MINIMAL GENERICS FROM SUBVARIETIES
OF THE CLONE EXTENSION
OF THE VARIETY OF BOOLEAN ALGEBRAS

BY

JERZY PŁONKA (Wrocław)

Abstract. Let $\tau$ be a type of algebras without nullary fundamental operation symbols. We call an identity $\varphi \approx \psi$ of type $\tau$ clone compatible if $\varphi$ and $\psi$ are the same variable or the sets of fundamental operation symbols in $\varphi$ and $\psi$ are nonempty and identical. For a variety $\mathcal{V}$ of type $\tau$ we denote by $\mathcal{V}^c$ the variety of type $\tau$ defined by all clone compatible identities from $\mathrm{Id}(\mathcal{V})$. We call $\mathcal{V}^c$ the clone extension of $\mathcal{V}$. In this paper we describe algebras and minimal generics of all subvarieties of $\mathcal{B}^c$, where $\mathcal{B}$ is the variety of Boolean algebras.

1. Preliminaries. Let $\tau : F \to \mathbb{N}$ be a type of algebras, where $F$ is the set of fundamental operation symbols and $\mathbb{N}$ is the set of positive integers. For a term $\varphi$ of type $\tau$, we denote by $\mathrm{Var}(\varphi)$ the set of variables occurring in $\varphi$ and by $F(\varphi)$ the set of fundamental operation symbols occurring in $\varphi$. For a variety $\mathcal{V}$ of type $\tau$ we denote by $\mathrm{Id}(\mathcal{V})$ the set of all identities of type $\tau$ satisfied in every algebra from $\mathcal{V}$. If $\Sigma$ is a set of identities of type $\tau$ we denote by $\mathrm{Mod}(\Sigma)$ the class of all algebras of type $\tau$ satisfying every identity from $\Sigma$. We shall use variables $x, y, z, u, v, x_1, \ldots, x_k, \ldots$, where $k < \omega$. An identity $\varphi \approx \psi$ of type $\tau$ is called clone compatible if $\varphi$ and $\psi$ are the same variable or $F(\varphi) = F(\psi) \neq \emptyset$. For a variety $\mathcal{V}$ of type $\tau$ we denote by $\mathcal{V}^c$ the variety of type $\tau$ defined by all clone compatible identities from $\mathrm{Id}(\mathcal{V})$. We call $\mathcal{V}^c$ the clone extension of $\mathcal{V}$ (see [2]–[9]). In [2], [4] and [6] some representation theorems for algebras from $\mathcal{V}^c$ were presented.

Let $\mathfrak{A} = (A; F^\mathfrak{A})$ be an algebra of type $\tau$. If $f^\mathfrak{A}$ is a fundamental operation from $F^\mathfrak{A}$ we shall often omit the upper index $\mathfrak{A}$ in $f^\mathfrak{A}$ when it is clear that we consider an operation in $\mathfrak{A}$. An endomorphism $r : A \to A$ of $\mathfrak{A}$ is called a splitting retraction of $\mathfrak{A}$ if it is idempotent ($r \circ r = r$) and for all $f \in F$, $a_1, \ldots, a_{\tau(f)} \in A$ and $k = 1, \ldots, \tau(f)$, we have

$$r(f^\mathfrak{A}(a_1, \ldots, a_{\tau(f)})) = f^\mathfrak{A}(a_1, \ldots, a_{k-1}, r(a_k), a_{k+1}, \ldots, a_{\tau(f)}).$$

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An algebra $\mathfrak{A}$ is called a generic of a variety $\mathcal{V}$ if $\text{HSP}(\mathfrak{A}) = \mathcal{V}$ (see [1, Appendix 4]). We call a generic $\mathfrak{A} = (A; F^\mathfrak{A})$ of $\mathcal{V}$ a minimal generic of $\mathcal{V}$ if for every generic $\mathfrak{B} = (B; F^\mathfrak{B})$ of $\mathcal{V}$ we have $|B| \geq |A|$. Let $\mathfrak{A} = (A; F^\mathfrak{A})$ be a minimal generic of $\mathcal{V}$. We put $g(\mathcal{V}) = |A|$. In the following we restrict our considerations to the type $\tau_b : \{+, ',\} \rightarrow \mathbb{N}$ where $\tau_b(+) = \tau_b(\cdot) = 2$ and $\tau_b(') = 1$. We denote by $\mathcal{B}$ the variety of Boolean algebras of type $\tau_b$.

