BIPARTITE COALGEBRAS AND A REDUCTION FUNCTOR FOR CORADICAL SQUARE COMPLETE COALGEBRAS

by

JUSTYNA KOSAKOWSKA and DANIEL SIMSON (Toruń)

Abstract. Let $C$ be a coalgebra over an arbitrary field $K$. We show that the study of the category $C$-Comod of left $C$-comodules reduces to the study of the category of (co)representations of a certain bicomodule, in case $C$ is a bipartite coalgebra or a coradical square complete coalgebra, that is, $C = C_1$, the second term of the coradical filtration of $C$. If $C = C_1$, we associate with $C$ a $K$-linear functor $H_C : C$-Comod $\rightarrow H_C$-Comod that restricts to a representation equivalence $H_C : C$-comod $\rightarrow H_C$-comod$^\bullet$, where $H_C$ is a coradical square complete hereditary bipartite $K$-coalgebra such that every simple $H_C$-comodule is injective or projective. Here $H_C$-comod$^\bullet$ is the full subcategory of $H_C$-comod whose objects are finite-dimensional $H_C$-comodules with projective socle having no injective summands of the form $\left[ S(i') \atop 0 \right]$ (see Theorem 5.11). Hence, we conclude that a coalgebra $C$ with $C = C_1$ is left pure semisimple if and only if $H_C$ is left pure semisimple. In Section 6 we get a diagrammatic characterisation of coradical square complete coalgebras $C$ that are left pure semisimple. Tameness and wildness of such coalgebras $C$ is also discussed.

1. Introduction. Throughout this paper we fix an arbitrary field $K$ and we use the coalgebra representation theory notation and terminology introduced in [14], [29]–[35]. The reader is referred to [1], [2], [12], [27], [37], and [38] for the representation theory terminology and notation, and to [16], [39] for the coalgebra and comodule terminology. In particular, given a finite-dimensional $K$-algebra $R$, we denote by mod$(R)$ the category of all finite-dimensional $R$-modules.

Let $C$ be a $K$-coalgebra with comultiplication $\Delta$ and counit $\varepsilon$. We recall that a left $C$-comodule is a $K$-vector space $X$ together with a $K$-linear map $\delta_X : X \rightarrow C \otimes X$ such that $(\Delta \otimes \text{id}_X) \delta_X = (\text{id}_C \otimes \delta_X) \delta_X$ and $(\varepsilon \otimes \text{id}_X) \delta_X$ is the canonical isomorphism $X \cong K \otimes X$, where $\otimes = \otimes_K$. Given a left $C$-comodule $X$, we denote by $X_0 = \text{soc} X$ the socle of $X$, that is, the sum of all simple $C$-subcomodules of $X$.
A $K$-linear map $f : X \rightarrow Y$ between two left $C$-comodules $X$ and $Y$ is a $C$-comodule homomorphism if $\delta_Y f = (\text{id}_C \otimes f)\delta_X$. The $K$-vector space of all $C$-comodule homomorphisms from $X$ to $Y$ is denoted by $\text{Hom}_C(X, Y)$. The $K$-algebra of all $C$-comodule endomorphisms of $X$ is denoted by $\text{End}_C X$.

We denote by $C$-$\text{Comod}$ the category of all left $C$-comodules, and by $C$-$\text{comod}$ the full subcategory of $C$-$\text{Comod}$ formed by $C$-comodules of finite $K$-dimension.

We recall that a $K$-coalgebra $C$ is semisimple (resp. hereditary) if $\text{Ext}^1_C(M, N) = 0$ (resp. $\text{Ext}^2_C(M, N) = 0$) for all $M$ and $N$ in $C$-$\text{Comod}$, or equivalently, if $M = \text{soc} M$ for all $M$ in $C$-$\text{Comod}$ (resp. if epimorphic images of injective $C$-comodules are injective $C$-comodules). A $K$-coalgebra $C$ is said to be indecomposable (or connected) if $C$ is not a product of two subcoalgebras, or equivalently, if $C$-$\text{Comod}$ is not a direct sum of two non-trivial subcategories.

Given a coalgebra $C$, we denote by $C_0 \subseteq C_1 \subseteq \cdots \subseteq C$ the coradical filtration of $C$, where $C_0 = \text{soc} C$ (or equivalently, the sum of all simple subcoalgebras of $C$), $C_1 = C_0 \wedge C_0$ is the wedge of two copies of $C_0$, and $C_{m+1} = C_0 \wedge C_m$ for $m \geq 1$.

We call $C$ basic if there is a decomposition $\text{soc} C = \bigoplus_{j \in I_C} S(j)$ such that $\{S(j); j \in I_C\}$ is a complete set of pairwise non-isomorphic simple left $C$-comodules (see [4], [6], [26] and [29]).

One of the aims of this paper is to study the comodule categories and the valued Gabriel quiver of the following class of coalgebras that are topologically dual (see [29]) to the class of (Jacobson) radical square zero algebras.

**Definition 1.1.** A $K$-coalgebra $C$ is defined to be coradical square complete if $C = C_1 = C_0 \wedge C_0$.

Following an idea of Gabriel [10] (see also [2, Section X.2]), we reduce the study of $C$-comodules over any coradical square complete coalgebra $C$ to the study of comodules over a coradical square complete hereditary coalgebra $H_C$ which is a bipartite coalgebra in the sense of Definition 2.0 below. Moreover, every simple subcomodule of $H_C$ is projective or injective. This is one of the motivations for our investigations in this paper, because the representation theory of hereditary coalgebras is well understood by a reduction to the study of nilpotent representations of quivers or $K$-species (see [14], [20], [29]–[35]), and therefore we get an efficient tool for the study of $C$-comod.

We recall from [1], [2], [10], [12], [15], [27], [37], and [38] that triangular matrix algebras play an important role in the representation theory of finite-dimensional algebras. In particular, we know from [10] and [2, Section X.2] that the representation theory of radical square zero algebras of finite $K$-dimension reduces to the representation theory of hereditary triangular matrix algebras. In Section 2 we follow this idea and, in analogy to
triangular matrix algebras and bipartite rings [27, Section 17.4], we introduce a concept of a bipartite $K$-coalgebra
\[
H = \begin{bmatrix}
H' & H'UH'' \\
0 & H''
\end{bmatrix},
\]
where $(H', \Delta', \varepsilon')$ and $(H'', \Delta'', \varepsilon'')$ are $K$-coalgebras and $H'UH''$ is a $H'$-$H''$-bicomodule, that is, $H'UH''$ is a left $H'$-comodule $(U, \delta_U : U \to H' \otimes U)$ equipped with a right $H''$-comodule structure given by a right $H''$-comodule homomorphism $\delta''_U : U \to U \otimes H''$, which is a homomorphism of left $H'$-comodules. Moreover, given $H$ as above, we define an equivalence of categories between $H$-$\text{Comod}$ and the category $\text{Rep}_{\square}(H'UH'')$ of (co)representations of $H'UH''$.

In Section 4, following Gabriel [10], with each coradical square complete coalgebra $C$ we associate a coradical square complete hereditary bipartite $K$-coalgebra $H_C$ and a $K$-linear functor
\[
(1.2) \quad \mathbb{H} : C-\text{Comod} \to H_C-\text{Comod}.
\]
We prove in Theorem 5.11 that $\mathbb{H}$ is full, carries injectives to injectives, does not vanish on non-zero comodules, but vanishes on the $C$-comodule homomorphisms $f : X \to Y$ such that $f(\text{soc } X) = 0$. Moreover, $\mathbb{H}$ restricts to a representation equivalence of categories (i.e. it is full, dense, and reflects isomorphisms, see [27], [28], and [38])
\[
(1.3) \quad \mathbb{H}_C : C-\text{comod} \to H_C-\text{comod}_{sp}^\bullet,
\]
where $H_C-\text{comod}_{sp}^\bullet$ is the full subcategory of $H_C-\text{comod}$ whose objects are the finite-dimensional $H_C$-comodules with projective socle having no injective summands of the form $[S(i')]$ (see Theorem 5.11). It follows that $C$ is left pure semisimple if and only if $H_C$ is. Hence, by applying [14], [20] and [29], we get in Section 6 a diagrammatic characterisation of coradical square complete coalgebras $C$ that are left pure semisimple.

Following an idea of trivial extension algebra (see [2] and [13]), and in connection with the reduction functor (1.2), we study in Section 4 the trivial extension coalgebra $D \ltimes _D U_D$ (see (4.8)) of a given coalgebra $D$ by a $D$-$D$-bicomodule $D U_D$, the repetitive coalgebra $R(D, D U_D)$ (see (4.15)), and the covering functor (see (4.17))
\[
f : R(D, D U_D)-\text{Comod} \to (D \ltimes _D U_D)-\text{Comod}
\]
induced by the canonical coalgebra surjection
\[
f : R(D, D U_D) \to D \ltimes _D U_D.
\]
Also we complete the results given in [3], [14], [17], [32], and [41] by presenting three alternative descriptions of the left valued Gabriel quiver of a
given basic coalgebra

\[ C = \bigoplus_{a \in I_C} E(a), \]

with indecomposable left coideals \( E(a), a \in I_C \). The descriptions are given by the \( F_a-F_b \)-bimodule isomorphisms (see (3.6)),

\[(1.4) \quad \text{Hom}_{F_a} (\text{Ext}^1_C (S(a), S(b)), F_a) \to \text{Irr}_C (E(b), E(a)) \to a(C_1/C_0)b,\]

where \( S(j) = \text{soc} E(j) \) and \( F_j = \text{End}_C S(j) \) for \( j \in I_C \).

Throughout this paper, by a quiver we mean a pair \( Q = (Q_0, Q_1) \), where \( Q_0 \) is the set of vertices of \( Q \) and \( Q_1 \) is the set of arrows of \( Q \). By a valued quiver we mean a pair \( (Q, d) \), where \( Q \) is a quiver such that each arrow \( \beta \in Q_1 \) is equipped with a pair \( (d'_\beta, d''_\beta) \) of positive integers; we visualise \( \beta \) as the valued arrow

\[ a \xrightarrow{(d'_\beta, d''_\beta)} b. \]

If \( d'_\beta = d''_\beta = 1 \), then we simply write \( a \to b \) instead of \( a \xrightarrow{(d'_\beta, d''_\beta)} b \).

By a valued quiver dual to \( (Q, d) \) we mean the valued quiver \( (Q^\circ, d^\circ) \), where \( Q^\circ_0 = Q_0 \) and, for each valued arrow \( a \xrightarrow{(d'_\beta, d''_\beta)} b \) in \( (Q, d) \), we define the unique valued arrow \( \beta^\circ \) in \( (Q^\circ, d^\circ) \) to be \( b \xrightarrow{(d''_\beta, d'_\beta)} a \).

Let \( X \) be a right \( C \)-comodule and \( Y \) be a left \( C \)-comodule. We recall from [9] that the cotensor product \( X \square Y \) is the \( K \)-vector space

\[(1.5) \quad X \square Y = \text{Ker} (X \otimes Y \xrightarrow{\delta_X \otimes \text{id}_Y - \text{id}_X \otimes \delta_Y} X \otimes C \otimes Y).\]

It is known that \( X \square C \cong X, C \square Y \cong Y \), the functors

\[ X \square - : \text{C-Comod} \to \text{mod} K \quad \text{and} \quad - \square Y : \text{Comod-C} \to \text{mod} K \]

are left exact, commute with arbitrary direct sums, and there is a functorial isomorphism

\[ X \square Y \cong \text{Hom}_C (Y^*, X) \]

for any \( X \) in \( \text{Comod-C} \) and any \( Y \) in \( C \)-comod, where \( Y^* = \text{Hom}_K (Y, K) \) is equipped with the \( K \)-dual right \( C \)-comodule structure (see [8] and [39]).

2. Bipartite coalgebras and representations of bicomodules. In this section we introduce a concept of a bipartite coalgebra (see (2.1)) in an analogy with the notion of a (generalised) triangular matrix algebra (see [1, Appendix 2.7], [27], and [38, Section VX.1]). We prove that, for a bipartite coalgebra \( H \), the category \( H\text{-Comod} \) is equivalent to the category of (co)representations of the bicomodule defining \( H \).

Bipartite coalgebras. In analogy with [1, Appendix 2.7], [27, Section 17.4], and [38, Section VX.1], we introduce the following definition.
Definition 2.0. Let $H'$ and $H''$ be $K$-coalgebras, and let $H'UH''$ be a non-zero $H'-H''$-bicomodule. We associate with $H'UH''$ the bipartite $K$-coalgebra

\[(2.1)\]

\[
H = \begin{bmatrix}
H' & H'UH'' \\
0 & H''
\end{bmatrix}
\]

consisting of all formal matrices $h = \begin{bmatrix} h' & u \\ 0 & h'' \end{bmatrix}$, where $h' \in H'$, $h'' \in H''$ and $u \in U$. We make the following identification:

\[(2.2)\]

\[
H \otimes H \equiv \begin{bmatrix}
H' \otimes H' & H' \otimes U & H' \otimes H'' \\
U \otimes H' & U \otimes U & U \otimes H'' \\
H'' \otimes H' & H'' \otimes U & H'' \otimes H''
\end{bmatrix}
\]

The comultiplication $\Delta : H \rightarrow H \otimes H$ of $H$ and the counit $\varepsilon : H \rightarrow K$ of $H$ are defined by the following formulae:

\[(2.3)\]

\[
\Delta(h) = \Delta'(h') + \Delta''(h'') + \delta_U'(u) + \delta_U''(u)
\]

\[
\varepsilon(h) = \varepsilon'(h') + \varepsilon''(h'').
\]

It is easy to check that $H$ is a $K$-coalgebra, the $K$-subspaces

\[(2.4)\]

\[
\begin{bmatrix} H' \\ 0 \end{bmatrix} \equiv \begin{bmatrix} H' \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U \\ H'' \end{bmatrix} \equiv \begin{bmatrix} 0 & H'UH'' \\ H'' \end{bmatrix}
\]

of $H$ are left coideals and, under the above identification, the left $H$-comodule $H$ has a direct sum decomposition

\[(2.5)\]

\[
H = \begin{bmatrix} H' & H'UH'' \\
0 & H''
\end{bmatrix} = \begin{bmatrix} H' \\ 0 \end{bmatrix} \oplus \begin{bmatrix} U \\ H'' \end{bmatrix}
\]

Moreover, the canonical projection $\pi : H \rightarrow H' \oplus H''$, defined by the formula $\pi \begin{bmatrix} h' & u \\ 0 & h'' \end{bmatrix} = (h', h'')$, is a $K$-coalgebra homomorphism and induces a faithful $K$-linear embedding

\[(2.6)\]

\[
\pi^\circ : H\text{-Comod} \rightarrow (H' \oplus H'')\text{-Comod}
\]

associating to each left $H$-comodule $(X, \delta_X)$ the left $(H' \oplus H'')$-comodule $(X, \widehat{\delta}_X)$ with comultiplication $\widehat{\delta}_X = (\pi \otimes \text{id}_X) \circ \delta_X : X \rightarrow (H' \oplus H'') \otimes X$. Denote by $\pi_{H'} : H \rightarrow H'$ and $\pi_{H''} : H \rightarrow H''$ the obvious projections.

**Representations of bicomodules.** In analogy with [1, Appendix 2.7] and [38, Section VX.1], we introduce the following definition.
DEFINITION 2.7. Let $H'$ and $H''$ be $K$-coalgebras. Given an $H'-H''$-bicomodule $H'U_{H''}$, we define the category $\text{Rep}_{\square}(H'U_{H''})$ of left (co)representations of $H'U_{H''}$ as follows.

(a) The objects of $\text{Rep}_{\square}(H'U_{H''})$ are triples $(X', X'', \varphi)$, where $X'$ is a left $H'$-comodule, $X''$ is a left $H''$-comodule and $\varphi : X' \rightarrow U \square X''$ is a homomorphism of left $H'$-comodules.

(b) A morphism from $(X', X'', \varphi)$ to $(Y', Y'', \psi)$ in $\text{Rep}_{\square}(H'U_{H''})$ is a pair $(f', f'')$, where $f' \in \text{Hom}_{H'}(X', Y')$, $f'' \in \text{Hom}_{H''}(X'', Y'')$ and $(\text{id}_U \square f'') \varphi = \psi f'$. The composition of morphisms in $\text{Rep}_{\square}(H'U_{H''})$ is componentwise.

(c) The representation $(X', X'', \varphi)$ is called finite-dimensional if the comodules $X'$ and $X''$ are of finite $K$-dimension.

(d) We denote by $\text{rep}_{\square}(H'U_{H''})$ the full subcategory of $\text{Rep}_{\square}(H'U_{H''})$ formed by the finite-dimensional representations.

It is clear that $\text{Rep}_{\square}(H'U_{H''})$ and $\text{rep}_{\square}(H'U_{H''})$ are abelian $K$-categories. We show below that there is an equivalence of categories $H\text{-Comod} \cong \text{Rep}_{\square}(H'U_{H''})$. For this, we define a pair of $K$-linear functors

$$\text{rep}_{\square}(H'U_{H''}) \xrightarrow{\Phi} \text{Rep}_{\square}(H'U_{H''})$$

as follows.

