\delta \cap \ker\iota_\nu.$$

It is also convenient to put $\chi_{q,l}^0 = \{0\}$ if either $q < 0$ or $l < 0$. Manifestly, $\mathfrak{H}_k^0 = \chi_{0,k}^0 = \mathcal{H}_k$ and $\mathfrak{H}_0^p = \Lambda_0^p$. Next define

$$I_L(p,k) = \varepsilon_\nu - c_L(p,k) r^2 d : \Lambda_k^p \to \Lambda_{k-1}^{p-1},$$

where

$$c_L(p,k) = \begin{cases} \dfrac{1}{2}\dfrac{2b-(b-a)(n-p+k)}{a(p+k-2)+b(n-p+k-2)} & \text{if } k\ge 2,\ 0<p\le n,\\ 0 & \text{otherwise.}\end{cases} \tag{1.3.1}$$

Notice that our assumption ($a, b > 0$ and $n \ge 3$) ensures that $c_L(p,k)$ is well-defined. Observe that in the very special case $a = b = 1$, i.e., $L = \Delta$, the above constant is

$$c_\Delta(p,k) = \begin{cases} 1/(n+2k-4) & \text{if } k\ge 2,\ 0<p\le n,\\ 0 & \text{otherwise.}\end{cases} \tag{1.3.2}$$

The decomposition below is the key step in the proof of the main theorem. For the proof see [5, Theorem 3.3.1]. For any $0 \le p \le n$ and $k \ge 0$ the space $\mathfrak{L}^p_k$ is the direct sum of four mutually orthogonal $\mathsf{SO}(n)$-invariant subspaces:

$$\mathfrak{L}^p_k = \chi^0_{p,k} \oplus^\perp d\chi^0_{p-1,k+1} \oplus^\perp \varepsilon_\nu d\chi^0_{p-2,k} \oplus^\perp I_L(p,k)\chi^0_{p-1,k-1}. \tag{1.3.3}$$

Moreover, $\chi^0_{p,k}$, $d\chi^0_{p-1,k+1}$ and $\varepsilon_\nu d\chi^0_{p-2,k}$ are subspaces of $\mathfrak{H}^p_k$. In particular,

$$\mathfrak{H}^p_k = \chi^0_{p,k} \oplus^\perp d\chi^0_{p-1,k+1} \oplus^\perp \varepsilon_\nu d\chi^0_{p-2,k} \oplus^\perp I_\Delta(p,k)\chi^0_{p-1,k-1}. \tag{1.3.4}$$

Note that in (1.3.3) or (1.3.4) some subspaces may degenerate or split into finer $\mathsf{SO}(n)$-subspaces ([5, §5.2]). In particular,

$$\chi^0_{p,0} = \{0\} \quad \text{for } p > 0, \tag{1.3.5}$$

$$I_L(n,2)(\chi^0_{n-1,1}) = \{0\}. \tag{1.3.6}$$

Remark. In view of (1.3.6), a natural question is whether $\chi^0_{n-1,1} = \{0\}$. The answer is: No. Namely, it is easy to observe that the form $\omega = \iota_\nu(dx^1 \wedge \cdots \wedge dx^n)$ is in $\chi^0_{n-1,1}$. In fact, $\chi^0_{n-1,1}$ is one-dimensional and is spanned by ω. To see this, we use property (i) from §2.1 below. Since $d : \chi^0_{n-1,1} \to \mathfrak{H}^n_0$ is one-to-one and obviously $\dim \mathfrak{H}^n_0 = 1$, we conclude that $\chi^0_{n-1,1}$ must be one-dimensional. Therefore, $\chi^0_{n-1,1} = \operatorname{span}\{\omega\}$.

2. Dirichlet boundary problem

2.1. *Bases of $\mathfrak{L}^p_k$ and $L^{2,p}(\Sigma)$ and relations between them.* Assume that $0 \le p \le n$ and $k \ge 0$. We identify the spaces Λ^p_k and $\{\omega|\Sigma : \omega \in \Lambda^p_k\}$, for any $\omega \in \Lambda^p_k$ is uniquely determined by its restriction to Σ. As a direct consequence of (1.1.1) we see that for any $\alpha, \beta \in \mathfrak{H}^p_k$,

$$(\alpha,\beta)_{\sigma,p} = s_k(\alpha,\beta)_{p.k}, \qquad s_k = (n-2)\prod_{j=0}^{k}\frac{1}{2j+n-2}. \tag{2.1.1}$$

In particular, the decomposition (1.3.4) is orthogonal in $L^{2,p}(\Sigma)$.