Let us consider the following six algebras:

- $\mathfrak{A}_1 = (\{a_1, b_1\}; +, ',\prime)$ where for $x, y \in \{a_1, b_1\}$ we have
  
  $x + y = \begin{cases} x & \text{if } x = y, \\ b_1 & \text{otherwise}, \end{cases}$

  $a'_1 = b_1, \quad b'_1 = a_1$;

- $\mathfrak{A}_2 = (\{a_2, b_2\}; +, ',\prime)$ where
  
  $x + y = \begin{cases} x & \text{if } x = y, \\ b_2 & \text{otherwise}, \end{cases}$

  $x \cdot y = x' = b_2$ for every $x, y \in \{a_2, b_2\}$;

- $\mathfrak{A}_3 = (\{a_3, b_3\}; +, ',\prime)$ where
  
  $x \cdot y = \begin{cases} x & \text{if } x = y, \\ b_3 & \text{otherwise}, \end{cases}$

  $x + y = x' = b_3$ for every $x, y \in \{a_3, b_3\}$;

- $\mathfrak{A}_4 = (\{a_4, b_4\}; +, ',\prime)$ where
  
  $x + y = x \cdot y = x' = b_4$ for every $x, y \in \{a_4, b_4\}$;

- $\mathfrak{A}_5 = (\{a_5, b_5\}; +, ',\prime)$ where
  
  $x' = x, \quad x + y = x \cdot y = b_5$ for every $x, y \in \{a_5, b_5\}$;

- $\mathfrak{A}_6 = (\{a_6, b_6, c_6\}; +, ',\prime)$ where
  
  $a'_6 = c_6, \quad c'_6 = a_6, \quad b'_6 = b_6$;

  $x + y = x \cdot y = b_6$ for every $x, y \in \{a_6, b_6, c_6\}$.

We see that no two of these algebras are isomorphic and $\mathfrak{A}_1$ is a 2-element Boolean algebra.

It follows from [3, Theorem 2.10 and remarks on p. 190] that

(1.1) An algebra $\mathfrak{A}$ of type $\tau_b$ belongs to $\mathcal{B}^c$ and is subdirectly irreducible iff $\mathfrak{A}$ is isomorphic to one of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_6$.

Define $\text{Ir}(\mathcal{B}^c) = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_6\}$. If $\mathcal{V}$ is a subvariety of $\mathcal{B}^c$ and an algebra $\mathfrak{B}$ belongs to $\mathcal{V}$ and is subdirectly irreducible then by (1.1) it has to be isomorphic to some algebra from $\text{Ir}(\mathcal{B}^c)$. Since by Birkhoff’s theorem (see
1, Theorem 20.3}) every variety is uniquely determined by its subdirectly irreducible algebras, by (1.i) we have

(1.ii) Every subvariety \( V \) of \( B^c \) is uniquely determined by the set \( \text{Ir}(V) = V \cap \text{Ir}(B^c) \), namely \( V = \text{HSP}(\text{Ir}(V)) \).

If \( V \) is a subvariety of \( B^c \) and \( S = V \cap \text{Ir}(B^c) \) we shall write \( V = \mathcal{V}(S) \). So one wishes to determine which subsets of \( \text{Ir}(B^c) \) are of the form \( \text{Ir}(V) \) for some \( V \in L(B^c) \), where \( L(B^c) \) is the lattice of subvarieties of \( B^c \).

It was shown in [5] that

(1.iii) The family \( S \) of all sets \( \text{Ir}(V) \) with \( V \in L(B^c) \) consists of the following 29 sets: \( \{\mathcal{A}_1, \ldots, \mathcal{A}_6\} \), \( \{\mathcal{A}_2, \ldots, \mathcal{A}_6\} \), \( \{\mathcal{A}_3, \ldots, \mathcal{A}_6\} \), \( \{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\} \), \( \{\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\} \), \( \{\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \), \( \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \), \( \{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4\} \), \( \{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4\} \), \( \{\mathcal{A}_5\} \), \( \{\mathcal{A}_6\} \), \( \{\mathcal{A}_1, \mathcal{A}_5, \mathcal{A}_6\} \), \( \{\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\} \), \( \{\mathcal{A}_1, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\} \), \( \{\mathcal{A}_5\} \), \( \{\mathcal{A}_4, \mathcal{A}_5\} \), \( \emptyset \). Moreover, the lattice \( L(B^c) \) is isomorphic to \( (S; \subseteq) \).

Also in [5, p. 164] we showed that

(1.iv) \( \mathcal{A}_5 \in \text{HSP}((\mathcal{A}_6)) \).

(1.v) If \( i, j \in \{2, 3, 5, 6\}, i \neq j \) and \( \{i, j\} \neq \{5, 6\} \),
then \( \mathcal{A}_4 \in \text{HSP}((\mathcal{A}_i, \mathcal{A}_j)) \).

(1.vi) \( \mathcal{A}_6 \in \text{HSP}((\mathcal{A}_1, \mathcal{A}_5)) \).

By (1.i) we have

(1.vii) \( B^c = \mathcal{V}((\mathcal{A}_1, \ldots, \mathcal{A}_6)) \).