The functor $\Phi$. Before we define the functor $\Phi$ (see (2.11)), we need a preparation. Given a left $H$-comodule $(X, \delta_X)$, we decompose the $K$-vector space $X$ as $X = X' \oplus X''$, where

$$X' = \hat{\delta}_X^{-1}(H' \otimes X) \quad \text{and} \quad X'' = \hat{\delta}_X^{-1}(H'' \otimes X).$$

It is easy to see that $X' = (X', \hat{\delta}_X' = (\hat{\delta}_X)|_{X'})$ and $X'' = (X'', \hat{\delta}_X'' = (\hat{\delta}_X)|_{X''})$ are a left $H'$-comodule and a left $H''$-comodule, respectively. We denote by $\tilde{\varphi} : X \rightarrow U \otimes X''$ the composite $K$-linear map

$$X \xrightarrow{\delta_X} H \otimes X \xrightarrow{\pi_U \otimes \pi_{X''}} U \otimes X'',$$

where $\pi_U : H \rightarrow U$ is the canonical projection defined by $\pi_U \begin{bmatrix} h' & u \end{bmatrix} = u$, and $\pi_{X''} : X \rightarrow X''$ is the obvious projection.

LEMMA 2.10. If $\tilde{\varphi} : X \rightarrow U \otimes X''$ is the map defined above then $\text{Im} \tilde{\varphi} \subseteq U \square X''$.

Proof. Note that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\delta_X} & H \otimes X \\
\downarrow{\delta_X} & & \downarrow{\text{id} \otimes \delta_X} \\
H \otimes X & \xrightarrow{\Delta \otimes \text{id}} & H \otimes H \otimes X \\
\downarrow{\delta_X} & & \downarrow{\text{id} \otimes \delta_X} \\
U \otimes H \otimes X & \xrightarrow{\text{id} \otimes \pi_{H''} \otimes \pi_{X''}} & U \otimes H'' \otimes X''
\end{array}$$
is commutative. Indeed, by the definition of $\tilde{\delta}_X$, the right square commutes. Moreover, $(\text{id} \otimes \delta_X)\delta_X = (\Delta \otimes \text{id})\delta_X$, because $X$ is a left $H$-comodule.

The commutativity of this diagram yields

$$(\text{id} \otimes \tilde{\delta}_X)\tilde{\varphi} = (\pi_U \otimes \pi_{H''} \otimes \pi_X)(\Delta \otimes \text{id})\delta_X.$$ 

Since the definition (2.3) of $\Delta$ yields $(\pi_U \otimes \pi_{H''})(\Delta = \delta_U' \pi_U$, we obtain

$$(\text{id} \otimes \tilde{\delta}_X)\tilde{\varphi} = (\pi_U \otimes \pi_{H''} \otimes \pi_X')(\Delta \otimes \text{id})\delta_X = ((\pi_U \otimes \pi_{H''})(\Delta \otimes \pi_X')\delta_X = (\delta_U' \pi_U \otimes \pi_X')\delta_X = (\delta_U' \otimes \text{id})\delta_X \tilde{\varphi}.$$ 

Hence, the required inclusion $\text{Im} \tilde{\varphi} \subseteq U \Box X''$ follows. ■

Denote by $\varphi : X' \to U \otimes X''$ the composite $K$-linear map

$$X' \hookrightarrow X \xrightarrow{\delta_X} H \otimes X \xrightarrow{\pi_U \otimes \pi_X'} U \otimes X''.$$ 

By Lemma 2.10, we have $\text{Im} \varphi \subseteq U \Box X'' \subseteq U \otimes X''$. Now we show that $\varphi$ is a homomorphism of left $H'$-comodules. Put $i_{X'} : X' \hookrightarrow X$ and note that

$$(\delta_U' \otimes \text{id})\varphi = (\delta_U' \otimes \text{id})(\pi_U \otimes \pi_X')\delta_X i_{X'} = \delta_U'(\pi_U \otimes \text{id})(\delta_U \otimes \text{id})(\delta_U i_{X'})$$

$$= (\pi_{H'} \otimes \pi_U)(\Delta \otimes \text{id})(\delta_U \otimes \text{id})(\delta_X i_{X'})$$

$$= (\pi_{H'} \otimes \pi_U)(\Delta \otimes \text{id})(\delta_U \otimes \text{id})(\delta_X i_{X'})$$

$$= (\pi_{H'} \otimes \tilde{\varphi})\delta_X i_{X'} = (\text{id} \otimes \tilde{\varphi})(\pi_{H'} \otimes \text{id})\delta_X i_{X'} = (\text{id} \otimes \varphi)\tilde{\delta}_X,$$

that is, $\varphi$ is a homomorphism of left $H'$-comodules.

To define the functor $\Phi$, we denote by $\varphi_X : X' \to U \Box X''$ the unique factorisation of $\varphi$ through the embedding $U \Box X'' \subseteq U \otimes X''$. It follows that $\varphi_X$ is a homomorphism of left $H'$-comodules and therefore $(X', X'', \varphi_X)$ is an object of the category $\text{Rep}_{\Box}(H' U_{H''})$. We set

$$(2.11) \quad \Phi(X) = (X', X'', \varphi_X).$$

Let $f : X \to Y$ be a homomorphism of left $H$-comodules, and let $X = X' \oplus X''$, $Y = Y' \oplus Y''$ be the decompositions defined by (2.9), where $X'$, $Y'$ are left $H'$-comodules and $X''$, $Y''$ are left $H''$-comodules. It is easy to see that $f(X') \subseteq Y'$ and $f(X'') \subseteq Y''$. Then the restrictions $f|_{X'}$ and $f|_{X''}$ induce $K$-linear maps $f' : X' \to Y'$ and $f'' : X'' \to Y''$, respectively. A straightforward calculation shows that $f'$ and $f''$ are homomorphisms of left $H'$-comodules and $H''$-comodules, respectively, such that the diagram

$$\begin{array}{ccc}
X' & \xrightarrow{\varphi_X} & U \Box X'' \\
f' \downarrow & & \downarrow \text{id}_U \otimes f'' \\
Y' & \xrightarrow{\varphi_Y} & U \Box Y''
\end{array}$$

in $H'$-Comod is commutative, that is, $(f', f'') : (X', X'', \varphi_X) \to (Y', Y''$, $\varphi_Y)$ is a morphism in the category $\text{Rep}_{\Box}(H' U_{H''})$. We define $\Phi(f) : \Phi(X) \to \Phi(Y)$
by setting $\Phi(f) = (f', f'').$ It is clear that we have defined a $K$-linear, faithful
and exact functor $\Phi : H\text{-Comod} \to \text{Rep}_{\square}(H'U_{H''})$.

**Example 2.12.** Let $H$ be a bipartite algebra of the form (2.1). Consider
the left $H$-comodules $[ [H']_0 ]$ and $[ [U_{H''}] ]$. To illustrate the definition of $\Phi,$
we compute the representations $\Phi([ [H']_0 ])$ and $\Phi([ [U_{H''}] ]).$ By (2.3) and (2.9),
we get $\Phi([ [H']_0 ] = (H', 0, 0)$ and $\Phi([ [U_{H''}] ] = (U, H'', \varphi).$ By the above con-
siderations and the definition of $\Phi,$ $\varphi = \delta'_U$ defines the right $H''$-comodule
structure on $U.$

The functor $\Psi.$ The functor $\Psi$ in (2.8) is defined by setting, for each
object $(X', X'', \varphi)$ in $\text{Rep}_{\square}(H'U_{H''}),$

$$\Psi(X', X'', \varphi) = (X, \delta_X),$$

where $X = X' \oplus X''$ and $\delta_X : X \to H \otimes X$ is the $K$-linear map defined by

$$\delta_X(x', x'') = \begin{bmatrix} \delta_{X'}(x') & \varphi(x') \\ 0 & \delta_{X''}(x'') \end{bmatrix} \in \begin{bmatrix} H' \otimes X' & H'U_{H''} \otimes X'' \\ 0 & H'' \otimes X'' \end{bmatrix} \subseteq H \otimes X.$$

Here we make the following identification of $K$-vector spaces:

$$H \otimes X = \begin{bmatrix} H' & H'U_{H''} \\ 0 & H'' \end{bmatrix} \otimes (X' \oplus X'')$$

$$\equiv \begin{bmatrix} H' \otimes (X' \oplus X'') & H'U_{H''} \otimes (X' \oplus X'') \\ 0 & H'' \otimes (X' \oplus X'') \end{bmatrix}.$$  

Now, we show that $(X, \delta_X)$ is a left $H$-comodule. The definition of $\delta_X$
yields

$$(\text{id}_H \otimes \delta_X) \circ \delta_X(x', x'') = (\text{id}_H \otimes \delta_X) \circ \begin{bmatrix} \delta_{X'}(x') & \varphi(x') \\ 0 & \delta_{X''}(x'') \end{bmatrix}$$

$$= \begin{bmatrix} (\text{id}_H \otimes \delta_X)\delta_{X'}(x') & (\text{id}_H \otimes \delta_X)\varphi(x') \\ 0 & (\text{id}_H \otimes \delta_X)\delta_{X''}(x'') \end{bmatrix}$$

$$= \begin{bmatrix} ((\Delta_{H'} \otimes \text{id}_{X'})\delta_{X'}(x'), (\text{id}_{H'} \otimes \varphi)\delta_{X'}(x')) & (\text{id}_{U} \otimes \delta_{X''})\varphi(x') \\ 0 & (\Delta_{H''} \otimes \delta_{X''})\delta_{X''}(x'') \end{bmatrix} = a.$$  

Since $X'$ is a left $H'$-comodule and $X''$ is a left $H''$-comodule, and $\varphi$ is
a homomorphism of $H'$-comodules with $\text{Im} \varphi \subseteq U \square X''$, it follows that

$$a = \begin{bmatrix} ((\Delta_{H'} \otimes \text{id}_{X'})\delta_{X'}(x'), (\text{id}_{U} \otimes \varphi)\delta_{X''}(x')) & (\text{id}_{U} \otimes \delta_{X''})\varphi(x') \\ 0 & (\Delta_{H''} \otimes \delta_{X''})\delta_{X''}(x'') \end{bmatrix}$$

$$= (\Delta_H \otimes \text{id}_X) \circ \begin{bmatrix} \delta_{X'}(x') & \varphi(x') \\ 0 & \delta_{X''}(x'') \end{bmatrix} = (\Delta_H \otimes \text{id}_X) \circ \delta_X(x', x''),$$

and our claim is proved.
We define $\Psi(f', f'') : \Psi(X', X'', \varphi) \to \Psi(Y', Y'', \psi)$ to be the homomorphism of left $H$-comodules given by $f = f' \oplus f'' : X' \oplus X'' \to Y' \oplus Y''$. We show that if $(f', f'') : (X', X'', \varphi) \to (Y', Y'', \psi)$ is a morphism in $\text{Rep}_K(H; U_{H''})$ then $f \circ f' = f' \oplus f'' : X' \oplus X'' \to Y' \oplus Y''$ defines a homomorphism of left $H$-comodules between $\Psi(X', X'', \varphi) = (X, \delta_X)$ and $\Psi(Y', Y'', \psi) = (Y, \delta_Y)$. Indeed, given $x' \in X'$ and $x'' \in X''$, we get

$$
\delta_Y \circ f(x', x'') = \delta_Y \circ (f'(x'), f''(x'')) = \begin{bmatrix}
\delta_Y \circ f'(x') & \psi(f'(x')) \\
0 & \delta_Y \circ f''(x'')
\end{bmatrix}
$$

and therefore $f$ is a homomorphism of left $H$-comodules.

It is clear that we have defined a $K$-linear, faithful and exact functor

$$
\Psi : \text{Rep}_K(H; U_{H''}) \to \text{H-Comod}.
$$

A straightforward computation shows that $\Psi$ is quasi-inverse to $\Phi$ and vice versa. Consequently, we get the following useful result.

**Theorem 2.14.** Let $H'$ and $H''$ be $K$-coalgebras, $H; U_{H''}$ a non-zero $H'$-$H''$-bicomodule, and $H$ the bipartite $K$-coalgebra (2.1). The $K$-linear functors $\Phi$ and $\Psi$ in (2.8) are $K$-linear equivalences of categories quasi-inverse to each other and they restrict to $K$-linear equivalences of categories

$$
(2.15) \quad \text{H-comod} \xrightarrow{\Phi'} \text{rep}_K(H; U_{H''}).
$$

By applying the equivalences (2.8) and (2.15), we are able to prove the following properties of the bipartite coalgebra $H$.

**Theorem 2.16.** Let $H'$ and $H''$ be basic $K$-coalgebras with the decompositions $\text{soc} H' = \bigoplus_{j' \in I_{H'}} S'(j')$ and $\text{soc} H'' = \bigoplus_{j'' \in I_{H''}} S''(j'')$ into direct sums of simple left comodules (and simple coalgebras). Let $H; U_{H''}$ be a non-zero $H'$-$H''$-bicomodule and $H$ the bipartite $K$-coalgebra (2.1).

(a) The coalgebra $H$ is basic and

$$
\text{soc}_HH = \begin{bmatrix} \text{soc} H' & 0 \\ 0 & \text{soc} H'' \end{bmatrix} = \begin{bmatrix} \text{soc} H' \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \text{soc} H'' \end{bmatrix}
$$

$$
= \bigoplus_{j' \in I_{H'}} S(j') \oplus \bigoplus_{j'' \in I_{H''}} S(j''),
$$

where $S(j') = \begin{bmatrix} S'(j') \\ 0 \end{bmatrix}$ if $j' \in I_{H'}$, and $S(j'') = \begin{bmatrix} S''(j'') \\ 0 \end{bmatrix}$ if $j'' \in I_{H''}$, in the notation (2.4) and (2.5).
(b) For each \( j' \in I_{H'} \), the left \( H \)-comodule \( E(j') = [E'(j')]_0 \) is the \( H \)-injective envelope of \( S(j') \), where \( E'(j') \) is the \( H' \)-injective envelope of \( S'(j') \).

(c) The left \( H \)-comodule \( [U_{H''}] \) in (2.5) is injective and has a decomposition

\[
\begin{pmatrix} U \\ H'' \end{pmatrix} = \bigoplus_{\nu' \in I_{H''}} \begin{pmatrix} H'U_{\nu''} \\ E''(t'' \nu') \end{pmatrix} = \bigoplus_{\nu' \in I_{H''}} E(t''),
\]

where \( E''(t'') \) is the \( H'' \)-injective envelope of \( S''(t'') \), \( H'U_{\nu''} = U \square E''(t'') \) is viewed as a left \( H' \)-subcomodule of \( H'U_{H''} \) and

\[
E(t'') = \begin{pmatrix} H'U_{\nu''} \\ E''(t'') \end{pmatrix} \subseteq \begin{pmatrix} H'U \\ H'' \end{pmatrix}
\]

is the \( H \)-injective envelope of \( S(t'') \).

(d) \( \max\{\text{gl.dim } H', \text{gl.dim } H''\} \leq \text{gl.dim } H \leq \text{gl.dim } H' + \text{gl.dim } H'' + 1 \).

(e) If \( H' \) and \( H'' \) are semisimple then

(e1) \( H' = \bigoplus_{j' \in I_{H'}} S'(j') \) and \( H'' = \bigoplus_{j'' \in I_{H''}} S''(j'') \) are direct sums of coalgebras and the \( H' \)-\( H'' \)-bicomodule \( H'U_{H''} \) has a \( K \)-vector space decomposition

\[
(2.16) \quad H'U_{H''} = \bigoplus_{s' \in I_{H'}} \bigoplus_{\nu'' \in I_{H''}} s'U_{\nu''},
\]

where \( s'U_{\nu''} = S'(s') \square_{H'} U_{H''} \square S''(t'') \) is viewed as an \( S'(s') \)-\( S''(t'') \)-bicomodule (and \( H' \)-\( H'' \)-bicomodule, in a natural way).

(e2) \( H \) is coradical square complete and every simple left \( H \)-comodule \( S \) is projective or injective.

(e3) \( \text{gl.dim } H = 1 \).

Proof. (a) Since \( H' \) and \( H'' \) are basic, by the definition (2.3) of the comultiplication in \( H, S(j') \) and \( S(j'') \) are simple subcoalgebras of \( H \) for all \( j' \in I_{H'} \) and \( j'' \in I_{H''} \), and \([\text{soc } H', 0 \atop 0 \atop \text{soc } H''] \subseteq \text{soc } H\).

To prove the opposite inclusion, we take a simple left subcomodule \( S \) of \( H \). In view of Theorem 2.14, we identify the category \( \text{H-Comod with Rep}_{\square} (H, U_{H''}) \) via the functor \( \Phi \) in (2.11). Then \( S \) has the form \( S = (S', S'', \varphi) \) and \((0, S'', 0)\) is a left subcomodule of \( S \). Hence, if \( S' \neq 0 \), then \( S'' = 0 \) and \( S = (S', 0, 0) \) is a simple left \( H' \)-comodule, and we are done; otherwise, \( S' = 0, S'' \neq 0 \), and \( S = (0, S'', 0) \) is a simple left \( H'' \)-comodule. This proves the required equality \([\text{soc } H', 0 \atop 0 \atop \text{soc } H''] = \text{soc } H\).