Consequently, by (1.1.2) we get

$$L^{2,p}(\Sigma) = \bigoplus_{k=0}^{\infty}{}^{\perp}(\chi^0_{p,k} \oplus^{\perp} d\chi^0_{p-1,k+1} \oplus^{\perp} \varepsilon_\nu d\chi^0_{p-2,k} \oplus^{\perp} I_{k,\Delta}\chi^0_{p-1,k-1}).$$

Consider now the subspace $\chi^0_{q,l}$ and put $\mu^q_l = \dim \chi^0_{q,l}$. Moreover, let $E^q_l = \{\eta^q_{l,i} : i = 1, \dots, \mu^q_l\}$ be an L^2-orthonormal basis of $\chi^0_{q,l}$. If $\chi^0_{q,l} = \{0\}$, it is convenient to treat E^q_l as a set which contains only the zero form. Now we are going to build an L^2-orthonormal basis of $\mathfrak{H}^p_k$ from E^q_l's. To do this we will need the following ([5, Proposition 3.2.1, Lemma 3.2.1]):

(i) If $0 < p \le n$ and $k \ge 0$ then for any $\eta', \eta'' \in \chi^0_{p-1,k+1}$,

$$(d\eta'|d\eta'') = (p+k)(\eta'|\eta'').$$

In particular, $d : \chi^0_{p-1,k+1} \to \chi_{p,k}$ is one-to-one.

(ii) If $2 \le p \le n$ and $k \ge 0$ then for any $\eta', \eta'' \in \chi^0_{p-2,k}$,

$$(\varepsilon_\nu d\eta'|\varepsilon_\nu d\eta'') = (n-p+k)(d\eta'|d\eta'').$$

In particular, $\varepsilon_\nu : d\chi^0_{p-2,k} \to \mathfrak{H}^p_k$ is one-to-one.

(iii) If $p \ge 1$ and $k \ge 1$ then for any $\eta', \eta'' \in \chi^0_{p-1,k-1}$,

$$(I_\Delta(p,k)\eta'|I_\Delta(p,k)\eta'') = \frac{(n+k-p-2)(n+2k-2)}{(n+2k-4)}(\eta'|\eta'').$$

In particular, if $p \ne n$ or $k \ne 2$ then $I_\Delta(p,k) : \chi^0_{p-1,k-1} \to \mathfrak{H}^p_k$ is one-to-one.

Remark. The constant in case (iii) is equal to 0 iff $n+k-p-2 = 0$. This implies that $p = n-1$ and $k = 1$, or $p = n$ and $k = 2$. In the case $p = n-1$ and $k = 1$, $I_\Delta(p,k)$ maps the space $\chi^0_{n-2,0}$. But by (1.3.5), $\chi^0_{n-2,0} = \{0\}$ for $n \ge 3$. Therefore, our map is one-to-one. In the case $p = n$ and $k = 2$ the situation is quite different. Namely, we have seen (1.3.6) that $I_\Delta(n,2)$ is the zero map, but $\chi^0_{n-1,1}$ is one-dimensional (remark below (1.3.6)). Therefore, the case $p = n$ and $k = 2$ must be excluded from the second part of (iii).

Points (a)–(c) below are direct consequences of (i)–(iii) and (2.1.1).

(a) Let $p > 0$ and $k \ge 0$. If $\mu^{p-1}_{k+1} \ge 1$ then the collection

$$\left\{\alpha^p_{k,j} = \frac{1}{\sqrt{(p+k)(2k+n)}}\, d\eta^{p-1}_{k+1,j} : \eta^{p-1}_{k+1,j} \in E^{p-1}_{k+1}\right\}$$

is an L^2-orthonormal basis of $d\chi^0_{p-1,k+1}$.

(b) Let $p \geq 2$, $k \geq 1$. If $\mu_k^{p-2} \geq 1$ then the collection

$$\left\{\beta_{k,j}^p = \frac{1}{\sqrt{(n-p+k)(p+k-1)}}\,\varepsilon_\nu d\eta_{k,j}^{p-2} : \eta_{k,j}^{p-2} \in E_k^{p-2}\right\}$$

is an L^2-orthonormal basis of $\varepsilon_\nu d\chi_{p-2,k}^0$.

(c) Let $p \geq 1$ and $k \geq 1$. If $\mu_{k-1}^{p-1} \geq 1$ then, except the case $p = n$ and $k = 2$, the collection

$$\left\{\gamma_{k,j}^p = \sqrt{\frac{n+2k-4}{n+k-p-2}}\, I_\Delta(p,k)\eta_{k-1,j}^{p-1} : \eta_{k-1,j}^{p-1} \in E_{k-1}^{p-1}\right\}$$

is an L^2-orthonormal basis of $I_\Delta(p,k)\chi_{p-1,k-1}^0$.