For an arbitrary variety \( V \) let \( \text{CL}(V) \) denote the set of all clone compatible identities from \( \text{Id}(V) \). The set \( \text{CL}(V) \) need not be an equational theory. It is if \( V \) is the variety of distributive lattices (see [8]). This is also the case for every variety \( \mathcal{V} \) of groupoids. However, \( \text{CL}(B) \) is not an equational theory. In fact, the identity \( x + x \cdot y \approx x + x \cdot z \) is clone compatible but its consequence \( x + x \cdot y \approx x + x \cdot y' \) is not; here we adopt the convention that \( \cdot \) binds stronger than + and we omit suitable parentheses.

In [9] we described forms of identities and we constructed equational bases of all subvarieties of \( B^c \).

2. Representations and minimal generics. By Birkhoff's subdirect irreducibility theorem and (1.i)-(1.iii) we already have:

If an algebra \( \mathcal{A} \) belongs to \( \mathcal{V}(S) \), where \( S \in S \), then \( \mathcal{A} \) is isomorphic to a subdirect product of some algebras from \( S \).

To get a more illustrative representation of algebras from subvarieties of \( B^c \) we need Theorem 1 below, which is in fact an application of more general
theorems (see [2, Section 3], [4, Section 2], [6, Section 3]) to the variety $\mathcal{B}^c$. However, in Theorem 1 we give more details specifically for the variety $\mathcal{B}^c$.

We put
\[
q_+(x) = x + x,
q_-(x) = x \cdot x,
q'(x) = x',
q''(x) = (x')',
q_b(x) = q_+(q_-(q''(q''(x)))),
\]

**Theorem 1.** If an algebra $\mathfrak{A} = (A; +, \cdot, ')$ belongs to $\mathcal{B}^c$, then the following conditions hold.

(2.i) Each of the mappings $q_+, q_-, q''$ is a splitting retraction of $\mathfrak{A}$ and any two of them commute.

(2.ii) Put $A_+ = q_+(A)$, $A_- = q_-(A)$, $A'' = q''(A)$, $A_b = q_b(A)$. Then $q_+\mid A_+$ is the identity on $A_+$, $q_-\mid A_-$ is the identity on $A_-$, $q''\mid A''$ is the identity on $A''$ and $q_b\mid A_b$ is the identity on $A_b$.

(2.iii) If $a \in A$, then $q^\mathfrak{A}_\alpha(q^\mathfrak{A}_\alpha(\ldots (q^\mathfrak{A}_{\alpha_1}(a))\ldots)) = q^\mathfrak{A}_\alpha(a)$ for every $\alpha_1, \ldots, \alpha_n$ in \{(+, -), (\cdot)\} with $|\{\alpha_1, \ldots, \alpha_n\}| > 1$.

(2.iv) $A_+ \cap A_- = A_+ \cap A'' = A_\cdot \cap A'' = A_b$.

(2.v) The algebra $\mathfrak{A}_+ = (A_+; +|A_+)$ is a $+$-semilattice, the algebra $\mathfrak{A}_- = (A_-; \cdot|A_-)$ is a $\cdot$-semilattice, the algebra $\mathfrak{A}'' = (A''; '\mid A''_b)$ is an algebra with involution, i.e. it satisfies $(x')' = x$, and the algebra $\mathfrak{A}_b = (A_b; \{+\cdot\}'|A_b)$ belongs to $\mathcal{B}$.

(2.vi) If $a, b \in A$, then $a + b = q^\mathfrak{A}_+(a) + q^\mathfrak{A}_+(b)$, $a \cdot b = q^\mathfrak{A}_-(a) \cdot q^\mathfrak{A}_-(b)$ and $a' = (q^\mathfrak{A}_+(a))'$.

The construction used in Theorem 1 was called a clone extension of an algebra $\mathfrak{A}$ in [2] and [4], and a clone network over a network of splitting retractions in [6].

**Example 1.** Let $a \in A_+$ and $b \in A_-$. Then by (2.vi), (2.ii), (2.iii), (2.i) we have:
\[
a + b = q^\mathfrak{A}_+(a) + q^\mathfrak{A}_+(b) = q^\mathfrak{A}_+(a) + q^\mathfrak{A}_+(q^\mathfrak{A}_+(b)) = q^\mathfrak{A}_+(a) + q^\mathfrak{A}_+(b) = q^\mathfrak{A}_+(a) + q^\mathfrak{A}_+(b).
\]

We also have $a' = (q^\mathfrak{A}_+(a))' = (q^\mathfrak{A}_+(q^\mathfrak{A}_+(a)))' = (q^\mathfrak{A}_+(a))'$.