(b) Since \( E'(j') \) is the \( H' \)-injective envelope of \( S'(j') \), it follows that \( E'(j') \) is a direct summand of \( H' \) and \( \text{soc } E'(j') = S'(j') \). Hence, \( E(j') = [E'(j')]_0 \) is a direct summand of \([H' \atop 0] \subseteq H \) (and of \( H \)), and \( \text{soc } E(j') = S(j') \). This means that \( E(j') \) is the \( H \)-injective envelope of \( S(j') \).
(c) We have the decompositions

\[ H' H' = \bigoplus_{s' \in I_{H'}} E'(s') \quad \text{and} \quad H'' H'' = \bigoplus_{t'' \in I_{H''}} E''(t'') \]

into direct sums of indecomposable injective left comodules. The decomposition of \( H'' \) yields the decomposition

\[ H' U \cong H' U \square H'' = H' U \square \bigoplus_{t'' \in I_{H''}} E''(t'') = \bigoplus_{t'' \in I_{H''}} H' U \square E''(t'') = \bigoplus_{t'' \in I_{H''}} H' U v'' \]

of \( U \), viewed as a left \( H' \)-comodule, where \( H' U v'' = H' U \square E''(t'') \) is viewed as a left \( H' \)-comodule. We set \( E(t'') = (H' U v'', E''(t''), \text{id}) \). It is clear that \( \bigoplus_{t'' \in I_{H''}} E(t'') \cong \left[ \begin{array}{c} U \\
\end{array} \right]_{H'} \subseteq H \), and hence \( E(t'') \) is an injective left \( H \)-comodule, as a direct summand of \( H H \). Since \( \text{soc} E(t'') = S(t'') \) we conclude that \( E(t'') \) is the \( H \)-injective envelope of \( S(t'') \).

(d) Each left \( H \)-comodule \( X \) is a triple \( X = (X', X'', \varphi_X) \) (see (2.11)). In particular, we get (cf. Example 2.12):

- \( \left[ \begin{array}{c} U \\
\end{array} \right]_{H''} \) = \( (U, H'', \delta_U) \), where \( \delta_U : H' U \to H' U \square H'' \) is the canonical isomorphism,
- \( S(i') = (S'(i'), 0, 0) \) for \( i' \in I_{H'} \),
- \( E(i') = (E'(i'), 0, 0) \) for \( i' \in I_{H'} \),
- \( S(t'') = (0, S''(t''), 0) \) for \( t'' \in I_{H''} \),
- \( E(t'') = (H' U v'', E''(t''), \text{id}) \) for \( t'' \in I_{H''} \), where \( \text{id} : H' U v'' \to H' U \square E''(t'') \) is the identity map.

We recall that \( \text{gl.dim} H \leq n \) if and only if \( \text{inj.dim}_H S \leq n \) for each simple left \( H \)-comodule \( S \) (see [18]). By (a), the comodules \( S(i') \) with \( i' \in I_{H'} \), and \( S(j'') \) with \( j'' \in I_{H''} \), form a complete set of pairwise non-isomorphic simple left \( H \)-comodules.

Given \( i' \in I_{H'} \), we fix a minimal injective resolution

\[ 0 \to S'(i') \to 0 E' \to 1 E' \to \cdots \to m E' \to \cdots \]

in \( H' \)-Comod of the simple left \( H' \)-comodule \( S'(i') \). Then the induced sequence

\[ 0 \to S(i') \to (0 E', 0, 0) \to (1 E', 0, 0) \to \cdots \to (m E', 0, 0) \to \cdots \]

in \( H \)-Comod = \( \text{Rep}_{\square}(H' U H'') \) is a minimal injective resolution of the left \( H \)-comodule \( (S(i'), 0, 0) \). It follows that \( \text{inj.dim}_H S(i') = \text{inj.dim}_{H'} S'(i') \) for each \( i' \in I_{H'} \), and so \( \text{gl.dim} H \geq \text{gl.dim} H' \).

Now fix \( t'' \in I_{H''} \). By (c), there is a non-split exact sequence

\[ 0 \to S(t'') \to E(t'') \to L_0(t'') \to 0 \]

in \( H \)-Comod = \( \text{Rep}_{\square}(H' U H'') \), where

\[ L_0(t'') = (H' U v'', L_0''(t''), \overline{\varphi}_{t''}) \quad \text{and} \quad L''_0(t'') = E''(t'')/S''(t'') \.]
Let 

\[ 0 \to L'_0(t'') \to 1E'' \to 2E'' \to \cdots \to mE'' \to \cdots \]

be a minimal injective resolution of \( L'_0(t'') \) in \( H''\)-Comod. If \( mE'' \neq 0 \) for all \( m \geq 1 \), then \( \text{gl.dim } H'' = \infty \) and the induced exact sequence

\[ 0 \to L_0(t'') \to (U \square 1E'', 1E'', 1h) \to \cdots \to (U \square mE'', mE'', mh) \to \cdots \]

in \( H\)-Comod = \( \text{Rep}_\mathbb{F}(H'U_{H''}) \), with \( mh = \text{id} : U \square mE'' \to U \square mE'' \) for \( m \geq 1 \), is a minimal injective resolution of \( L_0(t'') \). Hence \( \text{inj.dim } H'' \leq t'' = \infty \), and we are done.

Assume that \( m-1E'' \neq 0 \) and \( mE'' = 0 \) for some \( m \geq 1 \). Then the induced sequence

\[ 0 \to L_0(t'') \to (U \square 1E'', 1E'', 1h) \to \cdots \to (U \square m-1E'', m-1E'', m-1h) \]

\[ \to (mN, 0, 0) \to 0, \]

with \( jh = \text{id} : U \square jE'' \to U \square jE'' \) for \( j \geq 1 \), is exact. If \( mN = 0 \) then

\[ \text{inj.dim } H'' \leq t'' = m-1 = 1 + \text{inj.dim } H'' \leq t'' = \text{inj.dim } H'' \leq t'' \]

Assume that \( mN \neq 0 \). Let

\[ 0 \to mN \to mE' \to m+1E' \to \cdots \to m+rE' \to \cdots \]

be a minimal injective resolution of \( mN \) in \( H'\)-Comod. Then the induced sequence

\[ 0 \to (mN, 0, 0) \to (mE', 0, 0) \to \cdots \to (m+rE', 0, 0) \to \cdots \]

is a minimal injective resolution of \( (mN, 0, 0) \) in \( H\)-Comod. Therefore

\[ \text{inj.dim } H'' + \text{gl.dim } H' + 1 \geq \text{inj.dim } H'' \geq \text{inj.dim } H'' \geq \text{inj.dim } H'' \]

and (d) follows.

(e) Assume that the basic coalgebras \( H' \) and \( H'' \) are semisimple. Then we have decompositions \( H' = \bigoplus_{s' \in I_H} S'(s') \) and \( H'' = \bigoplus_{t'' \in I_{H''}} S''(t'') \) into direct sums of simple coalgebras. By (e), the semisimple decomposition of \( H'' \) yields the decomposition

\[ H'U \cong H'U \square H'' = \bigoplus_{t'' \in I_{H''}} H'U_{t''} \]

of \( U \), viewed as a left \( H'\)-comodule, where \( H'U_{t''} \) is viewed as an \( H'\)-\( S''(t'') \)-comodule. We note that \( E''(t'') = S''(t'') \) is a subcoalgebra of \( H'' \). Similarly, the semisimple decomposition of \( H' \) yields the \( H'\)-\( H'' \)-comodule decomposition

\[ H'U_{H''} \cong H'U \square U_{H''} = \bigoplus_{s' \in I_H} S'(s') \square U_{H''} = \bigoplus_{s' \in I_H} \bigoplus_{t'' \in I_{H''}} s'U_{t''}, \]

where \( s'U_{t''} = S'(s') \square U_{t''} \) is viewed as an \( S'(s')-S''(t'') \)-comodule, and hence as an \( H'\)-\( H'' \)-comodule. This proves (e1).
By (c), the left $H$-comodule $\left[ \frac{U}{H''} \right]$ is injective and has the decomposition

$$\begin{bmatrix} U \\ H'' \end{bmatrix} = \bigoplus_{t'' \in I_{H''}} \begin{bmatrix} H'U_{t''} \\ S''(t'') \end{bmatrix} = \bigoplus_{t'' \in I_{H''}} E(t''),$$

where $H'U_{t''} = H' \sqcap S''(t'')$ is viewed as a left $H'$-subcomodule of $H'U_{H''}$ and

$$E(t'') = \begin{bmatrix} H'U_{t''} \\ S''(t'') \end{bmatrix} \subseteq \begin{bmatrix} H' \sqcap \\ H'' \end{bmatrix}$$

is the injective envelope of $S(t'')$. Because (a) yields $\text{soc } H = \text{soc } H' \oplus \text{soc } H''$, the above considerations imply that $(\text{soc } H) \land (\text{soc } H) = H$, that is, $H$ is coradical square complete. The remaining statement of (e2) is easily seen by applying the identification $H\text{-Comod} = \text{Rep}_{\sqcap}(H'U_{H''})$.

By (d), $\text{gl.dim } H \leq 1$, because the coalgebras $H'$ and $H''$ are semisimple. Since $U \neq 0$, we have $\text{soc } H = \text{soc } H' \oplus \text{soc } H'' \subsetneq H$ and hence $\text{gl.dim } H \geq 1$. This completes the proof of (e3) and of the theorem. \hfill \Box

3. The valued Gabriel quiver of a bipartite coalgebra and of a coradical square complete coalgebra. Let $C$ be a basic coalgebra with a fixed left comodule decomposition

$$\text{soc}_C C = \bigoplus_{i \in I_C} S(i),$$

of the left socle where $S(i)$, for $i \in I_C$, are pairwise non-isomorphic simple left $C$-comodules (and simple subcoalgebras).

We recall that the left valued (Gabriel) quiver of $C$ is the valued quiver $(CQ, Cd)$, where $CQ_0 = I_C$ and, given two vertices $i, j \in CQ_0$, there exists a unique valued arrow

$$i \xrightarrow{(Cd'_{ij}, Cd''_{ij})} j$$

in $CQ_1$ if and only if $\text{Ext}^1_C(S(i), S(j)) \neq 0$ and

$$Cd'_{ij} = \dim \text{Ext}^1_C(S(i), S(j))_{F_i}, \quad Cd''_{ij} = \dim F_j \text{Ext}^1_C(S(i), S(j)),$$

where $F_a = \text{End}_C S(a)$ for any $a \in I_C$ (see [14, Definition 4.3]).

Now we recall from [14, Proposition 4.10] and [32] an equivalent definition of the left valued Gabriel quiver $(CQ, Cd)$ of a basic coalgebra $C$ by means of irreducible morphisms.

Assume that $C$ is a basic coalgebra with a fixed left comodule decomposition of $\text{soc}_C C$ as above. Given $a \in I_C$, we denote by $E(a)$ the injective envelope of $S(a)$. Denote by $C\text{-inj}$ the full subcategory of $C\text{-Comod}$ formed by socle-finite injective $C$-comodules, that is, a comodule $E$ lies in $C\text{-inj}$ if and only if $E$ is isomorphic to a finite direct sum of indecomposable injective $C$-comodules. Given $E'$ and $E''$ in $C\text{-inj}$, we define the radical
of \(\text{Hom}_C(E', E'')\) to be the \(K\)-subspace \(\text{rad}(E', E'') = \text{rad}_{\text{C-inj}}(E', E'')\) of \(\text{Hom}_C(E', E'')\) generated by all non-isomorphisms \(\varphi : E(i) \to E(j)\) between indecomposable summands \(E(i)\) of \(E'\) and \(E(j)\) of \(E''\), respectively. The square \(\text{rad}^2(E', E'')\) is defined to be the \(K\)-subspace of \(\text{rad}(E', E'')\) generated by all composite homomorphisms of the form

\[
E' \xrightarrow{f_j} E(j) \xrightarrow{f''_j} E'',
\]

where \(j \in I_C\), \(f'_j \in \text{rad}(E', E(j))\) and \(f''_j \in \text{rad}(E(j), E'')\). For any \(a, b \in I_C\), we set \(F_a = \text{End}_C S(a)\), \(F_b = \text{End}_C S(b)\) and we consider the \(K\)-vector space

\[
(3.1) \quad \text{Irr}_C(E(b), E(a)) = \text{rad}(E(b), E(a))/\text{rad}^2(E(b), E(a))
\]
as an \(F_a\)-\(F_b\)-bimodule. We call it the \textit{bimodule of irreducible morphisms} (see [14], [30] and [32]).

By [14, Proposition 4.7] and [32, Theorem 2.3], there exists a unique valued arrow \(a \xrightarrow{(d'_{ab}, d''_{ab})} b\) in \((C Q, C \mathbf{d})\) if and only if the \(F_a\)-\(F_b\)-bimodule \(\text{Irr}(E(b), E(a))\) is non-zero and

\[
(3.2) \quad d'_{ab} = \dim \text{Irr}_C(E(b), E(a))_{F_b}, \quad d''_{ab} = \dim F_a \text{Irr}_C(E(b), E(a)).
\]

The following proposition gives a description of the left valued Gabriel quiver of a coalgebra \(C\) in terms of the \(C_0\)-\(C_0\)-bicomodule

\[
(3.3) \quad C_0(C_1/C_0)_{C_0} = \bigoplus_{a, b \in I_C} a(C_1/C_0)_b,
\]

where the \((S(a)-S(b))\)-bicomodule \(a(C_1/C_0)_b = S(a) \Box (C_1/C_0) \Box S(b)\) is viewed as a rational \(F_a\)-\(F_b\)-bimodule. To see this we note that, in the notation of the proof of Proposition 3.5 below, there is an \(F_a\)-\(F_b\)-bimodule isomorphism \(a(C_1/C_0)_b \cong e_0(C_1/C_0) e_a\) (see (3.6)) and cf. [3], [17], and [41]).

To formulate the result, we assume that \(C\) is a basic coalgebra with a decomposition of \(\text{soc} \ C C\) as above. Given \(a \in I_C\), we denote by \(E(a) \supseteq E_1(a)\) the injective envelope of \(S(a)\) in \(C\)-Comod and \(C_1\)-Comod, respectively. Now, for \(a, b \in I_C\), we define an \(F_a\)-\(F_b\)-bimodule homomorphism

\[
(3.4) \quad \text{Irr}_C(E(b), E(a)) \overset{\text{res}_{ab}}{\longrightarrow} \text{Irr}_{C_1}(E_1(b), E_1(a))
\]

by associating to any non-isomorphism \(f : E(b) \to E(a)\) its restriction \(\text{res}_{ab}(f) : E_1(b) \to E_1(a)\) to \(E_1(b)\).

Now we complete [3], [14, Proposition 4.10], [17, Theorem 1.7] and [32, Theorem 2.5] as follows.

\textbf{Proposition 3.5.} Let \(C\) be a basic \(K\)-coalgebra with a left comodule decomposition \(\text{soc} \ C C = \bigoplus_{i \in I_C} S(i)\) as above, and let \(C_1 = C_0 \wedge C_0\).

(a) Given \(a, b \in I_C\), the \(F_a\)-\(F_b\)-bimodule homomorphism \(\text{res}_{ab}\) in (3.4) is an isomorphism.
(b) For any $a, b \in I_C$, there exist $F_a$-$F_b$-bimodule isomorphisms

$$\text{Hom}_{F_a}(\text{Ext}^1_C(S(a), S(b)), F_a) \cong \text{Irr}_C(E(b), E(a)) \cong a(C_1/C_0)_b.$$  
(3.6)  

(c) There exists a unique valued arrow $a \xrightarrow{(d_{ab}', d_{ab}'')} b$ in the left valued Gabriel quiver $(CQ_C, \text{d})$ of $C$ if and only if the $F_a$-$F_b$-bimodule $a(C_1/C_0)_b = S(a) \square (C_1/C_0) \square S(b)$ is non-zero and $d_{ab}' = \dim((a(C_1/C_0))_b F_a$, $d_{ab}'' = \dim F_b((a(C_1/C_0)_b)$.  
(3.7)  

(d) The left Gabriel quiver $C_1Q$ coincides with $CQ$.

Proof. (a) To show that $\text{res}_{ab}$ is bijective, we note that, given a non-isomorphism $f : E(b) \to E(a)$, the restriction $\text{res}_{ab}(f) : E_1(b) \to E_1(a)$ is obviously a non-isomorphism. Conversely, if $g : E_1(b) \to E_1(a)$ is a non-isomorphism of $C_1$-comodules then, by the injectivity of $E(a)$, $g$ uniquely extends to a non-isomorphism $f : E(b) \to E(a)$ such that $\text{res}_{ab}(f) = g$. This shows that (3.4) is bijective.

(b) The left-hand isomorphism in (3.6) is established in [14, Proposition 4.10]. To prove the right-hand one, we keep the notation of the proof of [14, Proposition 4.10]. Fix $a, b \in I_C$ and denote by $e_a$, $e_b$ the primitive idempotents in the pseudocompact $K$-algebra $C^* = \text{Hom}_K(C, K)$ that correspond to the direct summands $E(a)^*$ and $E(b)^*$ of $C^*$. Let $J(C^*)$ be the Jacobson radical of $C^*$. We recall that the functor $M \mapsto M^*$ defines a $K$-linear duality $C\text{-Comod} \cong C^*\text{-PC}$, where $C^*\text{-PC}$ is the category of pseudocompact left $C^*$-modules (see [29, 4.5]). Moreover, by [16, Proposition 5.2.9] there are isomorphisms $J(C^*)/J(C^*)^2 \cong C_0^+/C_1^+ \cong (C_1/C_0)^*$ of pseudocompact $C^*$-bimodules.