Moreover, we define $\alpha_{k,j}^p$, $\beta_{k,j}^p$ and $\gamma_{k,j}^p$ to be the zero form, in all the remaining cases. Summarizing we deduce

Corollary 2.1.1. *The collection of all nonzero p-forms $\eta_{k,j}^p$, $\alpha_{k,j}^p$, $\beta_{k,j}^p$ and $\gamma_{k,j}^p$ is an L^2-orthonormal basis of $L^{2,p}(\Sigma)$.*

Consequently, every $\omega \in L^{2,p}(\Sigma)$ has a unique expression as an L^2-orthogonal sum

$$\omega = \sum_{k=0}^{\infty}\sum_{j=1}^{\mu_k^p} u_{k,j}^p \eta_j^p + \sum_{k=0}^{\infty}\sum_{j=1}^{\mu_{k+1}^{p-1}} a_{k,j}^p \alpha_{k,j}^p + \sum_{k=0}^{\infty}\sum_{j=1}^{\mu_k^{p-2}} b_{k,j}^p \beta_j^p + \sum_{k=0}^{\infty}\sum_{j=1}^{\mu_{k-1}^{p-1}} c_{k,j}^p \gamma_{k,j}^p, \tag{2.1.2}$$

where, of course, the coefficients are given by

$$\begin{aligned} u_{k,j}^p &= (\omega|\eta_{k,j}^p)_{\sigma,p}, \quad & a_{k,j}^p &= (\omega|\alpha_{k,j}^p)_{\sigma,p},\\ b_{k,j}^p &= (\omega|\beta_{k,j}^p)_{\sigma,p}, \quad & c_{k,j}^p &= (\omega|\gamma_{k,j}^p)_{\sigma,p}. \end{aligned} \tag{2.1.3}$$

In particular, the L^2-norm $\|\omega\|_{\sigma,p}$ is

$$\|\omega\|_{\sigma,p}^2 = \sum_{k,j}|u_{k,j}^p|^2 + \sum_{k,j}|a_{k,j}^p|^2 + \sum_{k,j}|b_{k,j}^p|^2 + \sum_{k,j}|c_{k,j}^p|^2.$$

(d) Suppose $\mu_{k-1}^{p-1} \geq 1$. If $p < n$ or $k \neq 2$ then the collection

$$\{\varepsilon_\nu \eta_{k-1,j}^{p-1} : \eta_{k-1,j}^{p-1} \in E_{k-1}^{p-1}\}$$

is an L^2-orthonormal system in $L^{2,p}(\Sigma)$.

Proof of (d). Take any $\eta_{k-1,i}^{p-1}, \eta_{k-1,j}^{p-1} \in E_{k-1}^{p-1}$. Then on Σ we have

$$\varepsilon_\nu \eta_{k-1,i}^{p-1} = A_{p,k}\alpha_{k-2,i}^p + C_{p,k}\gamma_{k,i}^p, \qquad \varepsilon_\nu \eta_{k-1,j}^{p-1} = A_{p,k}\alpha_{k-2,j}^p + C_{p,k}\gamma_{k,j}^p,$$

where

$$A_{p,k} = \sqrt{\frac{p+k-2}{n+2k-4}}, \qquad C_{p,k} = \sqrt{\frac{n+k-p-2}{n+2k-4}}. \tag{2.1.4}$$

By the orthogonality relations (a) and (c), we obtain

$$\begin{aligned}(\varepsilon_\nu \eta^{p-1}_{k-1,i} | \varepsilon_\nu \eta^{p-1}_{k-1,j})_{\sigma,p} &= \frac{n+k-p-2}{n+2k-4}\,(\gamma^p_{k,i}|\gamma^p_{k,j})_{\sigma,p} \\ &\quad + \frac{(p+k-2)(n+2k-4)}{(n+2k-4)^2}\,(\alpha^p_{k-2,i}|\alpha^p_{k-2,j})_{\sigma,p} \\ &= \delta_{i,j} \quad \text{(the Kronecker symbol).}\end{aligned}$$

Our task is to build a collection of polynomial p-forms belonging to the kernel of L such that their restrictions to the unit sphere consist a complete basis in $L^{2,p}(\Sigma)$. By (1.3.3) and the decomposition of $L^{2,p}(\Sigma)$ we may suppose that all p-forms $\eta^p_{k,j}$, $\alpha^p_{k,j}$ and $\beta^p_{k,j}$ belong to our basis. The only thing we have to do is to slightly modify the forms $\gamma^p_{k,j}$.

(e) Suppose $\mu^{p-1}_{k-1} \geq 1$. If $p < n$ or $k \neq 2$ then the collection

$$\left\{\tau^p_{k,j} = \frac{1}{C_{p,k}}\, I_L(p,k)\eta^{p-1}_{k-1,j} + \frac{c(p,k)}{C_{p,k}}\, d\eta^{p-1}_{k-1,j} : \eta^{p-1}_{k-1,j} \in E^{p-1}_{k-1}\right\}$$

restricted to Σ is an L^2-orthonormal system in $L^{2,p}(\Sigma)$. More precisely, $\tau^p_{k,j} = \gamma^p_{k,j}$ on Σ. Each $\tau^p_{k,j}$ is a member of $\ker L$. Here $c(p,k) = (c_L(p,k) - c_\Delta(p,k))$, whereas $C_{p,k}$ is defined in (2.1.4).

(f) Suppose $\mu^{p-1}_{k-1} \geq 1$. If $p < n$ or $k \neq 2$ then the collection

$$\{\psi^p_{k,j} = \varepsilon_\nu \eta^{p-1}_{k-1,j} + c_L(p,k)(1-r^2)d\eta^{p-1}_{k-1,j} : \eta^{p-1}_{k-1,j} \in E^{p-1}_{k-1}\}$$

restricted to Σ is an L^2-orthonormal system in $L^2(\Sigma)$. More precisely, $\psi^p_{k,j} = \varepsilon_\nu \eta^{p-1}_{k-1,j}$ on Σ. Each $\psi^p_{k,i}$ is a member of $\ker L$.