By [14, p. 480], the equivalence $C\text{-Comod} \cong (C^*\text{-PC})^{\text{op}}$, $M \mapsto M^*$, induces isomorphisms

$$\text{Irr}_C(E_1(b), E_1(a)) \cong (e_a [J(C^*)/J(C^*)^2] e_b)^\circ \cong (e_a [(C_1/C_0)^*] e_b)^\circ$$

$$\cong e_b ((C_1/C_0)^*)^\circ e_a \cong e_b (C_1/C_0) e_a \cong a(C_1/C_0)_b$$

of $F_a$-$F_b$-bimodules. The final isomorphism is the inverse of the following composite one:

$$a(C_1/C_0)_b = S(a) \square (C_1/C_0) \square S(b)$$

$$\cong \text{Hom}_{C_0}(S(a)^*, (C_1/C_0) \square S(b))$$

$$\cong \text{Hom}_{C_0}(S(a)^*, \text{Hom}_{C_0}(S(b)^*, C_1/C_0))$$

$$\cong \text{Hom}_{C_0}(S(a)^*, e_b(C_1/C_0)) \cong e_b(C_1/C_0) e_a.$$  
(3.6')

Note also that, since the pseudocompact left $C^*$-modules $S(a)^* \cong (C_0)^* e_a$ and $S(b)^* \cong (C_0)^* e_b$ are finite-dimensional, they are discrete (= rational),
and therefore they are viewed as left $C$-comodules. Moreover, there are algebra isomorphisms $S(a)^* \cong e_a(C_0)^* e_a \cong F_a^{\text{op}}$, $S(b)^* \cong e_b(C_0)^* e_b \cong F_b^{\text{op}}$, and $F_a$-$F_b$-bimodule isomorphisms $a(C_1/C_0)_b = S(a) \square (C_1/C_0) \square S(b) \cong C_0 e_a \square (C_1/C_0) \square e_b C_0 \cong e_b(C_1/C_0)e_a$.

(c) Apply (a), (b) and (3.2).
(d) Apply (a) and (3.4). ■

**Corollary 3.8.** Let $C$ be a basic $K$-coalgebra. Then the left valued and right valued Gabriel quivers of $C$ are dual to each other.

**Proof.** It is well-known that there is a $K$-duality $D : C\text{-inj} \rightarrow \text{inj}-C$ between the categories of socle finite injective left $C$-comodules and socle finite injective right $C$-comodules (see [5, Proposition 3.1(c)]). Given an indecomposable $E(a)$ in $C\text{-inj}$, we denote by $E'(a)$ the indecomposable $DE(a)$ in $\text{inj}-C$. Obviously, the socle $S'(a)$ of $E'(a)$ is isomorphic to the right $C$-comodule $S(a)^*$. Since, for any $a, b \in I_C$, there are division ring isomorphisms

$$F_a' = \text{End}_C S'(a) \cong (\text{End}_C S(a))^{\text{op}} \cong F_a^{\text{op}},$$

$$F_b' = \text{End}_C S'(b) \cong (\text{End}_C S(b))^{\text{op}} \cong F_b^{\text{op}},$$

the $F_b'$-$F_a'$-bimodule $\text{Irr}(E'(a), E'(b))$ is viewed as an $F_a$-$F_b$-bimodule in a standard way. Moreover, the functor $D$ induces an isomorphism $\text{Irr}(E(b), E(a)) \cong \text{Irr}(E'(a), E'(b))$ of $F_a$-$F_b$-bimodules. Hence, in view of Proposition 3.5 and [32, Theorem 2.3], the corollary follows. ■

We end this section by a description of the Gabriel quiver of an arbitrary bipartite coalgebra.

**Corollary 3.9.** Let $H'$ and $H''$ be basic $K$-coalgebras, $H'U_{H''}$ a non-zero $H'$-$H''$-bicomodule, and $H$ the bipartite $K$-coalgebra (2.1). In the notation of Theorem 2.16 we have:

(a) $H$ is basic and the Gabriel quiver $(HQ, Hd)$ has the form [15]

$$HQ, Hd) = (H'Q, H'd) \square U (H''Q, H''d),$$

that is, $(HQ, Hd)$ is obtained from the disjoint union of $(H'Q, H'd)$ and $(H''Q, H''d)$ by adding, for each $s' \in H'Q = I_{H'}$ and each $t'' \in H''Q = I_{H''}$, the valued arrow

$$s' \overset{(d''_{s't''}, d'_{s't''})}{\longrightarrow} t''$$

from $s'$ to $t''$, provided that $s'U_{t''} \neq 0$, and $d''_{s't''} = \dim(s'U_{t''})F_{s'}$, $d'_{s't''} = \dim F_{s'}(s'U_{t''})$. Here the $S''(s')$-$S''(t'')$-bicomodule $s'U_{t''} = S'(s') \square U \square S''(t'')$ is viewed as a (rational) $F_{s'}$-$F_{t''}$-bimodule, in view of the division algebra isomorphisms $\text{End}_H S''(t'') \cong F_{t''}$ and $\text{End}_H S'(s') \cong F_{s'}$. 
(b) If $H'$ and $H''$ are semisimple then $(H'Q, H'd)$ and $(H''Q, H''d)$ have no arrow, and the only arrows in $(HQ, Hd)$ are of the form (3.11), where $s' \in I_{H'}$ and $t'' \in I_{H''}$. If $H'$ and $H''$ are simple and $H' U_{H''} \neq 0$, then $H$ is indecomposable and $(HQ, Hd)$ has the form $\bullet (d', d'') \bullet$ for some natural numbers $d'$ and $d''$.

Proof. Given $b \in I_H = I_{H'} \cup I_{H''}$, we set $E(b) = E(b)/S(b)$. Since $E(b)$ is an injective $H$-comodule, there is an isomorphism

$$\text{Ext}^1_H(S(a), S(b)) \cong \text{Hom}_H(S(a), E(b))$$

of right $\text{End}_H S(a)$-modules for each $a \in I_{H'} = I_{H'} \cup I_{H''}$ (see [14, p. 477]).

Since $H'$ and $H''$ are basic, so is $H$, by Theorem 2.16(a). We recall from Theorem 2.16 that, given $j' \in I_{H'}$ and $j'' \in I_{H''}$, we have

$$S(j') = \begin{bmatrix} S'(j') \\ 0 \end{bmatrix}, \quad E(j') = \begin{bmatrix} E'(j') \\ 0 \end{bmatrix},$$

$$S(j'') = \begin{bmatrix} 0 \\ S''(j'') \end{bmatrix}, \quad E(t'') = \begin{bmatrix} H' U_{t''} \\ E''(t'') \end{bmatrix},$$

in the notation of Theorem 2.16 and (2.5). Hence, for $s' \in I_{H'}$ and $t'' \in I_{H''}$,

$$E(t'') \cong \begin{bmatrix} H' U_{t''} \\ E''(t'') \end{bmatrix} \quad \text{and} \quad E(s') \cong \begin{bmatrix} E'(s') \\ 0 \end{bmatrix}.$$  

It follows that $\text{Ext}^1_H(S(a), S(b)) = 0$ if $a \in I_{H''}$ and $b \in I_{H'}$. Moreover, there are isomorphisms of $\text{End}_H S(b)$-$\text{End}_H S(a)$-bimodules

$$\text{Ext}^1_H(S(a), S(b))$$

\[= \begin{cases} 
\text{Hom}_{H'}(S'(a), E'(b)) \cong \text{Ext}^1_H(S'(a), S'(b)) & \text{if } a, b \in I_{H'}, \\
\text{Hom}_{H''}(S''(a), E''(b)) \cong \text{Ext}^1_H(S''(a), S''(b)) & \text{if } a, b \in I_{H''}, \\
\text{Hom}_{H'}(S'(a), H' U_b) \cong aU_b & \text{if } a \in I_{H'}, b \in I_{H''} 
\end{cases} \]

(see [14, p. 480] and [41, Proposition 4.9]). Hence, (a) follows. Since (b) easily follows from (a), the proof is complete.

Following a suggestion of the referee we include another proof of (a). Let $H$ be a bipartite coalgebra as in the corollary. We consider $\tilde{U} = H' \text{(soc } H') U \cap (\text{soc } U_{H''}) H''$ and we view it as an $H'$-$H''$-bicomodule. Note that, for all $a \in I_{H'}$ and $b \in I_{H''}$, there are isomorphisms of $(S(a)-S(b)$-bicomodules

$$S(a) \boxtimes U \boxtimes H'' S(b) \cong S(a) \boxtimes H'_0 U \boxtimes H''_0 S(b) \cong S(a) \boxtimes H'_0 \tilde{U} \boxtimes H''_0 S(b) = a\tilde{U}_b.$$  

By a straightforward calculation we show that $H_1 = H_0 \land H_0 = H'_1 \oplus \tilde{U} \oplus H''_1$, and hence $H_1/H_0 = H'_1/ H'_0 \oplus \tilde{U} \oplus H''_1/ H''_0$. Note also that $H^* = H'^* \oplus U^* \oplus H''^*$ is the upper triangular matrix algebra with the identity element $\varepsilon_H = \sum_{a \in I_{H'}} e'_a + \sum_{b \in I_{H''}} e''_b$, where $e'_a \cdot \begin{bmatrix} h' \\ 0 \end{bmatrix} = e'_a(h')$ and $e''_a \cdot \begin{bmatrix} h'' \\ 0 \end{bmatrix} = e''_a(h'').$
We also recall from \[16\] that
\[
e \rightarrow h = eh = (1 \otimes e) \circ \Delta_H(h) \quad \text{and} \quad h \leftarrow e = he = (e \otimes 1) \circ \Delta_H(h).
\]
Hence, for \(a, \overline{a} \in I_{H'}\) and \(b, \overline{b} \in I_{H''}\) we get
\[
\begin{align*}
\bullet \quad a(H_1/H_0)a &= e'_a(H_1/H_0)e'_a = e'_a(H_1'/H'_0)e'_a = a(H'_1/H'_0)a, \\
\bullet \quad a(H_1/H_0)b &= e''_a(H_1/H_0)e'_a = e''_a(H'_1/H'_0)e'_a = a(H'_1/H'_0)b, \\
\bullet \quad b(H_1/H_0)a &= e'_a(H_1/H_0)e''_a = 0, \\
\bullet \quad b(H_1/H_0)b &= e''_a(H_1/H_0)e''_a = e''_a(H'_1/H'_0)e''_b = b(H''_1/H''_0)b.
\end{align*}
\]
Now (a) follows by applying Proposition 3.5.


4. Loop representations and trivial extensions of coalgebras.

Let \(D\) be a \(K\)-coalgebra and \(pU_D\) be a \(D\)-bicomodule. We recall that the cotensor \(D\)-coalgebra on \(U\) is the positively graded \(K\)-vector space

\[
T_D^\square(U) = \bigoplus_{n=0}^{\infty} U^\square^n = D \oplus U \oplus U \square U \oplus \cdots \oplus U^\square^n \oplus \cdots,
\]

where \(U^\square^0 = D\), \(U^\square^1 = U\) and \(U^\square^n = U \square \cdots \square U\) (\(n\) times) for \(n \geq 2\), equipped with the \(K\)-coalgebra structure defined as follows (see \[10\], \[19\] and \[41\] for details).

The counit \(\varepsilon : T_D^\square(U) \to K\) of \(T_D^\square(U)\) vanishes on \(U^\square^n\) for all \(n \geq 1\), and \(\varepsilon|_D : D \to K\) is the counit of \(D\). Under the identification

\[
T_D^\square(U) \otimes T_D^\square(U) = \bigoplus_{n,m \geq 0} U^\square^n \otimes U^\square^m,
\]

for each \(n \geq 0\) the component \(\Delta_{n,i,j} : U^\square^n \to U^\square^i \otimes U^\square^j\) of the comultiplication of \(T_D^\square(U)\) is zero if \(i + j \neq n\). If \(i + j = n\) and \(i, j \geq 1\), then \(\Delta_{n,i,j}\) is the inclusion; if either \(i = 0\) or \(j = 0\), then \(\Delta_{n,i,j}\) is induced by the comultiplication on \(U\) (or on \(D\) if \(i = j = 0\)).

Following \[10\] and \[41\], we define the category \(\text{Rep}_{\square}(D U_D)\) of locally nilpotent loop (co)representations of the \(D\)-bicomodule \(pU_D\) to be the category of all pairs \((Y, \mu)\), where \(Y\) is a left \(D\)-comodule and \(\mu : Y \to U \square Y\) is a homomorphism of left \(D\)-comodules such that

\[
Y = \bigcup_{n=1}^{\infty} \ker(\mu^{(n)} : Y \to U^\square^n \otimes Y),
\]

where \(\mu^{(n)} : Y \to U^\square^n \otimes Y\) is the composite

\[
Y \xrightarrow{\mu'} U \otimes Y \xrightarrow{id_U \otimes \mu'} U^\square^2 \otimes Y \to \cdots \to U^\square^{n-1} \otimes Y \xrightarrow{id_U \otimes \mu^{(n-1)}} U^\square^n \otimes Y
\]

and \(\mu' : Y \to U \otimes Y\) is the composite \(Y \xrightarrow{\mu} U \square Y \xrightarrow{\mu} U \otimes Y\). The left \(D\)-comodule structure on \(U \square Y\) is induced from that of \(U\).
A morphism from \((Y, \mu)\) to \((Z, \nu)\) in \(\text{Rep}_D^\square(DUD)\) is a homomorphism \(f : Y \to Z\) of left \(D\)-comodules such that \(\nu \circ f = (\text{id}_Y \square f) \circ \mu\). It is clear that \(\text{Rep}_D^\square(DUD)\) is a Grothendieck \(K\)-category and its full subcategory \(\text{rep}_D^\square(DUD)\), consisting of all pairs \((Y, \mu)\) with \(Y\) finite-dimensional, is abelian and consists of objects of finite length.

**Theorem 4.4.** Let \(D\) be a \(K\)-coalgebra, \(DUD\) a \(D\)-\(D\)-bicomodule, and \(T^\square_D(U)\) the cotensor \(D\)-coalgebra.

(a) \(\text{soc} T^\square_D(U) = \text{soc} D\). As a consequence, \(T^\square_D(U)\) is basic if and only if \(D\) is basic.

(b) There is a \(K\)-linear equivalence of categories

\[
\Theta : T^\square_D(U)\text{-Comod} \to \text{Rep}_D^\square(DUD),
\]

which restricts to an equivalence \(\Theta' : T^\square_D(U)\text{-comod} \cong \text{rep}_D^\square(DUD)\).

(c) If \(D\) is semisimple, then \(T^\square_D(U)\) is hereditary and, given \(i \in I_D\), the vector subspace

\[
E(i) = S(i) \oplus (S(i) \square U) \oplus (S(i) \square U \square U) \oplus \cdots
\]

of \(T^\square_D(U)\) is the injective envelope of \(S(i)\).

**Proof.** For the proof of (a) the reader is referred to [41, Lemma 4.4].

(b) The equivalence (4.5) is proved in [41, Lemma 4.3]. Here, for the convenience of the reader, we recall the definition of \(\Theta\). Since the canonical projection \(\pi : T^\square_D(U) \to D\) is a coalgebra homomorphism, every left \(T^\square_D(U)\)-comodule \(Y\) is a \(D\)-comodule via \(\pi\). The functor \(\Theta\) is defined by associating with \((Y, \delta_Y)\) in \(T^\square_D(U)\text{-Comod}\) the pair

\[
\Theta(Y, \delta_Y) = (Y, \delta'),
\]

where \(Y\) is the underlying \(D\)-comodule and \(\delta' : Y \to U \square Y\) is the composition of \(\delta_Y : Y \to T^\square_D(U) \square Y\) with the canonical \(D\)-comodule projection \(T^\square_D(U) \square Y \to U \square Y\). If \(f : (Y, \delta_Y) \to (Z, \delta_Z)\) is a homomorphism in \(T^\square_D(U)\text{-Comod}\), we take for \(\Theta(f) : (Y, \delta') \to (Z, \delta')\) the morphism defined by \(f : Y \to Z\) in \(D\text{-Comod}\). By [41, Lemma 4.3], the functor \(\Theta\) is an equivalence of categories and obviously it restricts to an equivalence \(\Theta' : T^\square_D(U)\text{-comod} \cong \text{rep}_D^\square(DUD)\).

(c) Assume that \(D\) is semisimple. To prove the second part of (c), note that there is a decomposition \(DUD = D \square DU = \bigoplus_{i \in I_D} (S(i) \square DU)\) and, for any \(i \in I_D\), \(E(i)\) is a left submodule direct summand of \(T^\square_D(U)\); hence \(E(i)\) is injective. Since obviously \(\text{soc} E(i) = S(i)\), it follows that \(E(i)\) is the injective envelope of \(S(i)\).

To show that \(T^\square_D(U)\) is hereditary, it is enough to prove \(\text{inj.dim} \ T^\square_D(U)S \leq 1\) for each simple \(T^\square_D(U)\text{-comodule} \ S(i)\) (see [18]). Consider the exact
The trivial extension

\[ 0 \rightarrow S(i) \rightarrow E(i) \rightarrow \overline{E}(i) \rightarrow 0 \]

of left \( T_H^\square(U) \)-comodules, where \( \overline{E}(i) = E(i)/S(i) \). It follows that there are isomorphisms of left \( T_H^\square(U) \)-comodules

\[
\overline{E}(i) \cong (S(i) \square U) \oplus (S(i) \square U \square U) \oplus (S(i) \square U \square U \square U) \oplus \cdots
\]

\[
\cong [S(i) \oplus (S(i) \square U) \oplus (S(i) \square U \square U) \oplus (S(i) \square U \square U \square U) \oplus \cdots] \square U
\]

Since \( E(i) \square U \) is injective (see [8, Proposition 1]), so is \( \overline{E}(i) \). This shows that \( T_H^\square(U) \) is hereditary. ■

**Corollary 4.7.** Assume that \( H' \) and \( H'' \) are \( K \)-coalgebras and \( H'U_{H''} \) is an \( H' \cdot H'' \)-bicomodule. Let \( H = [H'_{0} H''_{U}] \) be the bipartite coalgebra (2.1) and let \( D = H' \oplus H'' \).