Points (e) and (f) are direct consequences of (c), (d) and the decomposition (1.3.3). Note that $\tau^p_{k,j}$ and $\psi^p_{k,j}$ are not homogeneous. Moreover, we define $\tau^p_{k,j}$ and $\beta^p_{k,j}$ to be the zero form in all the remaining cases.

Observe that

$$\begin{aligned}\psi^p_{k,j} &= C_{p,k}\gamma^p_{k,j} \\ &\quad + (c_L(p,k) - c(p,k)r^2)\sqrt{(p+k-2)(2k+n-4)}\,\alpha^p_{k-2,j}.\end{aligned} \tag{2.1.5}$$

Consequently, there exists a constant $N = N(n,p) > 0$ such that for any x and any coefficient $\psi^p_{k,j;i_1,\dots,i_p}$ of $\psi^p_{k,j}$,

$$|\psi^p_{k,j;i_1,\dots,i_p}(x)| \leq N|\gamma^p_{k,j;i_1,\dots,i_p}(x)| + Nk|\alpha^p_{k,j;i_1,\dots,i_p}(x)|. \tag{2.1.6}$$

Now we are in a position to introduce two collections of forms, $\mathcal{E}^p$ and $\mathcal{F}^p$, having properties described above. We do this as follows;

- $\mathcal{E}^p$ consists of all non-zero p-forms $\eta^p_{k,j}$, $\alpha^p_{k,j}$, $\beta^p_{k,j}$ and $\tau^p_{k,j}$.
- $\mathcal{F}^p$ consists of all non-zero p-forms $\eta^p_{k,j}$, $\alpha^p_{k,j}$, $\beta^p_{k,j}$ and $\psi^p_{k,j}$.

(A) The members of $\mathcal{E}^p$ lie in $\ker L$. The collection $\mathcal{E}^p$ restricted to Σ is an L^2-orthonormal basis of $L^{2,p}(\Sigma)$.

(B) The members of $\mathcal{F}^p$ lie in $\ker L$. The collection $\mathcal{F}^p$ restricted to Σ is an L^2-complete (but not orthogonal) basis in $L^{2,p}(\Sigma)$.

Proof. (A) is a direct consequence of (c) and (e). Consider (B). Clearly, each member of $\mathcal{F}^p$ lies in $\ker L$. We must show that the restrictions of members of $\mathcal{F}^p$ to Σ are linearly independent and that $\mathcal{F}^p$ restricted to Σ is a complete system in $L^{2,p}(\Sigma)$. By (2.1.5), on Σ we have

$$\psi^p_{k,j} = A_{p,k}\alpha^p_{k-2,j} + C_{p,k}\gamma^p_{k,j}.$$

On the other hand, directly by the definition of $\mathcal{E}^p$, we have $C_{p,k} \neq 0$, and moreover $\gamma^p_{k,j} = 0$ iff $\eta^{p-1}_{k-1,j} = 0$. This means that on Σ: (1) Each $\gamma^p_{k,j}$ is a linear combination of members of $\mathcal{F}^p$, thus $\mathcal{F}^p$ restricted to Σ is complete, for $\mathcal{E}^p$ is. (2) The members of $\mathcal{F}^p$ are linearly independent, for the members of $\mathcal{E}^p$ are. Points (1) and (2) together imply (B). ∎

Remark. The collection $\mathcal{F}^p$ is a generalization of the complete basis found by H. M. Reimann in [11]. Unfortunately, $\mathcal{F}^p$ restricted to Σ is not an orthonormal basis. Of course, any member $\mathcal{F}^p$ has unit length, but $\psi^p_{k,j}$ and $\alpha^p_{k-2,j}$ may not be perpendicular, for $(\psi^p_{k,j}|\alpha^p_{k-2,j})_{\sigma,p} = A_{p,k}\|\alpha^p_{k-2,j}\|^2_{\sigma,p}$.

Let us conclude this section with the following observation. Let $\omega \in L^{2,p}(\Sigma)$. Then comparing coefficients on each level of homogeneity we conclude that

$$\omega = \sum_{k=0}^{\infty}\sum_{j=1}^{\mu^p_k} u^p_{k,j}\eta^p_j + \sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k+1}} \widetilde{a}^p_{k,j}\alpha^p_{k,j} + \sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-2}_k} b^p_{k,j}\beta^p_j + \sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k-1}} \widetilde{c}^p_{k,j}\psi^p_{k,j}, \tag{2.1.7}$$

where $u^p_{k,j}$ and $b^p_{k,j}$ are defined by (2.1.3), whereas

$$\widetilde{c}^p_{k,j} = \frac{1}{C_{p,k}}\, c^p_{k,j}, \qquad \widetilde{a}^p_{k,j} = a^p_{k,j} - A_{p,k+2}\widetilde{c}^p_{k+2,j}, \tag{2.1.8}$$

i.e.,

$$\widetilde{c}^p_{k,j} = \sqrt{\frac{n+2k-4}{n+k-p-2}}\, c^p_{k,j}, \qquad \widetilde{a}^p_{k,j} = a^p_{k,j} - \sqrt{\frac{p+k}{n-p+k}}\, c^p_{k+2,j}.$$

Next by the estimate $2|st| \le s^2+t^2$, $s,t\in\mathbb{R}$, we see that (cf. [11, p. 169])

$$\Big(1-\sqrt{\frac{p+k}{n+2k}}\Big)(|\widetilde{a}^p_{k,j}|^2+|\widetilde{c}^p_{k+2,j}|^2) \le |a^p_{k,j}|^2+|c^p_{k+2,j}|^2 \le \Big(1+\sqrt{\frac{p+k}{n+2k}}\Big)(|\widetilde{a}^p_{k,j}|^2+|\widetilde{c}^p_{k+2,j}|^2).$$

Consequently, there exists a positive constant $M=M(n,p)$ (depending on p and n only) such that

$$\frac{1}{M}\,\|\omega\|^2_\sigma \le \sum_{k,j}|u^p_{k,j}|^2+\sum_{k,j}|\widetilde{a}^p_{k,j}|^2+\sum_{k,j}|b^p_{k,j}|^2+\sum_{k,j}|\widetilde{c}^p_{k,j}|^2 \le M\|\omega\|^2_\sigma. \tag{2.1.9}$$

2.2. *L^2-data Dirichlet boundary problem.* Suppose η is any p-form in the unit ball B. For any $0<t\le 1$ let η_t be the *t-dilatation* of η, i.e., $(\eta_t)_{i_1,\dots,i_p}(x)=\eta_{i_1,\dots,i_p}(tx)$. Clearly η_t is defined in the open ball $(1/t)B$. In particular, if η is a (p/k)-form then $\eta_t=t^k\eta$. We say that $\omega\in L^{2,p}(\Sigma)$ is the *L^2-boundary value* of η, and we write

$$\eta|\Sigma=\omega \quad \text{in } L^{2,p}(\Sigma),$$

if (1) $\eta_t\in L^{2,p}(\Sigma)$ for any $0\le t<1$ and (2) $\lim_{t\to1}\|\eta_t-\omega\|_{p,\sigma}=0$.

Fix $\omega\in L^{2,p}(\Sigma)$. Let $a^p_{k,j}$, $\widetilde{a}^p_{k,j}$, $b^p_{k,j}$, $c^p_{k,j}$, $\widetilde{c}^p_{k,j}$ and $u^p_{k,j}$ be defined as in (2.1.3) and (2.1.8). By (2.1.2) and (2.1.7), ω can be expressed as a sum of four series that converge in $L^{2,p}(\Sigma)$:

$$\omega=\eta+\widetilde{\alpha}+\beta+\psi=\eta+\alpha+\beta+\tau,$$

where

$$\eta=\sum_{k=0}^{\infty}\sum_{j=1}^{\mu^p_k}u^p_{k,j}\eta^p_{k,j},\quad \alpha=\sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k+1}}a^p_{k,j}\alpha^p_{k,j},\quad \widetilde{\alpha}=\sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k+1}}\widetilde{a}^p_{k,j}\alpha^p_{k,j},$$

$$\beta=\sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-2}_k}b^p_{k,j}\beta^p_{k,j},\quad \tau=\sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k-1}}c^p_{k,j}\tau^p_{k,j},\quad \psi=\sum_{k=0}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k-1}}\widetilde{c}^p_{k,j}\psi^p_{k,j}.$$

By Lemma 1.1.2 and the local description of d, δ and ε_ν it follows that each of the above series converges uniformly on compact sets in the unit ball B to a smooth form belonging to the kernel of $L_{a,b}$. In fact, α, $\widetilde{\alpha}$, η, β are even harmonic forms in B. Thus for any $0\le t<1$, $\eta_t=\sum_{k,j}u^p_{k,j}\eta^p_{k,j,t}$, $\alpha_t=\sum_{k,j}a^p_{k,j}\alpha^p_{k,j,t}$, etc., while the series converge uniformly on $\overline{B}$. Clearly,

$$\omega_t=\eta_t+\widetilde{\alpha}_t+\beta_t+\psi_t=\eta_t+\alpha_t+\beta_t+\tau_t. \tag{2.2.1}$$

On the other hand, $\omega_t\in L^{2,p}(\Sigma)$, so in view of (2.1.2) and (2.1.7), ω_t can be expressed as a sum of four series

$$\omega_t = \eta(t) + \widetilde{\alpha}(t) + \beta(t) + \psi(t) = \eta(t) + \alpha(t) + \beta(t) + \tau(t). \tag{2.2.2}$$