(a) The \( H' \cdot H'' \)-bicomodule structure on \( H'U_{H''} \) defines a \( D \cdot D \)-bicomodule structure on \( U \) such that \( DU \square DU_{D} = 0 \), \( T_H^\square(D) = D \oplus D_{UD} \), and \( [h', u] \mapsto (h', h'', u) \) defines an isomorphism \( H \cong T_H^\square(U) \) of coalgebras.

(b) There are \( K \)-linear equivalences of categories

\[
\begin{align*}
\text{H-Comod} & \cong \text{Rep}_{\square}(H'U_{H''}) & \Rightarrow & \text{Rep}_{\square}(DU_{D}) & \cong & T_H^\square(U)-\text{Comod} \\
\text{H-comod} & \cong \text{rep}_{\square}(H'U_{H''}) & \Rightarrow & \text{rep}_{\square}(DU_{D}) & \cong & T_H^\square(U)-\text{comod}
\end{align*}
\]

where \( \Phi \) and \( \Theta \) are the equivalences (2.11) and (4.5), respectively.

**Proof.** (a) The first part of (a) is obvious. The equality \( DU \square DU_{D} = 0 \) follows immediately from the definition of the cotensor product, because of the definition of the right coaction of \( H' \) on \( DU_{D} \) and the left coaction of \( H'' \) on \( DU_{D} \). Now the remaining part of (a) easily follows.

(b) By (a), the coalgebras \( H \) and \( T_H^\square(U) \) are isomorphic. Hence we get \( H-\text{Comod} \cong T_H^\square(U)-\text{Comod} \). Since, according to Theorems 2.14 and 4.4, the functors \( \Phi \) and \( \Theta \) are \( K \)-linear equivalences of categories, they imply the equivalence \( \text{Rep}_{\square}(H'U_{H''}) \cong \text{Rep}_{\square}(DU_{D}) \) required in (b). ■

Let us now introduce the notion of trivial extension of a coalgebra.

**Definition 4.8.** Let \( D \) be a \( K \)-coalgebra and \( DU_{D} \) a \( D \cdot D \)-bicomodule. The **trivial extension** of \( D \) by \( DU_{D} \) is the coalgebra \( D \times DU_{D} = (D \oplus U, \Delta, \varepsilon) \), where \( \Delta(d, u) = (\Delta_{D}(d), \delta_{D}(u), \delta_{U}(u), 0) \) and \( \varepsilon(d, u) = (\varepsilon_{D}(d), 0) \) for all \( d \in D \) and \( u \in U \). Here we make the identification \( (D \oplus U) \otimes (DU_{D}) \cong (D \otimes D) \oplus (D \otimes U) \oplus (U \otimes D) \oplus (U \otimes U) \).
Note that the $K$-linear map $(d, u) \mapsto \begin{bmatrix} d & u \\ 0 & d \end{bmatrix}$ defines an isomorphism

$$D \ltimes_{D} DU_D \cong \begin{bmatrix} D & U \\ 0 & D \end{bmatrix} = \left\{ \begin{bmatrix} d & u \\ 0 & d \end{bmatrix} : d \in D, u \in U \right\} \subseteq \begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix}$$

of vector spaces. However, unless $U = 0$, $\begin{bmatrix} DU_D \\ 0 & D \end{bmatrix}$ is not a subcoalgebra of the bipartite coalgebra $\begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix}$.

We denote by $\text{Rep}_{\square}^{(2)}(DU_D)$ the full subcategory of $\text{Rep}_{\square}^{(2)}(DU_D)$ whose objects are the pairs $(Y, \mu)$ such that $\mu^{(2)} = 0$.

To describe the left valued Gabriel quiver of the trivial extension coalgebra $D \ltimes_{D} DU_D$, we define

$$(dQ, dD) \mapsto_U (dQ, dD)$$

(4.9)

to be the quiver obtained from the valued quiver $(dQ, dD) \mapsto_U (dQ, dD)$ (see (3.10)) of the bipartite coalgebra $\begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix}$ by the identification of the left copy of $(dQ, dD)$ in $(dQ, dD) \mapsto_U (dQ, dD)$ with the right one, via the identification of the vertex $s'$ with $s''$ and the arrow $s' \to t'$ with $s'' \to t''$, for all $s, t \in DU_D$. This operation is illustrated in Example 4.13 below.

Now we list some of the main properties of the coalgebra $C = D \ltimes_{D} DU_D$.

**PROPOSITION 4.10.** Let $C = D \ltimes_{D} DU_D$ be the trivial extension of a $K$-coalgebra $D$ by a $D$-$D$-bicomodule $DU_D$.

(a) $C$ is isomorphic to the subcoalgebra $D \oplus DU_D$ of $T_D(U)$, $D = D \ltimes 0$ is a subcoalgebra of $C = D \ltimes_{D} DU_D$, $\text{soc} C = \text{soc} D$, and $C_1 = D_1 \oplus U_1$, where $U_1 = \text{soc} D U \cap \text{soc} U D$. If $D$ is semisimple then $C$ is coradical square complete.

(b) If $C$ is basic then the left valued Gabriel quiver $(cQ, cD)$ has the form

$$(cQ, cD) = (dQ, dD) \mapsto_U (dQ, dD).$$

(c) The canonical coalgebra embedding $C \hookrightarrow T_D(U)$ induces an embedding $C \text{-Comod} \subseteq T_D(U)\text{-Comod}$ and the equivalence $\Theta$ of (4.5) restricts to a $K$-linear equivalence of categories

$$(cQ, cD) \mapsto \text{Rep}_{\square}^{(2)}(DU_D) \subseteq \text{Rep}_{\square}^{(2)}(DU_D).$$

(d) The $K$-linear map $\theta : \begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix} \to D \ltimes_{D} DU_D$, given by the formula $\begin{bmatrix} d' & u \\ 0 & d'' \end{bmatrix} \mapsto (d' + d'', u)$, is a coalgebra surjection. If

$$\Theta_+ : C\text{-Comod} \to \begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix}\text{-Comod}$$

(4.12)
is the composite $K$-linear functor

$$C\text{-Comod} \xrightarrow{\Theta} \text{Rep}_2(DU_D) \subseteq \text{Rep}_3(DU_D) \cong \begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix} \text{-Comod}$$

then $\Theta_+$ is a full, faithful, and exact embedding such that, for each $Y$ in $C\text{-Comod}$, $\Theta_+(Y) = (Y, \mu : Y \rightarrow U \square Y)$ and $\mu^{(2)} = 0$.

**Proof.** (a) It is easy to see that the canonical inclusion $C = D \times DU_D \hookrightarrow T_D^\square(U)$ is a coalgebra embedding and defines a coalgebra isomorphism of $C$ with the $D$-subcoalgebra $D \oplus DU_D$ of $T_D^\square(U)$ consisting of the sums of elements of degree 0 and 1 (see (4.1)). Hence the first part of (a) easily follows.

Now we show that $C_1 = D_1 \oplus U_1$, where $U_1 = \text{soc} DU \cap \text{soc} U_D$. We recall that $C_1 = \Delta^{-1}(C_0 \otimes C \otimes C \otimes C_0)$ and $C_0 = D_0 \oplus 0$. Then Definition 4.8 yields

$$\Delta(d) = \Delta_D(d) \in D \otimes D \quad \text{for } d \in D$$

$$\Delta(u) = (\delta'(u), \delta''(u)) \in D \otimes U \oplus U \otimes D \quad \text{for } u \in U.$$  

Hence $C_1 = D_1 \oplus U_1$. The final part of (a) follows from the previous one.

(b) We apply Proposition 3.5. By (a), $C_1/C_0 \cong (D_1/D_0) \oplus U_1$. Let $H = \begin{bmatrix} D & DU_D \\ 0 & D \end{bmatrix}$ be the bipartite coalgebra and

$$H_0 = \begin{bmatrix} D_0 & 0 \\ 0 & D_0 \end{bmatrix} = \bigoplus_{j' \in I_D} S(j') \oplus \bigoplus_{j'' \in I_D} S(j''),$$

Note that $H_1 = \begin{bmatrix} D_1 & U_1 \\ 0 & D_1 \end{bmatrix}$ and $C_0 = \bigoplus_{a \in I_D} S(a) = \bigoplus_{a \in I_C} S(a)$. It follows from the definition that

$$\{a; a \in I_D\} \quad \text{and} \quad \{a'; a' \in I_D\} \cup \{a''; a'' \in I_D\}$$

are the sets of vertices of the left valued Gabriel quivers of $C$ and $H$, respectively. To describe the set of arrows of the quiver $(CQ, \text{cd})$, given a pair $a, b \in I_D = I_C$, we consider the vector space

$$a(C_1/C_0)_b = S(a) \boxdot (C_1/C_0) \boxdot S(b)$$

$$\cong (S(a) \boxdot (D_1/D_0) \boxdot S(b)) \oplus (S(a) \boxdot U_1 \boxdot S(b))$$

By the definition of comultiplication in $C$ and $H$, we have

$$a(D_1/D_0)_b = S(a) \boxdot (D_1/D_0) \boxdot S(b) \cong S(a') \boxdot (D_1/D_0) \boxdot S(b')$$

$$\cong S(a'') \boxdot (D_1/D_0) \boxdot S(b'') = a''(D_1/D_0)_{b''},$$

and

$$S(a) \boxdot U_1 \boxdot S(b) \cong S(a') \boxdot U_1 \boxdot S(b'').$$

Hence, by applying Proposition 3.5, we get (b).
(c) Note that the canonical coalgebra embedding
\[ D \ltimes D U D = D \oplus D U D \hookrightarrow T_D(U) \]
induces an embedding \( D \ltimes D U D \text{-Comod} \subseteq T_D(U)\text{-Comod} \). By applying the definitions, it is easy to check that the equivalence
\[ \Theta : (D \ltimes D U D)\text{-Comod} \cong \text{Rep}_G(D U D) \]
(see (4.5)) restricts to the required \( K \)-linear equivalence of categories (4.11).

(d) The first statement follows by a direct calculation, and the second follows easily from the definitions.

**Example 4.13.** Let \( C = K \Box Q \) be the hereditary path coalgebra of the infinite linear quiver
\[ Q : 1 \to 2 \to \cdots \to s - 1 \to s \to s + 1 \to \cdots \]
and let \( H = \begin{bmatrix} C & \overline{C} \overline{C} \\ \overline{C} \overline{C} & C \end{bmatrix} \) be the bipartite coalgebra (2.1), where we set \( H' = H'' = C \) and \( \overline{C} \overline{C} = C \overline{C} \). Here \( C \overline{C} \) is viewed as a \( C-C \)-bicomodule in the obvious way. It follows from Corollary 3.9 that the left Gabriel quiver of \( H \) has the form
\[
\begin{array}{cccccccc}
1' & \to & 2' & \to & \cdots & \to & (s - 1)' & \to & s' & \to & (s + 1)' & \to & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
1'' & \to & 2'' & \to & \cdots & \to & (s - 1)'' & \to & s'' & \to & (s + 1)'' & \to & \cdots
\end{array}
\]

By Proposition 4.10, the left Gabriel quiver of \( D \ltimes D U D \) has the form
\[
\begin{array}{cccccccc}
\bigcirc & \to & \bigcirc & \to & \cdots & \to & s - 1 & \to & \bigcirc & \to & s & \to & \bigcirc & \to & s + 1 & \to & \cdots
\end{array}
\]

By applying the results in [31] and [33], one can show that there is a coalgebra isomorphism \( H \cong K \Box I_Q \), where \( I_Q \) is viewed as a poset and \( K \Box I_Q \) is its incidence coalgebra. Hence, \( H\text{-comod} \cong K \Box I_Q \text{-comod} \) is equivalent to the category \( \text{rep}_K(I_Q) \) of finite-dimensional \( K \)-linear representations of the poset \( I_Q \).

Now, following [36] and [13], we define the repetitive coalgebra and its connection with the trivial extension coalgebra (4.8).

**Definition 4.14.** Let \( (D, \Delta_D, \varepsilon_D) \) be a coalgebra and \( U = (D U_D, \delta_U', \delta_U'') \) be a \( D-D \)-bicomodule.

(a) The **repetitive coalgebra** of the pair \( (D, D U_D) \) is the \( \mathbb{Z} \)-graded \( K \)-vector space
\[(4.15) \quad \mathcal{R}(D, D U_D) = \bigoplus_{m \in \mathbb{Z}} (D^{(m)} \oplus U^{(m)})\]

with \(D^{(m)} = D\) and \(U^{(m)} = D U_D\) in the \(m\)th row, for all \(m \in \mathbb{Z}\), equipped with the coalgebra structure maps

\[\hat{\Delta} : \mathcal{R}(D, D U_D) \to \mathcal{R}(D, D U_D) \otimes \mathcal{R}(D, D U_D)\]

and \(\hat{\varepsilon} : \mathcal{R}(D, D U_D) \to K\) defined by:

- \(\hat{\Delta}(d) = \Delta_D(d) \in D^{(i)} \otimes D^{(i)}, \quad \hat{\varepsilon}(d) = \varepsilon_D(d), \) for \(d \in D^{(i)}\), and
- \(\hat{\Delta}(u) = (\delta_U(u), \delta'_U(u)) \in D^{(i)} \otimes U^{(i)} \oplus U^{(i)} \otimes D^{(i+1)}, \quad \hat{\varepsilon}(u) = 0, \) for \(u \in U^{(i)}\).

(b) The group \(\mathbb{Z}\) of integers acts on \(\mathcal{R}(D, D U_D)\) as a group of coalgebra automorphisms by the shift

\[\nu : \mathcal{R}(D, D U_D) \to \mathcal{R}(D, D U_D), \quad D^{(m)} \oplus U^{(m)} \mapsto D^{(m+1)} \oplus U^{(m+1)},\]

called the Nakayama automorphism of \(\mathcal{R}(D, D U_D)\).

It is easy to check that the \(K\)-linear map

\[(4.16) \quad f : \mathcal{R}(D, D U_D) \to D \ltimes D U_D\]

defined by the formula

\[f(\ldots, (d^{(-1)}, u^{(-1)}), (d^{(0)}, u^{(0)}), (d^{(1)}, u^{(1)}), \ldots) = \left(\sum_{m \in \mathbb{Z}} d^{(m)}, \sum_{m \in \mathbb{Z}} u^{(m)}\right) \in D \ltimes D U_D,\]

with \((d^{(m)}, u^{(m)}) \in D^{(m)} \oplus U^{(m)}\), is a coalgebra surjection, and induces a pair of \(K\)-linear functors

\[(4.17) \quad \mathcal{R}(D, D U_D)\text{-Comod} \xrightarrow{f_*} (D \ltimes D U_D)\text{-Comod}\]

defined as follows. We define \(f_*\) by setting \(f_*(-) = \hat{D} \square (-)\). Here the repetitive coalgebra \(\hat{D} = \mathcal{R}(D, D U_D)\) is viewed as a right \(D \ltimes D U_D\)-comodule and as a left \(D \ltimes D U_D\)-comodule with comultiplications

\[\hat{\delta}_r = (\text{id} \otimes f) \hat{\Delta} : \hat{D} \to \hat{D} \otimes (D \ltimes D U_D),\]
\[\hat{\delta}_l = (f \otimes \text{id}) \hat{\Delta} : \hat{D} \to (D \ltimes D U_D) \otimes \hat{D},\]
respectively. The functor $f^\triangledown$ associates to any left $\hat{D}$-comodule $(X, \delta_X)$ the left $(D \times DU_D)$-comodule $f^\triangledown(X, \delta_X) = (X, (f \otimes \text{id})\delta_X)$. Given $h \in \text{Hom}(X, Y)$, we set $f^\triangledown(h) = h : f^\triangledown(X) \to f^\triangledown(Y)$.

Now we collect some of the main properties of the functors (4.17). In particular, $f$ is a Galois $\mathbb{Z}$-covering homomorphism and $f^\triangledown$ plays the role of a covering functor for comodule categories (see [11] and [29, (10.7)]).

**Proposition 4.18.** Let $D$ be a coalgebra, $U = DU_D$ a $D-D$-bicomodule, $D \ltimes U$ the trivial extension coalgebra (4.8), and $\mathbb{R}(D, DU_D)$ the $\mathbb{Z}$-graded repetitive coalgebra (4.15) with the $\mathbb{Z}$-action defined above.

(a) The $K$-linear space $\mathbb{R}(D, DU_D)/\mathbb{Z}$ of $\mathbb{Z}$-orbits has a canonical coalgebra structure such that the $\mathbb{Z}$-invariant coalgebra surjection (4.16) induces a coalgebra isomorphism $\tilde{f} : \mathbb{R}(D, DU_D)/\mathbb{Z} \xrightarrow{\tilde{\sim}} D \ltimes U$.

(b) The $K$-linear functor $f_\ast$ in (4.17) is right adjoint to $f^\triangledown$.

(c) The $K$-linear functor $f^\triangledown$ in (4.17) is exact and faithful.

**Proof.** For simplicity of notation, we set $\hat{D} = \mathbb{R}(D, DU_D)$. The fact that (4.16) is a coalgebra surjection follows by a direct calculation, and we leave it to the reader.

(a) We define a coalgebra structure on $D/\mathbb{Z}$ by the linear maps $\Delta : D/\mathbb{Z} \to D/\mathbb{Z} \otimes D/\mathbb{Z}$ and $\varepsilon : D/\mathbb{Z} \to K$ given by $\varepsilon(\mathbb{Z} \ast c) = \varepsilon(c)$ and $\Delta(\mathbb{Z} \ast c) = \sum \mathbb{Z} \ast c(1) \otimes \mathbb{Z} \ast c(2)$, where $c \in \hat{D}$ and $\Delta(c) = \sum c(1) \otimes c(2)$. It is straightforward to check that $\Delta$ and $\varepsilon$ are well-defined and define a coalgebra structure on $D/\mathbb{Z}$.

A direct check shows that the coalgebra surjection $f : \hat{D} \to D \ltimes U$ is $\mathbb{Z}$-invariant. Hence it easily follows that $f$ induces the required coalgebra isomorphism $\tilde{f}$.

(b) It follows from [40, Proposition 1.10] that $f_\ast$ has a left adjoint functor. Given a left $\hat{D}$-comodule $X$ and a left $(D \ltimes U)$-comodule $Z$, the $K$-linear map

$\hat{\varepsilon}_\ast : \text{Hom}_D(X, \hat{D} \square \mathbb{Z}) \to \text{Hom}_{D \ltimes U}(f^\triangledown(X), Z)$

that associates to any $h \in \text{Hom}_D(X, \hat{D} \square \mathbb{Z})$ the homomorphism

$\hat{\varepsilon}_\ast(h) = ((\varepsilon_{D \ltimes U} \circ f) \square \text{id}_{\mathbb{Z}}) \circ h : f^\triangledown(X) \to Z$

of left $(D \ltimes U)$-comodules, is an isomorphism. The inverse $F$ of $\hat{\varepsilon}_\ast$ is defined by the formula

$F(h') = (\text{id}_D \otimes h') \circ \delta_X : X \to \hat{D} \square \mathbb{Z}$

for $h' \in \text{Hom}_{D \ltimes U}(f^\triangledown(X), Z)$ (see [7, Theorem 1.5] for a proof). Since $\hat{\varepsilon}_\ast$ is functorial with respect to comodule homomorphisms $X \to X'$ and $Z \to Z'$, the functor $f^\triangledown$ is the right adjoint of $f_\ast$, and (b) follows.

Since (c) follows from the definition of $f^\triangledown$, the proof is complete. ■
5. A reduction functor for coradical square complete coalgebras. Assume that $C$ is a coradical square complete $K$-coalgebra, that is, $C = C_1 = C_0 \wedge C_0$, where $C_0 = \text{soc } C$. Following an idea of Gabriel [10], we associate with $C$ the bipartite coalgebra

$$H_C = \begin{bmatrix} C_0 & C \\ 0 & C_0 \end{bmatrix} \quad \text{with} \quad \overline{C} = C/C_0$$

(see (2.1)) and a $K$-linear reduction functor

$$\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod}$$

(5.2)

defined as follows. We view $\overline{C} = C/C_0$ as a $C_0$-$C_0$-bicomodule and we make the identification $H_C\text{-Comod} = \text{Rep}_C(C_0 \overline{C} C_0)$ via the functor $\Phi$ (see (2.8) and (2.15)). Then each left $H_C$-comodule $X$ is a triple $X = (X', X'', \varphi_X)$ as in (2.11), where $X', X''$ are left $C_0$-comodules and $\varphi_X : X' \to \overline{C} \square X''$ is a homomorphism of left $C_0$-comodules. In particular, we make the identification

$$\begin{bmatrix} \overline{C} \\ C_0 \end{bmatrix} = (\overline{C}, C_0, j),$$

where $j : \overline{C} \to \overline{C} \square C_0$ is the canonical isomorphism.

Note that, given $(X, \delta_X)$ in $C\text{-Comod}$, $X_0 = \delta_X^{-1}(C_0 \otimes X)$ is the socle of $X$. If $\delta_0$ is the restriction of $\delta_X$ to $X_0$ and $\pi : X \to \overline{X} = X/X_0$ is the projection on the quotient $C$-comodule $(\overline{X}, \delta_{\overline{X}})$, then the diagram of left $C$-comodules

$$
\begin{array}{c}
0 \longrightarrow X_0 \longrightarrow X \xrightarrow{\delta_0} \overline{X} \longrightarrow 0 \\
0 \longrightarrow C_0 \square X \longrightarrow C \square X \xrightarrow{\pi \square \text{id}} \overline{C} \square X
\end{array}
$$

(5.3)

with exact rows is commutative, where $\pi_C$ is the canonical projection and $\delta_{\overline{X}}$ is induced by $\delta_X$. It follows that

$$\delta_0(X_0) \subseteq C_0 \square X_0 \subseteq C_0 \square X \quad \text{and} \quad X = \delta_{\overline{X}}^{-1}((C_0 \otimes X) + (C \otimes X_0)),$$

because $C = C_0 \wedge C_0$. Consequently, $\overline{X}$ is a semisimple $C$-comodule and has a left $C_0$-comodule structure $\delta_{\overline{X}} : \overline{X} \to C_0 \square \overline{X}$. Hence, we also conclude that $(\pi_C \square \pi)\delta_X = 0$ and $(\text{id} \square \pi)\delta_{\overline{X}} = 0$, because

$$X = \delta_{\overline{X}}^{-1}((C_0 \otimes X) + (C \otimes X_0)), \quad (\text{id} \square \pi)\delta_{\overline{X}} = (\pi_C \square \pi)\delta_X = 0$$

and $\pi$ is surjective. Since the row of the commutative diagram

$$
\begin{array}{c}
0 \longrightarrow \overline{C} \square X_0 \xrightarrow{\text{id} \square \pi} \overline{C} \square X \xrightarrow{\text{id} \square \pi} \overline{C} \square \overline{X}
\end{array}
$$

(5.4)
is exact and \((\text{id} \circ \varphi)\hat{\varphi}_X = 0\), there is a unique map \(\varphi_X : \overline{X} \to \overline{C} \boxtimes X_0\) of left \(C\)-comodules such that \(\overline{\varphi}_X = (\text{id} \circ u)\varphi_X\), where \(u : X_0 \to X\) is the inclusion. The left \(C\)-comodules \(\overline{C}\) and \(\overline{X}\) are semisimple, so they are left \(C_0\)-comodules and therefore \(\varphi_X\) is a map of left \(C_0\)-comodules. Note that \(\overline{C} \boxtimes C_0 X_0 = \overline{C} \boxtimes C_0 X_0 = \overline{C} \boxtimes X_0\) and there is a \(K\)-vector space decomposition \(X \cong X_0 \oplus \overline{X}\) of \(X\).

The following lemma is of importance.

**Lemma 5.5.** Let \(C\) be a coradical square complete coalgebra and \((X, \delta_X)\) be a left \(C\)-comodule. Under the identification \(X = X_0 \oplus \overline{X}\) and the notation above, the \(C\)-comodule structure map \(\delta_X : X_0 \oplus \overline{X} \to (C \otimes X_0) \oplus (C \otimes \overline{X})\) of \(X\) has the form

\[
\delta_X = \begin{bmatrix}
\delta_0 & \varphi_X \\
0 & \delta_{\overline{X}}
\end{bmatrix},
\]

where \(\varphi_X : \overline{X} \to C \otimes X_0\) is the composite \(K\)-linear map

\[
\overline{X} \leftarrow X_0 \oplus \overline{X} \xrightarrow{\delta_X} C \boxtimes X \xrightarrow{\text{id} \circ \pi X_0} C \boxtimes C_0 X_0 \leftarrow C \otimes X_0
\]

and \((\pi_C \otimes \text{id})\varphi_X = \varphi_X\). Moreover, \(\text{Im} \varphi_X \cap (C_0 \otimes X) = (0)\).

**Proof.** Consider the \(K\)-linear map

\[
\delta_X = \begin{bmatrix}
(\delta_X)_{1,1} & (\delta_X)_{1,2} \\
(\delta_X)_{2,1} & (\delta_X)_{2,2}
\end{bmatrix} : X_0 \oplus \overline{X} \to (C \otimes X_0) \oplus (C \otimes \overline{X}).
\]

Since \(\delta_X(X_0) \subseteq C_0 \otimes X_0\), we have \((\delta_X)_{1,1} = \delta_0\) and \((\delta_X)_{2,1} = 0\). By the definition of \(\overline{X}\), we have \(\delta_X \pi = (\text{id} \circ \pi)\delta_X\) and therefore \((\delta_X)_{2,2} = \delta_{\overline{X}}\). Finally, if \(\varphi_X = (\delta_X)_{1,2} : \overline{X} \to C \otimes X_0\) and \(i : \overline{X} \to X\) is the inclusion, then the equality \(X_0 = \delta_X^{-1}(C_0 \otimes X)\) and the commutativity of the diagrams (5.3) and (5.4) yield

\[
(\pi_C \otimes \text{id})\varphi_X = (\pi_C \otimes \text{id})(\text{id} \otimes \pi X_0)\delta_X i = (\text{id} \otimes \pi X_0)(\pi_C \otimes \text{id})\delta_X i = (\text{id} \otimes \pi X_0)\overline{\delta}_X \pi i = (\text{id} \otimes \pi X_0)(\text{id} \otimes u)\varphi_X = \varphi_X.
\]

**Definition 5.6.** We assume that \(C = C_1\) and use the notation introduced above. We define the reduction functor (5.2) by associating with each left \(C\)-comodule \((X, \delta_X)\) the left \(H_C\)-comodule

\[
H_C(X) = (X', X'', \varphi_X),
\]

where \(X'' = X_0 = \delta_X^{-1}(C_0 \otimes X) = \text{soc} X\) and \(X' = \overline{X} = X/X_0\) are viewed as left \(C_0\)-comodules (see (5.3)), and \(\delta_X = \begin{bmatrix} \delta_0 & \varphi_X \end{bmatrix}\) is as in Lemma 5.5.

Given \(f \in \text{Hom}_C(X, Y)\), we define \(H_C(f) : H_C(X) \to H_C(Y)\) to be the pair \(H_C(f) = (f', f'')\), where \(f'' : X_0 \to Y_0\) is the restriction of \(f\) and \(f' : \overline{X} \to Y\) is induced by \(f\).
We show that $\mathbb{H}_C(f)$ is an $H_C$-comodule homomorphism, by proving that the pair $(f', f'')$ is a morphism in the category $\text{Rep}_\odot(C, \overline{C}_C)$. We make the identifications $X = X_0 \oplus \overline{X}$ and $Y = Y_0 \oplus \overline{Y}$. Since $f : X_0 \oplus \overline{X} \to Y_0 \oplus \overline{Y}$ is a $C$-comodule homomorphism and $f(X_0) \subseteq Y_0$, $f$ has the matrix form

$$f = \begin{bmatrix} f'' & f_{1,2} \\ 0 & f' \end{bmatrix}$$

and $\delta_Y f = (\text{id} \otimes f) \delta_X$. By Lemma 5.5, we have $\delta_0 f'' = (\text{id} \otimes f'') \delta_0$, $\delta_Y f' = (\text{id} \otimes f') \delta_X$ and $\delta_0 f_{1,2} + \overline{\varphi}_X f' = (\text{id} \otimes f'') \overline{\varphi}_X + (\text{id} \otimes f_{1,2}) \delta_X$, and therefore $f'$ and $f''$ are $C_0$-comodule homomorphisms. Since $\text{Im}(\delta_0 f_{1,2}) \subseteq C_0 \otimes Y$, $\text{Im}(\text{id} \otimes f_{1,2}) \delta_X \subseteq C_0 \otimes Y$, $\text{Im}(\overline{\varphi}_X f') \cap (C_0 \otimes Y) = (0)$, and $\text{Im}(\text{id} \otimes f'') \overline{\varphi}_X \cap (C_0 \otimes Y) = (0)$, the final equality yields $\overline{\varphi}_X f' = (\text{id} \otimes f'') \overline{\varphi}_X$ and our claim is proved.

The main properties of the functor $\mathbb{H}_C$ are collected in Theorem 5.11 below. To formulate it, we need the following definition (cf. Gabriel [10]).

**Definition 5.8.** Let $C$ be a basic coalgebra and let $(\mathcal{C} Q, \mathcal{C} \text{d})$ be the left valued Gabriel quiver of $C$. The **left separated valued quiver** $(\mathcal{C} Q, \mathcal{C} \text{d})$ of $C$ is defined as follows. The set $\mathcal{C} Q_0$ of vertices is the disjoint union $\mathcal{C} Q'_0 \cup \mathcal{C} Q''_0$ of two copies of $\mathcal{C} Q_0$, where $\mathcal{C} Q'_0 = \{i' ; i \in I_C\}$ and $\mathcal{C} Q''_0 = \{j'' ; j \in I_C\}$. Given two vertices $a, b \in \mathcal{C} Q'_0 = \mathcal{C} Q'_0 \cup \mathcal{C} Q''_0$, there exists a unique valued arrow $a \xrightarrow{(\mathcal{C} d'_{ab}^{s}, \mathcal{C} d''_{ab}^{s})} b$ if and only if $a = i'$ with $i' \in \mathcal{C} Q'_0$, $b = j''$ with $j'' \in \mathcal{C} Q''_0$, and there exists a valued arrow $i \xrightarrow{(\mathcal{C} d'_{ij}^{s}, \mathcal{C} d''_{ij}^{s})} j$ in $(\mathcal{C} Q, \mathcal{C} \text{d})$. We set $\mathcal{C} d'_{ab}^{s} = \mathcal{C} d'_{ij}^{s}$ and $\mathcal{C} d''_{ab}^{s} = \mathcal{C} d''_{ij}^{s}$.

It follows that the valued quiver $(\mathcal{C} Q, \mathcal{C} \text{d})$ has no loops, no valued arrows between the vertices in $\mathcal{C} Q'_0$, between the vertices in $\mathcal{C} Q''_0$, and no valued arrow from a vertex $a \in \mathcal{C} Q''_0$ to $b \in \mathcal{C} Q'_0$.

To formulate the next result, we define the **stable categories** of $C$-$\text{Comod}$ and $C$-$\text{comod}$ to be the quotient categories

$$(5.9) \quad C-\text{Comod} = C-\text{Comod}/\mathcal{I} \quad \text{and} \quad C-\text{comod} = C-\text{comod}/\mathcal{I}$$

modulo the ideal $\mathcal{I}$ in $C$-$\text{Comod}$ and $C$-$\text{comod}$, respectively, consisting of all $C$-comodule homomorphisms $f : X \to Y$ having a factorisation through an injective comodule $E$ in $C$-$\text{Comod}$. More precisely, the objects of $C$-$\text{Comod}$ and $C$-$\text{comod}$ are the same as in $C$-$\text{Comod}$ and $C$-$\text{comod}$, respectively, and the space of morphisms from $X$ to $Y$ in the quotient category is the quotient $K$-vector space

$$(5.10) \quad \text{Hom}_C(X, Y) = \text{Hom}_C(X, Y)/\mathcal{I}(X, Y),$$
where $I(X,Y)$ is formed by all $f : X \to Y$ that have a factorisation through an injective in $C$-Comod (see [2]).

We denote by $H_C$-Comod$_{sp}$ the full subcategory of $H_C$-Comod whose objects are $H_C$-comodules $X$ such that $\text{soc } X$ is projective and has no injective summands of the form $\left[ \begin{array}{c} S(i') \\ 0 \end{array} \right]$, where $S(i')$ is a simple $C_0$-comodule.

**Theorem 5.11.** Assume that $C$ is a basic coradical square complete $K$-coalgebra. Let

$$H_C = \left[ \begin{array}{cc} C_0 & \overline{C} \\ 0 & C_0 \end{array} \right]$$

be the associated bipartite coalgebra (5.1), with $C_0 = \text{soc } C$ and $\overline{C} = C/C_0$.

(a) $H_C$ is basic, hereditary, coradical square complete, and every simple $C$-comodule is projective or injective.

(b) The reduction functor $H_C : C$-Comod $\to H_C$-Comod of (5.2) is $K$-linear, full, additive, commutes with arbitrary direct sums and has the following properties:

(b1) Given a $C$-comodule homomorphism $f : X \to Y$, we have $H_C(f) = 0$ if and only if $f(\text{soc } X) = 0$. In particular, the kernel of the algebra surjection $\text{End}_C X \to \text{End}_{H_C} H_C(X)$, $f \mapsto H_C(f)$, equals $\text{Hom}_C(X/\text{soc } X, X)$. If $X, Y$ have no injective direct summands then $H_C(f) = 0$ if and only if $f \in I(X,Y)$.

(b2) $H_C$ does not vanish on non-zero comodules, carries $\mathcal{C}C$ to the left coideal $\left[ \begin{array}{c} \overline{C} \\ 0 \end{array} \right]$ of $H_C = \left[ \begin{array}{cc} C_0 & \overline{C} \\ 0 & C_0 \end{array} \right]$ and carries simple comodules to simple ones.

(b3) A comodule $X = (X', X'', \varphi)$ in $H_C$-comod lies in $\text{Im } H_C$ if and only if $\varphi : X' \to \overline{C} \square X''$ is a monomorphism.

(b4) An indecomposable comodule $X$ in $H_C$-comod does not belong to $\text{Im } H_C$ if and only if $X$ is simple injective of the form $\left[ \begin{array}{c} S'(i') \\ 0 \end{array} \right]$, where $S'(i')$ is a simple subcomodule of $C$.

(b5) $\text{Im } H_C = H_C$-Comod$_{sp}$.