Let $u^p_{k,j}(t)$, $a^p_{k,j}(t)$ etc. denote the corresponding coefficients of $\eta(t)$, $\alpha(t)$ etc. At first sight it would seem that $u^p_{k,j}(t) = t^k u^p_{k,j}$, $a^p_{k,j}(t) = t^k a^p_{k,j}$ etc. But the inhomogeneity of $\tau^p_{k,j}$ and $\psi^p_{k,j}$ provents that from being the case. Comparing the coefficients of series in (2.2.1) and (2.2.2), we find

$$u^p_{k,j}(t) = t^k u^p_{k,j}, \quad b^p_{k,j}(t) = t^k b^p_{k,j}, \quad c^p_{k,j} = t^k c^p_{k,j} \quad \widetilde{c}^p_{k,j}(t) = t^k \widetilde{c}^p_{k,j},$$
$$a^p_{k,j}(t) = t^k a^p_{k,j} + t^k(1-t^2)c^p_{k+2,j}\sqrt{(p+k)(n+2k)}\, c(p,k+2)C^{-1}_{p,k+2},$$
$$\widetilde{a}^p_{k,j}(t) = t^k \widetilde{a}^p_{k,j} + t^k(1-t^2)\widetilde{c}^p_{k+2,j}\sqrt{(p+k)(n+2k)}\, c_L(p,k+2),$$

where as in (e), $c(p,k+2) = c_L(p,k+2) - c_\Delta(p,k+2)$. Moreover, we put

$$\alpha'(t) = \sum_{k,j} t^k a^p_{k,j}\alpha^p_{k,j}, \quad \widetilde{\alpha}'(t) = \sum_{k,j} t^k \widetilde{a}^p_{k,j}\alpha^p_{k,j}.$$

In the proof of the main theorem we will need to show that some series of p-forms has zero L^2-boundary value. To do this we will follow Reimann's approach (cf. [11, p. 171]).

LEMMA 2.2.1. *Let (z_k) be any sequence of real or complex numbers. Then*

$$(1-t^2)^2\sum_{k=0}^{\infty} t^{2k-2}k^2|z_k|^2 \le \sum_{k=0}^{\infty}|z_k|^2 \quad \textit{for any } 0 < t \le 1.$$

THEOREM 2.2.1 (Dirichlet boundary problem). *There exists a unique solution φ of $L_{a,b}\varphi = 0$ in the unit ball B with the boundary condition $\varphi|\Sigma = \omega$ in $L^{2,p}(\Sigma)$. The solution may be expressed as follows*:

$$\varphi = \eta + \alpha + \beta + \tau = \eta + \widetilde{\alpha} + \beta + \psi.$$

Proof. Let $\varphi' = \eta + \alpha + \beta + \tau$ and $\varphi'' = \eta + \widetilde{\alpha} + \beta + \psi$. As we have seen, φ' and φ'' satisfy the differential equation $L_{a,b}\varphi = 0$. We show that they have the same L^2-boundary value ω.

Standard arguments show that in $L^{2,p}(\Sigma)$ we have

$$\lim_{t\to 1}\eta(t) = \eta, \quad \lim_{t\to 1}\beta(t) = \beta, \quad \lim_{t\to 1}\psi(t) = \psi,$$
$$\lim_{t\to 1}\tau(t) = \tau, \quad \lim_{t\to 1}\alpha'(t) = \alpha, \quad \lim_{t\to 1}\widetilde{\alpha}'(t) = \widetilde{\alpha}.$$

Next we show that

$$\lim_{t\to 1}\alpha(t) = \alpha, \quad \lim_{t\to 1}\widetilde{\alpha}(t) = \widetilde{\alpha}.$$

Since $\lim_{t\to 1}\alpha'(t)=\alpha$ and $\lim_{t\to 1}\widetilde{\alpha}'(t)=\widetilde{\alpha}$, it suffices to prove that

$$\lim_{t\to 1}\Big(\sum_{k,j} t^k(1-t^2)c^p_{k+2,j}\sqrt{(p+k)(n+2k)}c(p,k+2)C^{-1}_{p,k+2}\alpha^p_{k,j}\Big)=0,$$

$$\lim_{t\to 1}\Big(\sum_{k,j} t^k(1-t^2)\widetilde{c}^p_{k+2,j}\sqrt{(p+k)(n+2k)}c_L(p,k+2)\alpha^p_{k,j}\Big)=0.$$

For brevity we will only compute the first limit; the second is analogous. For simplicity, let $D(t)$ and $D(t,k,j)$ denote the expression inside the bracket and its components, respectively. Clearly, there exists $N>0$ such that

$$c^p_{k+2,j}\sqrt{(p+k)(n+2k)}\,c(p,k+2)C^{-1}_{p,k+2}\le Nk.$$

Now by Lemma 2.2.1 we easily see that for any positive integer k_0,

$$\Big\|\sum_{k=k_0+1}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k+1}} D(t,k,j)\Big\|^2_{\sigma,p}\le N^2\sum_{k=k_0+1}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k+1}}|c^p_{k+2,j}|^2.$$

Take now any $\varepsilon>0$. There exists an integer $K>0$ such that

$$\sum_{k=K+1}^{\infty}\sum_{j=1}^{\mu^{p-1}_{k+1}}|c^p_{k+2,j}|^2<(\varepsilon/N)^2.$$