(c) The functor $H_C$ defines a representation equivalence (see [27], [38])

$H_C : C$-Comod $\to H_C$-Comod$_{sp} \subseteq H_C$-Comod

and carries indecomposable $C$-comodules to indecomposable ones.

(d) A $C$-comodule $E$ is injective if and only if $H_C(E)$ is an injective $H_C$-comodule. Moreover, the functor $H_C$ induces

- an isomorphism $F_a = \text{End}_C S(a) \cong \text{End}_{H_C} H_C(S(a))$ of division rings for each $a \in I_C$,
- equivalences of stable categories

$$C\text{-Comod} \cong H_C\text{-Comod} \quad \text{and} \quad C\text{-comod} \cong H_C\text{-comod}.$$
(e) The left valued Gabriel quiver of the hereditary coalgebra $H_C$ is the left separated valued quiver $(\tilde{Q}, \tilde{d})$ of $C$.

Proof. Throughout the proof, we make the identification $H_C\text{-Comod} = \text{Rep}_\Phi(C_0, \mathcal{C})$ via the functor $\Phi$ of (2.8) and (2.15) (see Theorem 2.14).

(a) Apply Theorem 2.16.

(b) That $\mathbb{H}_C$ is additive and commutes with arbitrary direct sums follows immediately from its definition.

Now we prove that $\mathbb{H}_C$ is full. Let $X, Y$ be $C$-comodules and $\mathbb{H}_C(X) = (\overline{X}, X_0, \varphi_X)$, $\mathbb{H}_C(Y) = (\overline{Y}, Y_0, \varphi_Y)$. Given a homomorphism $(f', f'') : \mathbb{H}_C(X) \to \mathbb{H}_C(Y)$ of $H_C$-comodules, we define a $K$-linear map

$$f = \begin{bmatrix} f'' & 0 \\ 0 & f' \end{bmatrix} : X \cong X_0 \oplus \overline{X} \to Y \cong Y_0 \oplus \overline{Y}.$$ 

We claim that $f$ is a $C$-comodule homomorphism such that $\mathbb{H}_C(f) = (f', f'')$. Indeed,

$$\delta_Y \circ f = \begin{bmatrix} \delta_0 & \varphi_Y \\ 0 & \delta_Y \end{bmatrix} \circ \begin{bmatrix} f'' & 0 \\ 0 & f' \end{bmatrix} = \begin{bmatrix} \delta_0 f'' & \varphi_Y f' \end{bmatrix}.$$ 

On the other hand,

$$(I \otimes f) \circ \delta_X = \begin{bmatrix} I \otimes f'' & 0 \\ 0 & I \otimes f' \end{bmatrix} \circ \begin{bmatrix} \delta_0 & \varphi_X \\ 0 & \delta_X \end{bmatrix} = \begin{bmatrix} (I \otimes f'') \delta_0 & (I \otimes f'') \varphi_X \\ 0 & (I \otimes f') \delta_X \end{bmatrix}.$$ 

Since $(f', f'')$ is an $H_C$-comodule homomorphism, $\delta_Y \circ f = (I \otimes f) \circ \delta_X$ and our claim follows, because the equality $\mathbb{H}_C(f) = (f', f'')$ is obvious. This shows that $\mathbb{H}_C$ is full.

(b1) If $X$ is a non-zero $C$-comodule, then $X_0 = \text{soc} X \neq 0$ and therefore $\mathbb{H}_C(X) \neq 0$.

Let $f : X \to Y$ be a non-zero $C$-homomorphism such that $\mathbb{H}_C(f) = 0$. By the definition of $\mathbb{H}_C$, we get $X_0 \subseteq \text{Ker} f$. Conversely, let $X_0 \subseteq \text{Ker} f$; then $f|_{X_0} = 0$. Since $C = C_0 \wedge C_0$, the left $C$-comodule $X/X_0$ is semisimple. Therefore $\text{Im} f \cong X/\text{Ker} f$ is semisimple and $\text{Im} f \subseteq Y_0$. Consequently, $\overline{f} = 0$ and $\mathbb{H}_C(f) = (\overline{f}, f|_{X_0}) = 0$.

To prove the second statement in (b1), assume that $X$ and $Y$ are $C$-comodules having no injective direct summands. Let $f \in \mathcal{I}(X, Y)$, that is, $f : X \to Y$ is a $C$-comodule homomorphism that factorises through an injective $C$-comodule $E$. Let $g : X \to E$ and $h : E \to Y$ be $C$-comodule homomorphisms such that $f = hg$. Assume, to the contrary, that $\mathbb{H}_C(f) \neq 0$. By the above considerations, $f(X_0) \neq 0$ and therefore $h|_{E_0} \neq 0$. Since $g(X_0) \subseteq E_0$, we have $0 \neq h(E_0) \subseteq Y$. There exists an indecomposable direct summand $E' \subseteq E$ such that $0 \neq h(E'_0) \subseteq Y$. If $\text{Ker} h|_{E'} \neq 0$ then the simple $C$-comodule $E'_0$ is contained in $\text{Ker} h|_{E'}$ and therefore $h(E'_0) = 0$, 


a contradiction. This proves that \( h_{|E'} : E' \to Y \) is a monomorphism. Since \( E' \) is injective, it is a direct summand of \( Y \), contrary to our assumption. Consequently, \( \mathbb{H}_C(f) = 0 \).

Conversely, let \( f : X \to Y \) be such that \( \mathbb{H}_C(f) = 0 \). Let \( \pi : X \to X/X_0 \) be the natural projection. By the first part of (b1) we have \( f(X_0) = 0 \). Therefore \( f = g\pi \) for some homomorphism \( g : X/X_0 \to Y \). Assume that \( j : X \to E(X) \) is the injective envelope of \( X \). Applying standard arguments we can construct commutative diagram with exact rows

\[
\begin{array}{c}
0 \to X_0 \to X \to \bar{X} \to 0 \\
| & & \downarrow \pi & & \\
0 \to X_0 \to E(X) \to \bar{E}(X)/X_0 \to 0
\end{array}
\]

where \( h \) is a monomorphism and the comodules \( \bar{X} = X/X_0 \), \( E(X)/X_0 \) are semisimple (because \( C \) is coradical square complete). Therefore there exists a homomorphism \( h_1 : E(X)/X_0 \to \bar{X} \) such that \( h_1h = \id_{\bar{X}} \), and hence \( f = g\pi = gh_1h\pi = gh_1\pi_1j \in \mathcal{I}(X,Y) \).

(b2) It was shown in the proof of (b1) that \( \mathbb{H}_C(X) \neq 0 \) if \( X \neq 0 \). By the definition of \( \mathbb{H}_C \), we know that \( \mathbb{H}_C(C) = \left[ \frac{C}{C_0} \right] \). Moreover, for any simple \( C \)-comodule \( S, \mathbb{H}_C(S) = (0, S, 0) = \left[ \begin{smallmatrix} 0 \\ S \end{smallmatrix} \right] \) is a simple \( H_C \)-comodule, by Theorem 2.16.

(b3) Take a \( C \)-comodule \( X \) and consider \( \mathbb{H}_C(X) = (\bar{X}, X_0, \varphi_X) \). Note that \( \bar{\delta}_X \) (defined in (5.3)) is a monomorphism. Indeed, assume that \( \bar{\delta}_X(x) = 0 \) for some \( x \in \bar{X} \). Then there exists \( y \in X \) such that \( \pi(y) = x \) and \( (\pi_C \Box \id)\delta_X(y) = \bar{\delta}_X(y) = 0 \). It follows that \( \delta_X(y) \in C_0 \Box X \) and \( y \in X_0 \). Finally, \( 0 = \pi(y) = x \) and \( \delta_X \) is a monomorphism. Therefore, by the definition, \( \varphi_X \) is a monomorphism. Conversely, let \( (X', X'', \varphi) \) be an \( H_C \)-comodule such that \( \varphi \) is a monomorphism. Let \( X \) be the \( K \)-vector space \( X = X'' \oplus X' \).

Note that there is an isomorphism of vector spaces \( C \cong C_0 \oplus C/C_0 \). It is easy to see that the \( K \)-linear map

\[
\delta_X = \begin{bmatrix} \delta_{X''} & \varphi \\ 0 & \delta_{X'} \end{bmatrix} : X'' \oplus X' \to (C \otimes X'') \oplus (C \otimes X')
\]

defines a \( C \)-comodule structure on \( X \). Since \( \varphi \) is a monomorphism, we have \( \soc X = X'' \) and therefore \( \mathbb{H}_C(X) = (X', X'', \varphi) \) (see Lemma 5.5).

(b4) The proof above shows that the \( H_C \)-comodules of the form \( (X', 0, 0) \), where \( X' \neq 0 \), are not in \( \text{Im} \mathbb{H}_C \). Conversely, let \( (X', X'', \varphi) \) be an \( H_C \)-comodule such that \( \varphi \) is not a monomorphism. Then there exists a non-zero direct summand of \( (X', X'', \varphi) \) of the form \( (Y', 0, 0) \), namely \( (\ker \varphi, 0, 0) \). Hence (b4) follows, because \( C_0 \) is a semisimple \( K \)-coalgebra.

(b5) follows from (b3), (b4), and Theorem 2.16.
(c) We recall that an additive functor is said to be a representation equivalence (or epivalence, see [12]) if it is full, dense, and respects isomorphisms (see [27], [28], and [38]). By (b), the functor $\mathbb{H}_C : C\text{-comod} \to H_C\text{-comod}_{sp}$ is full and dense. To show that $\mathbb{H}_C$ reflects isomorphisms, assume that $f : X \to Y$ is a $C$-homomorphism in $C\text{-Comod}$ such that $\mathbb{H}_C(f) = (f', f'')$ is an isomorphism. It follows that $f'' : X_0 \to Y_0$ and $f' : \bar{X} \to \bar{Y}$ are isomorphisms. Hence, in view of the Snake Lemma, $f$ is an isomorphism and the first part of (c) follows.

To finish the proof of (c), assume that $X$ is an indecomposable $C$-comodule but $\mathbb{H}_C(X) \cong \bar{Y} \oplus \bar{Z}$ decomposes. By (b4), the $H_C$-comodules $\bar{Y}$ and $\bar{Z}$ lie in the image of $\mathbb{H}_C$. Therefore there exist $C$-comodules $Y$ and $Z$ such that $\bar{Y} \cong \mathbb{H}_C(Y)$ and $\bar{Z} \cong \mathbb{H}_C(Z)$. Hence $\mathbb{H}_C(X) \cong \mathbb{H}_C(Y \oplus Z)$, because $\mathbb{H}_C$ is additive. Since we have shown that $\mathbb{H}_C$ reflects isomorphisms, the $C$-comodule $X \cong Y \oplus Z$ decomposes, a contradiction.

(d) Let $E$ be an indecomposable injective $C$-comodule. There exists a $C$-comodule $E'$ such that $E \oplus E' \cong C$. Then $\mathbb{H}_C(C) \cong \mathbb{H}_C(E \oplus E') \cong \mathbb{H}_C(E) \oplus \mathbb{H}_C(E')$ and $\mathbb{H}_C(E)$ is a direct summand of $\mathbb{H}_C(C)$. By (b2) and (2.5) the $H_C$-comodule $\mathbb{H}_C(E)$ is injective.

Conversely, let $\mathbb{H}_C(E)$ be an indecomposable injective $H_C$-comodule. By (b4), there exists an $H_C$-comodule $\bar{X}$ such that $\mathbb{H}_C(E) \oplus \bar{X} \cong \frac{C}{C_0}$ and there exists a $C$-comodule $X$ such that $\mathbb{H}_C(X) \cong \bar{X}$. Therefore $\mathbb{H}_C(C) \cong \mathbb{H}_C(E \oplus X)$. Since $\mathbb{H}_C$ reflects isomorphisms, we have $C \cong E \oplus X$, and hence $E$ is injective.

The first item in the final part of (d) follows from the first one and (b). To finish the proof of (d), we note that $\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod}$ induces the functors

$$
\mathbb{H}_C : C\text{-Comod} \longrightarrow H_C\text{-Comod} \quad \text{and} \quad \mathbb{H}_C : C\text{-comod} \longrightarrow H_C\text{-comod}
$$

that are full (by (c)) and dense, because $\mathbb{H}_C$ carries injectives to injectives and all non-injective comodules in $H_C$-Comod are in $\text{Im} \mathbb{H}_C$, by (b4). It remains to show that $\mathbb{H}_C$ is faithful. Let $\bar{f} : \bar{X} \to \bar{Y}$ be a morphism in $C$-Comod with $f \in \text{Hom}_C(X, Y)$ such that $\mathbb{H}_C(\bar{f}) = 0$. We can assume that $X$ and $Y$ have no non-zero injective summands. Then $\mathbb{H}_C(f) : \mathbb{H}_C(X) \to \mathbb{H}_C(Y)$ has a factorisation $\mathbb{H}_C(X) \xrightarrow{g_1} Z \xrightarrow{g_2} \mathbb{H}_C(Y)$, where $Z$ is an injective $H_C$-comodule. By (c) and the first part of (d), $Z \cong \mathbb{H}_C(E)$, where $E$ is an injective $C$-comodule, and there exist $C$-comodule homomorphisms $X \xrightarrow{f_1} E \xrightarrow{f_2} Y$ such that $\mathbb{H}_C(f_1) = g_1$ and $\mathbb{H}_C(f_2) = g_2$. It follows that $\mathbb{H}_C$ vanishes on $h = f - g_2g_1 : X \to Y$ and, by (b1), $h \in I(X, Y)$. Hence $f = h + g_2g_1 \in I(X, Y)$ and therefore $\bar{f}$ is zero in the quotient category $C$-Comod. This shows that the functor $\mathbb{H}_C$ is faithful, and consequently, it is an equivalence of categories.
(e) We apply Corollary 3.8 to $H = H_C$. In this case, we have

$$H' = C_0, \quad H'' = C_0, \quad U = \overline{C} = C/C_0, \quad I_{H'} = I_C, \quad I_{H''} = I_C.$$ 

In the notation of (3.11), given $s' = s \in I_{H'} = I_C$ and $s' = s \in I_{H''} = I_C$, we have $s' U_{s'} = s(C/C_0)$. Hence, (e) follows from Corollary 3.8, Proposition 3.5 and the definition of the separated Gabriel valued quiver of $C$. 

Following [38, Remark XIX.1.13] and the proof of the previous theorem, we construct a functor

$$\mathbb{H}_C^*: H_C\text{-comod}^{\ast}_{\text{sp}} \to C\text{-comod}$$

as follows. Given an $H_C$-comodule $(X', X'', \varphi)$ in $H_C\text{-comod}^{\ast}_{\text{sp}} = \text{Im} \mathbb{H}_C$, we set

$$\mathbb{H}_C^*(X', X'', \varphi) = \left( X'' \oplus X', \begin{bmatrix} \delta X'' & \varphi \\ 0 & \delta X' \end{bmatrix} \right),$$

and given a homomorphism $(f', f''): (X', X'', \varphi) \to (Y', Y'', \varphi)$ in the category $H_C\text{-comod}^{\ast}_{\text{sp}}$, we set $\mathbb{H}_C^*(f', f'') = \begin{bmatrix} f'' & 0 \\ 0 & f' \end{bmatrix}$. It is clear that we have defined a covariant $K$-linear functor $\mathbb{H}_C^*$. Now we collect its main properties.

**Corollary 5.13.** Assume that $C$ is a basic coradical square complete $K$-coalgebra. Under the notation and assumptions of Theorem 5.11, the functor $\mathbb{H}_C^*: H_C\text{-comod}^{\ast}_{\text{sp}} \to C\text{-Comod}$ has the following properties.

(a) $\mathbb{H}_C \circ \mathbb{H}_C^*$ is isomorphic to the identity functor on $H_C\text{-comod}^{\ast}_{\text{sp}}$.

(b) $\mathbb{H}_C^*$ is faithful, exact, carries indecomposables to indecomposables, and non-isomorphic comodules to non-isomorphic ones.

**Proof.** (a) This follows from the proof of Theorem 5.11(b).

(b) Obviously, $\mathbb{H}_C^*$ is faithful and exact. Let $(X', X'', \varphi)$ be an object in $H_C\text{-comod}^{\ast}_{\text{sp}} = \text{Im} \mathbb{H}_C$ and assume that $X = \mathbb{H}_C^*((X', X'', \varphi)) \cong Y \oplus Z$ for some non-zero $C$-comodules $Y$ and $Z$. By (a), we have

$$(X', X'', \varphi) \cong \mathbb{H}_C \circ \mathbb{H}_C^*((X', X'', \varphi)) \cong \mathbb{H}_C(Y \oplus Z) \cong \mathbb{H}_C(Y) \oplus \mathbb{H}_C(Z).$$

It follows that $(X', X'', \varphi)$ is decomposable, because by Theorem 5.11(b2) the functor $\mathbb{H}_C$ does not vanish on non-zero objects. Since the final part of (b) is a consequence of (a), the proof is complete. 

6. Applications. We recall from [20], [29] and [30] that a $K$-coalgebra $C$ is said to be left pure semisimple if every left $C$-comodule is a direct sum of finite-dimensional $C$-comodules (see also [23], [24], and [25]).

The following characterisation of left pure semisimple coalgebras is of importance.
Theorem 6.1. Assume that $C$ is a $K$-coalgebra. The following conditions are equivalent.

(a) $C$ is left pure semisimple.

(b) For every infinite sequence $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots$ of non-zero monomorphisms between indecomposables in $C$-comod there exists $m_0 \geq 1$ such that $f_j$ is an isomorphism for all $j \geq m_0$.

(c) For every infinite sequence $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_1} \cdots$ of non-zero non-isomorphisms between indecomposables in $C$-comod there exists $m_0 \geq 1$ such that $f_j \ldots f_1 = 0$ for all $j \geq m_0$.

Proof. Apply [21, Theorem 3.1] and [22, Theorem 6.3] to $A = C$-Comod (see also [29, Theorem 7.2]).

The following result shows that the reduction functor $H_C$ respects pure semisimplicity.

Proposition 6.2. Assume that $C$ is a basic coradical square complete $K$-coalgebra and let $H_C = \left[ \begin{smallmatrix} C_0 & \mathcal{C} \\ 0 & C_0 \end{smallmatrix} \right]$ be the associated bipartite hereditary coalgebra, with $C_0 = \text{soc} \ C$ and $\mathcal{C} = C/\text{soc} \ C$. The following conditions are equivalent.

(a) $C$ is left pure semisimple.

(b) $H_C$ is left pure semisimple.

(c) $H_C$ is a direct sum of finite-dimensional coalgebras of finite comodule type.

(d) The left separated valued quiver $(\mathcal{C}Q, \mathcal{C}d)$ is a disjoint union of Dynkin valued quivers, that is, finite valued quivers whose underlying graphs are Dynkin diagrams of one of the types $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$ (see [14, Table 2]).

Proof. We prove that (a) implies (b) by applying Theorem 6.1. Assume that $C$ is a basic left pure semisimple coalgebra and

$$Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} \cdots$$

is a sequence of non-zero non-isomorphisms between finite-dimensional indecomposable left $H_C$-comodules. We may assume that no $Y_i$ is simple injective, because otherwise some $f_i$ is zero or an isomorphism, contrary to assumption.

By Theorem 5.11(b), this sequence lies in $H_C$-comod$^{\bullet}_{sp} = \text{Im} \ H_C$. By Theorem 5.11(c), for each $i \geq 1$, there exists an indecomposable $C$-comodule $X_i$ in $C$-comod and a non-zero non-isomorphism $f_i \in \text{Hom}_C(X_i, X_{i+1})$ such
that \( \mathbb{H}_C(X_i) = Y_i \) and \( \mathbb{H}_C(f_i) = \bar{f}_i \). Thus we have a sequence

\[
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots
\]

of non-zero non-isomorphisms between finite-dimensional indecomposable \( C \)-comodules. Since \( C \) is left pure semisimple, there exists \( m_0 \geq 1 \) such that \( f_j \ldots f_1 = 0 \) for all \( j \geq m_0 \); hence \( \bar{f}_j \ldots \bar{f}_1 = 0 \) for all \( j \geq m_0 \). Then, in view of Theorem 6.1, \( H_C \) is left pure semisimple.

To prove that (b) implies (a), assume that \( H_C \) is left pure-semisimple. Let

\[
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots
\]

be a sequence of non-zero monomorphisms between finite-dimensional indecomposable \( C \)-comodules. It follows that \( f_m \ldots f_1(\text{soc} \, X_1) \neq 0 \) for each \( m \geq 1 \), and, according to Theorem 5.10, \( \mathbb{H}_C(f_m \ldots f_1) = \mathbb{H}_C(f_m) \ldots \mathbb{H}_C(f_1) : \mathbb{H}_C(X_1) \rightarrow \mathbb{H}_C(X_m) \) is non-zero. By Theorem 5.10, the sequence

\[
Y_1 \xrightarrow{\bar{f}_1} Y_2 \xrightarrow{\bar{f}_2} \ldots
\]

with \( Y_i = \mathbb{H}_C(X_i) \), \( \bar{f}_i = \mathbb{H}_C(f_i) \) in \( H_C \)-comod\(^\bullet_{sp} \), consists of indecomposable comodules connected by non-zero homomorphisms. The observation made above yields \( \bar{f}_n \ldots \bar{f}_1 \neq 0 \) for each \( n \geq 1 \). Since \( H_C \) is pure semisimple, there exists \( i_0 \) such that \( \bar{f}_n \) is an isomorphism for any \( n \geq i_0 \). Hence, \( f_n \) is an isomorphism for any \( n \geq i_0 \), because \( \mathbb{H}_C \) reflects isomorphisms by Theorem 5.10(c). Consequently, \( C \) is left pure semisimple by Theorem 6.1, and therefore (a) and (b) are equivalent.

To prove \( (b) \iff (c) \), it is sufficient to show that the left pure semisimplicity of \( H_C \) implies (c), because the converse follows from [29, Theorem 7.5].

Assume that \( H_C \) is left pure semisimple and decompose it into a direct sum

\[
H_C = \bigoplus_{\beta \in T} H_\beta
\]

of indecomposable coalgebras \( H_\beta \). It follows that, for each \( \beta \in T \), the left valued Gabriel quiver \( (H_\beta Q, H_\beta d) \) is a connected component of \( (H_C Q, H_C d) \) (see [29, Corollary 8.7] and [32, Corollary 2.8]). Since \( H_C \) is hereditary and left pure semisimple, so is \( H_\beta \) for each \( \beta \in T \). Then, according to [14, Theorem 4.14] (see also [20] and [29]), either the quiver \( (H_\beta Q, H_\beta d) \) is one of the infinite pure semisimple locally Dynkin valued quivers \( \mathbb{A}_{\infty}^{(s)}, \mathbb{A}_{\infty}^{(s)}, \mathbb{B}_{\infty}^{(s)}, \mathbb{C}_{\infty}^{(s)} \) or \( \mathbb{D}_{\infty}^{(s)} \), with \( s \geq 0 \), presented in [14, Table 1], or \( (H_\beta Q, H_\beta d) \) is finite and its underlying valued graph is one of the Dynkin valued diagrams \( \mathbb{A}_n \) \( (n \geq 1) \), \( \mathbb{B}_n \) \( (n \geq 2) \), \( \mathbb{C}_n \) \( (n \geq 3) \), \( \mathbb{D}_n \) \( (n \geq 4) \), \( \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4 \) or \( \mathbb{G}_2 \) presented in [14, Table 2].
Since every infinite pure semisimple locally Dynkin valued quiver contains an infinite chain of the form $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots$, it follows that $(H_\beta Q, H_\beta d)$ is not infinite, because $(H_C Q, H_C d)$ is the separated valued quiver $(\hat{C} Q, \hat{d} d)$ of $C$, by Theorem 5.11(d), the quiver $(H_\beta Q, H_\beta d)$ is a connected subquiver of $(H_C Q, H_C d) = (\hat{C} Q, \hat{d} d)$, and it follows from the definition of separated valued quiver that it does not contain infinite chains of the above form. Consequently, $(H_\beta Q, H_\beta d)$ is finite and the underlying valued graph of $(H_\beta Q, H_\beta d)$ is one of the Dynkin valued diagrams. It follows that $\dim_K H_\beta$ is finite and, by [29, Theorem 7.5], the coalgebra $H_\beta$ is of finite comodule type for each $\beta \in T$. This finishes the proof of (b) $\Leftrightarrow$ (c). Since this also shows that (c) and (d) are equivalent, the proposition is proved.

**Corollary 6.3.** Let $C = D \ltimes_D U_D$ be the trivial extension of a basic semisimple coalgebra $D$ by a $D$-$D$-bicomodule $U_D$.

(a) $C$ is coradical square complete, the associated bipartite coalgebra $H_C$ is the hereditary coalgebra $[\begin{array}{c} D \\ D D_D \end{array}]_0$ and the reduction functor $H_C$ is a representation equivalence.

(b) The left valued Gabriel quiver of $C$ has the form $(D Q, D d) (D Q, D d)$ (see (4.9)), that is, it is obtained from the valued quiver $(D Q, D d) (D Q, D d)$ (see (3.10)) of the bipartite coalgebra $[\begin{array}{c} D \\ D D_D \end{array}]_0$ by the identification of the vertex $s'$ with the vertex $s''$ and the arrow $s' \rightarrow t'$ with the arrow $s'' \rightarrow t''$ in $(D Q, D d) (D Q, D d)$, for all $s, t \in D Q_0 = I_D$.

(c) $C$ is left pure semisimple if and only if $[\begin{array}{c} D \\ D D_D \end{array}]_0$ is left pure semisimple, and if and only if the left separated valued quiver of $C$ is a disjoint union of Dynkin valued quivers.

**Proof.** Apply Proposition 4.10, Theorem 5.11, and Proposition 6.2.

**Example 6.4.** Let $\mathbb{N}$ be the set of positive integers and let

$$C = \bigoplus_{n \in \mathbb{N}} Ke_n \oplus \bigoplus_{m \in \mathbb{N}} K \eta_m$$

be a $K$-vector space with a countable basis $\{e_n, \eta_m\}_{n, m \in \mathbb{N}}$ equipped with the comultiplication $\Delta : C \rightarrow C \otimes C$ and the counit $\varepsilon : C \rightarrow K$, defined by the formulae:

- $\Delta(e_n) = e_n \otimes e_n$ and $\Delta(\eta_m) = e_m \otimes \eta_m + \eta_m \otimes e_{m+1}$,
- $\varepsilon(e_n) = 1$ and $\varepsilon(\eta_m) = 0$ for $n, m \in \mathbb{N}$.

It is straightforward to check that $C = (C, \Delta, \varepsilon)$ is a basic $K$-coalgebra, $C_0 = \text{soc} C = \bigoplus_{n \in \mathbb{N}} S(n)$, where $S(n) = Ke_n$, is a simple subcoalgebra of $C$, and $C = C_1 = C_0 \wedge C_0$, that is, $C$ is coradical square complete.

It is easy to check that, for each $i \in \mathbb{N}$, we have $\text{Ext}_C^i(S(i), S(i+1)) \cong K$ and $\text{Ext}_C^i(S(i), S(j)) = 0$ for $j \neq i + 1$. It follows that the separated valued
quiver \((C^dQ, C^d\mathfrak{d})\) has the form

\[
\begin{array}{cccccccc}
1' & 2' & 3' & 4' & 5' & 6' & \cdots \\
\downarrow & \downarrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
1'' & 2'' & 3'' & 4'' & 5'' & 6'' & \cdots
\end{array}
\]

and, by Proposition 6.2, \(C\) is left pure semisimple.

Note also that \(C\) is isomorphic to the trivial extension coalgebra \(D \ltimes DU_D\), where \(D = \text{soc} \, C\) is a basic semisimple subcoalgebra of \(C\) and \(DU_D = \bigoplus_{m \in \mathbb{N}} K \eta_m \subseteq C\) is viewed as a \(D\)-\(D\)-bicomodule in the obvious way.

It follows from Theorem 5.11 and Corollary 6.3 that the left Gabriel quiver of the bipartite coalgebra \(H_C\) is the quiver presented above, whereas the left Gabriel quiver of \(C \cong D \ltimes DU_D\) is the infinite linear quiver

\[
Q : 1 \xrightarrow{\beta_1} 2 \xrightarrow{\beta_2} \cdots \to s - 1 \xrightarrow{\beta_{s-1}} s \xrightarrow{\beta_s} s + 1 \xrightarrow{\beta_{s+1}} \cdots
\]

obtained from the above by the identification \(n \equiv n' \equiv n''\) for each \(n \in \mathbb{N}\).

Let \(K^\square Q\) be the path coalgebra of the quiver \(Q\). One can show that there is a coalgebra isomorphism \(C \cong (K^\square Q)_1 = KQ_0 \oplus KQ_1\) given by \(e_n \mapsto \hat{e}_n\) (the stationary path at the vertex \(n \in Q_0\)) and \(\eta_n \mapsto \beta_n \in KQ_1\). Hence, by applying the results in [29], [31] and [33], one can show that \(C\) is isomorphic to the path coalgebra \(K^\square (Q, \Omega) = C(Q, \Omega)\) with the ideal \(\Omega \subseteq KQ\) of relations generated by all compositions \(\beta_n \beta_{n+1}\) with \(n \in \mathbb{N}\). Consequently, the category \(C\text{-comod} \cong K^\square (Q, \Omega)\text{-comod}\) is equivalent to the category \(\text{rep}_K(Q, \Omega)\) of finite-dimensional representations of \(Q\) satisfying the relation \(\beta_n \beta_{n+1} = 0\) for each \(n \in \mathbb{N}\).

We finish the paper by a discussion of tame and wild comodule type of any basic coalgebra \(C\) by means of its separated valued quiver. For the definition of tame and wild comodule type the reader is referred to [29, Definition 6.6], [30], and [31]. In particular, the tame-wild dichotomy for coalgebras over an algebraically closed field is discussed in [31].

**Proposition 6.5.** Assume that \(K\) is an algebraically closed field. Let \(C\) be a basic \(K\)-coalgebra, \(C_1\) the first term of the coradical filtration of \(C\), and \(H = H_{C_1}\) the associated hereditary bipartite coalgebra.

(a) The quiver \(HQ\) coincides with the left separated quiver \(C^dQ\).

(b) If \(H_{C_1}\) is of wild comodule type, then so is \(C\).

(c) If \(C\) is of tame comodule type, then so is \(H = H_{C_1}\), and the underlying non-oriented graph of each of the connected components of \(HQ\) (= \(C^dQ\)) is of one of the types:
\begin{itemize}
  \item the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.
  \item the Euclidean diagrams $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, $\tilde{\mathbb{E}}_8$.
  \item the infinite locally Dynkin diagrams (see [14], [29]–[31]),
  \begin{align*}
    \mathbb{A}_\infty & : \circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots, \\
    \infty \mathbb{A}_\infty & : \cdots \rightarrow \circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots, \\
    \mathbb{D}_\infty & : \circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots.
  \end{align*}
\end{itemize}

Proof. We recall from Theorem 5.11 that $H_{C_1}$ is hereditary.

(a) By Proposition 3.5, the left Gabriel quiver $C_1Q$ coincides with $CQ$. Then (a) follows from Theorem 5.11(d).

(b) Assume that $H_{C_1}$ is of wild comodule type. Then there exists a $K$-linear representation embedding functor $T : \text{mod}\, \Gamma_3(K) \to H_{C_1}\text{-comod}$, where $\Gamma_3(K) = [K, K^3]$ is the path $K$-algebra of the wild quiver $\circ \rightarrow \cdots \rightarrow \circ$. By [38, Corollary XVIII.4.2], there exists a full, faithful, exact $K$-linear endofunctor $F : \text{mod}\, \Gamma_3(K) \to \text{mod}\, \Gamma_3(K)$ such that $\text{Im}\, F$ is contained in the category $\text{add}\, R(\Gamma_3(K))$ of all regular $\Gamma_3(K)$-modules. It follows that the image of $T \circ F : \text{mod}\, \Gamma_3(K) \to H_{C_1}\text{-comod}$ does not contain simple comodules. Indeed, given a non-zero module $X$ in $\text{mod}\, \Gamma_3(K)$, the module $F(X)$ is regular, and hence not simple. It follows that there exists a non-split exact sequence $0 \to Y' \to F(X) \to Y'' \to 0$ in $\text{mod}\, \Gamma_3(K)$, where $Y'$ and $Y''$ are non-zero. Since $T$ is exact, we derive the exact sequence $0 \to T(Y') \to T(F(X)) \to T(Y'') \to 0$ in $H_{C_1}\text{-comod}$, where $T(Y')$ and $T(Y'')$ are non-zero. This shows that $\dim_K T(F(X)) \geq 2$, and consequently $T(F(X))$ lies in $H_{C_1}\text{-comod}^{\bullet}$. It follows that $T \circ F : \text{mod}\, \Gamma_3(K) \to H_{C_1}\text{-comod}$ defines a representation embedding $(T \circ F)' : \text{mod}\, \Gamma_3(K) \to H_{C_1}\text{-comod}^{\bullet}$. Since, by Corollary 5.13, $H_{C_1}^{\bullet} : H_{C_1}\text{-comod}^{\bullet} \to C_1\text{-comod}$ is a representation embedding, so is $H_{C_1}^{\bullet} \circ (T \circ F)' : \text{mod}\, \Gamma_3(K) \to C_1\text{-comod} \hookrightarrow C\text{-comod}$.

This shows that $C$ is of wild comodule type.

(c) Assume that $C$ is of tame comodule type. By [29, Theorem 6.11(a)] and its proof, the subcoalgebra $C_1$ of $C$ is also of tame comodule type. Suppose that $H_{C_1}$ is not tame. Since, by [31, Theorem 5.12], the tame-wild dichotomy holds for hereditary basic coalgebras, $H_{C_1}$ is of wild comodule type. Hence, by (b), $C$ is of wild comodule type and, according to [31, Corollary 5.6] (a weak version of tame-wild dichotomy for coalgebras), we get a contradiction.
We recall that $H_{C_1}$ is hereditary. Since it is of tame comodule type, every indecomposable coalgebra direct summand $H'$ of $H_{C_1}$ is also of tame comodule type and, obviously, the left Gabriel quiver $Q'$ of $H'$ is a connected component of $H_Q$. Then, by [29, Theorem 9.4] and [31, Theorem 5.12], the underlying unoriented graph of $Q'$ is of one of the types listed in (c).

Acknowledgments. The authors would like to thank the referee for corrections, useful suggestions and an improvement of the original text. They are also indebted to Dr. Gabriel Navarro for his valuable remarks and comments.

REFERENCES


[34] —, *Localising embeddings of comodule categories with applications to tame and Euler coalgebras*, J. Algebra 312 (2007), 455–494.


Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: justus@mat.uni.torun.pl
simson@mat.uni.torun.pl

*Received 22 March 2007; revised 8 August 2007*