Since $\lim_{t\to 1}\|D(t,k,j)\|_{\sigma,p}=0$, we obtain

$$\limsup_{t\to 1}\|D(t)\|^2_{\sigma,p}\le\lim_{t\to 1}\Big(\sum_{k=0}^{K}\sum_{j=1}^{\mu^{p-1}_{k+1}}\|D(t,k,j)\|_{\sigma,p}+\varepsilon\Big)=\varepsilon.$$

Summarizing,

$$\lim_{t\to 1}\varphi'_t=\lim_{t\to 1}\varphi''_t=\omega\quad\text{ in } L^{2,p}(\Sigma).$$

Now we will show that the solution is unique. Suppose φ is a smooth p-form in $\overline{B}$ with $L\varphi=L_{a,b}\varphi=0$ and $\varphi|\Sigma=0$. We will show that $\varphi=0$. Since $\varphi|\Sigma=0$, we have $\iota_\nu\varphi|\Sigma=0$. Therefore, applying the integral formula (1.2.3) we obtain

$$\int_B(L\varphi)\varphi\,dx=a\int_B(\delta\varphi)^2\,dx+b\int_B(d\varphi)^2\,dx.$$

This implies that φ is both closed and co-closed, so harmonic. Hence each coefficient of φ is a harmonic function in B vanishing on Σ, thus it is the zero function, by the maximum principle. So $\varphi=0$ in $\overline{B}$.

Suppose now that φ is a solution to the Dirichlet problem with zero L^2-boundary value. We have

$$\lim_{t\to 1}\|\varphi_t\|^2_{\sigma,p}=0.$$

Since for each $0 < t < 1$, φ_t is smooth up to the boundary and satisfies $L\varphi_t = 0$, it can be expressed as a sum of four series from the theorem: $\varphi_t = \eta(t) + \widetilde{\alpha}(t) + \beta(t) + \psi(t)$. Denote their coefficients by $u^p_{k,j}(t)$, $\widetilde{\alpha}^p_{k,j}(t)$ etc., respectively. Then by (2.1) we obtain

$$\lim_{t\to 1}\Big(\sum_{k,j}|u^p_{k,j}(t)|^2 + \sum_{k,j}|\widetilde{a}^p_{k,j}(t)|^2 + \sum_{k,j}|b^p_{k,j}(t)|^2 + \sum_{k,j}|\widetilde{c}^p_{k,j}(t)|^2\Big) = 0.$$

Take any $0 < R' < R < 1$. We show that $|\varphi|'_\infty = |\varphi|^{R'}_\infty = 0$, where $|\cdot|^{R'}_\infty$ is the norm of uniform convergence, i.e.,

$$|\varphi|'_\infty = |\varphi|^{R'}_\infty = \sup_{i_1,\ldots,i_p}\sup_{|x|\le R'}|\varphi_{i_1,\ldots,i_p}(x)|.$$

We have

$$|\varphi|'_\infty \le |\varphi - \varphi_t|'_\infty + |\eta(t)|'_\infty + |\widetilde{\alpha}(t)|'_\infty + |\beta(t)|'_\infty + |\psi(t)|'_\infty. \tag{2.2.3}$$

Since φ is uniformly continuous in the ball $|x| \le R'$, $|\varphi - \varphi_t|'_\infty$ tends to zero. It remains to estimate $|\eta(t)|'_\infty$, $|\widetilde{\alpha}(t)|'_\infty$, $|\beta(t)|'_\infty$, $|\psi(t)|'_\infty$.

We will use the notation from the end of §1.1. Fix a multi-index $(i_1,\ldots,i_p)$. Take any coefficient $\eta^p_{k,j;i_1,\ldots,i_p}$ of $\eta^p_{k,j}$ and put $g_{k,j} = \eta^p_{k,j;i_1,\ldots,i_p}$. Then $(g_{k,j})$ is a series of spherical harmonics and $\|g_{k,j}\|_\sigma \le \|\eta^p_{k,j}\|_{\sigma,p} = 1$. So for each $|x| < R'$, by Lemma 1.1.1 we have

$$\begin{aligned}\Big|\sum_{k,j}u^p_{k,j}(t)g_{k,j}(x)\Big| &\le \sum_{k,j}|u^p_{k,j}(t)g_{k,j}(x)|\\ &\le C(R)\Big(\sum_{k,j}|u^p_{k,j}(t)|^2\Big)^{1/2}\Big(\sum_{k,j}\varepsilon^{2k}\Big)^{1/2}.\end{aligned}$$

This means that

$$\lim_{t\to 1}|\eta(t)|'_\infty \le C(R)\Big(\sum_{k,j}\varepsilon^{2k}\Big)^{1/2}\lim_{t\to 1}\Big(\sum_{k,j}|u^p_{k,j}(t)|^2\Big)^{1/2} = 0.$$

The same arguments show that $\lim_{t\to 1}|\widetilde{\alpha}(t)|'_\infty = 0$ and $\lim_{t\to 1}|\beta(t)|'_\infty = 0$.

To prove that $\lim_{t\to 1}|\psi(t)|'_\infty = 0$ it suffices to apply (2.1.6) and preceding arguments (Lemma 1.1.1 with the sequences $w_k = 1$ and $w_k = k$, resp.).

Consequently, $|\varphi|'_\infty = 0$, by (2.2.3). Since $0 < R' < 1$ was arbitrary, φ is the zero form. ∎

As announced in the Introduction, under the natural duality given by the canonical inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$, we may identify 1-forms with vector fields. Then we have

$$d \simeq \operatorname{grad}, \qquad \delta \simeq -\operatorname{div}, \qquad S^\star S \simeq -L_{(n-1)/n,1/2},$$

where grad, div and $S^\star S$ denote the gradient, divergence and the Ahlfors–Laplace operator, respectively. Denote by $\vec{\mathcal{H}}^k$ the space of vector fields in

$\mathbb{R}^n$ with coefficients from $\mathcal{H}^k$ and by $\mathcal{L}^2(\Sigma)$ the space of vector fields on Σ whose coefficients belong to $L^2(\Sigma)$. Moreover, let $\langle\cdot|\cdot\rangle_\sigma$ denote the inner product in $\mathcal{L}^2(\Sigma)$ induced from $L^2(\Sigma)$, i.e.,

$$\langle V|W\rangle_\sigma = \int_\Sigma \langle V(x), W(x)\rangle \, d\sigma(x)$$

for vector fields $V, W \in \mathcal{L}^2(\Sigma)$. Let, as before, $(h_{k,j})$ be an L^2-orthonormal basis of $\mathcal{H}^k$. The spaces $\mathcal{Q}^k$, $\mathcal{M}^k$ and $\mathcal{N}^k$, and the vector fields q_j^k, m_j^k, n_j^k and p_j^k constructed in [11] correspond to the following:

$$\begin{aligned}
\mathcal{Q}^k &= \{H = (h_1, \dots, h_n) \in \vec{\mathcal{H}}^k : \operatorname{div} H = 0,\ \langle H(x), x\rangle = 0\} \simeq \chi^0_{1,k},\\
\mathcal{M}^k &= \{H = \operatorname{grad} h \in \vec{\mathcal{H}}^k : h \in \mathcal{H}_{k+1}\} \simeq d\chi^0_{0,k+1},\\
\mathcal{N}^k &= \{H = (n+2k-4)xh - r^2 \operatorname{grad} h \in \vec{\mathcal{H}}^k : h \in \mathcal{H}_{k-1}\} \simeq I_\Delta(1,k)\chi^0_{0,k-1},\\
q_j^k &\simeq \eta^1_{k,j},\\
m_j^k &= \frac{1}{\sqrt{(k+1)(n+2k)}} \operatorname{grad} h_{k+1,j} \simeq \alpha^1_{k,j},\\
n_j^k &= \sqrt{\frac{n+2k-4}{n+k-3}}\, x h_{k-1,j} - \frac{r^2}{\sqrt{(n+2k-4)(n+k-3)}} \operatorname{grad} h_{k-1,j} \simeq \gamma^1_{k,j},\\
p_j^k &= x h_{k-1,j} + c_{S^\star S}(n,k)(1-r^2) \operatorname{grad} h_{k-1,j} \simeq \psi^1_{k,j}.
\end{aligned}$$

The constant $c_{S^\star S}(n,k)$ is equal to $c_L(1,k)$ with $L = L_{(n-1)/n,1/2}$. The following theorem ([11, Theorem 3]) is now a direct consequence of Theorem 2.2.1 and the identity $\beta^1_{k,j} = 0$.

THEOREM 2.2.2 (Reimann). *Given $V \in \mathcal{L}^2(\Sigma)$, $n \geq 3$, there exists a unique solution Φ of $S^\star S\Phi = 0$ in the ball B with L^2-boundary value V:*

$$\lim_{t\to 1} \int_\Sigma |\Phi(tx) - V(x)|^2 \, d\sigma(x) = 0.$$

This solution is given by the formula

$$\Phi = \sum_{k=0}^{\infty} \sum_{j=1}^{d_{k+1}} \widetilde{a}_{k,j} m_j^k + \sum_{k=1}^{\infty} \sum_{j=1}^{d_{k-1}} \widetilde{c}_{k,j} p_j^k + \sum_{k=1}^{\infty} \sum_{j=1}^{d_k} u_{k,j} q_j^k,$$

with

$$\begin{aligned}
\widetilde{a}_{k,j} &= \langle V|m_j^k\rangle_\sigma - \sqrt{\frac{k+1}{n+k-1}}\, \langle V|n_j^{k+2}\rangle_\sigma,\\
\widetilde{c}_{k,j} &= \sqrt{\frac{n+2k-4}{n+k-3}}\, \langle V|n_j^k\rangle_\sigma,\\
u_{k,j} &= \langle V|q_j^k\rangle_\sigma.
\end{aligned}$$

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