(2.vii) $g(\mathcal{B}^c) = 6$. Moreover, the subdirect product
\[
\mathfrak{A}(1, 2, 3, 5) = \{\langle a_1, a_2, b_3, b_5 \rangle, \langle a_1, b_2, a_3, b_5 \rangle, \langle a_1, b_2, b_3, b_5 \rangle,
\langle b_1, b_2, b_3, b_5 \rangle, \{a_1, b_2, b_3, a_5 \}, \{b_1, b_2, b_3, a_5 \} \}; +, \cdot, ' \}
\]
of the direct product $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_5$ is a minimal generic of $\mathcal{B}^c$. 
Proof. The first statement of (2.vii) holds by Theorem 4 from [4]. By (1.v) and (1.vi) we have \( \{ \mathfrak{A}_1, \ldots, \mathfrak{A}_6 \} \subseteq \text{HSP}(\mathfrak{A}(1, 2, 3, 5)) \), so by (1.vii) we have \( \mathcal{B}^c \subseteq \text{HSP}(\mathfrak{A}(1, 2, 3, 5)) \). But \( \mathfrak{A}(1, 2, 3, 5) \in \mathcal{B}^c \) by (1.i), so \( \text{HSP}(\mathfrak{A}(1, 2, 3, 5)) = \mathcal{B}^c \). □

To find minimal generics of proper subvarieties of \( \mathcal{B}^c \) we need some lemmas.

From now on we assume that \( \mathfrak{A} = (A; +, \cdot, ') \) belongs to \( \mathcal{B}^c \) so it is of the form described in Theorem 1.

Let us record the following obvious observation. If \( e \) is an identity of type \( \tau_b \) and \( \mathfrak{A} \) is a generic of \( \mathcal{V}(S) \), \( S \in \mathcal{S} \), then \( e \in \text{Id}(\mathfrak{A}) \) iff \( e \in \text{Id}(\mathcal{V}(S)) \) iff for every \( \mathfrak{A}_k \in \mathcal{S} \) we have \( e \in \text{Id}(\mathfrak{A}_k) \) iff there is \( \mathfrak{A}_k \in \mathcal{S} \) with \( e \not\in \text{Id}(\mathfrak{A}_k) \). This observation will be useful in the proofs of some of the corollaries below.

**Lemma 1.** \(|A_b| = 1\) iff \( \mathfrak{A} \) satisfies

\[
q_b(x) \approx q_b(y).
\]

\((1)\)

\(q_b(x) \Rightarrow \) Follows from the fact that for every \( a, b \in A \) we have \( q_b^A(a), q_b^A(b) \in A_b \) by (2.ii).

\(q_b(x) \Leftarrow \) If \( a, b \in A_b \), then by (2.ii) and (1) we have \( a = q_b^A(a) = q_b^A(b) = b \). □

**Lemma 2.** \( A_{(+)} \setminus A_b = \emptyset \), i.e. \( A_{(+)} = A_b \), iff \( \mathfrak{A} \) satisfies \n
\[
q_{(+)}(x) \approx q_b(x).
\]

\((2)\)

\(q_{(+)}(x) \Rightarrow \) By (2.iv) we have \( A_b \subseteq A_{(+)} \), so \( A_{(+)} \setminus A_b = \emptyset \) iff \( A_{(+)} = A_b \). So if \( A_{(+)} = A_b \), then for \( a \in A \) we have \( q_{(+)}(a) \in A_b \). Then by (2.ii) and (2.iii) we have \( q_{(+)}^A(a) = q_b^A(q_{(+)}^A(a)) = q_b^A(a) \).

\(q_{(+)}(x) \Leftarrow \) Obvious. □

The proofs of the next two lemmas are analogous to that of Lemma 2. It is enough to replace \((+)\) by \((\cdot)\) and \((+)\) by \("\)\) respectively.

**Lemma 3.** \( A_{(\cdot)} = A_b \) iff \( \mathfrak{A} \) satisfies

\[q_{(\cdot)}(x) \approx q_b(x).
\]

\((3)\)

**Lemma 4.** \( A_{(\cdot\cdot)} = A_b \) iff \( \mathfrak{A} \) satisfies

\[q_{(\cdot\cdot)}(x) \approx q_b(x).
\]

\((4)\)

**Corollary 1.** If \( S \in \mathcal{S} \), \( \mathfrak{A} \in \mathcal{V}(S) \) and \( \mathfrak{A}_1 \not\in \mathcal{S} \), then \(|A_b| = 1\).

\(Proof.\) If \( k \neq 1 \) and \( \mathfrak{A}_k \in \text{Ir}(\mathcal{B}^c) \), then \( \mathfrak{A}_k \) satisfies (1). By (1.ii) we have \( \mathcal{V}(S) = \text{HSP}(S) \), so \( \mathcal{V}(S) \) satisfies (1) and consequently \( \mathfrak{A} \) satisfies (1). Now by Lemma 1, \( A_b \) from \( \mathfrak{A} \) is 1-element. □

**Corollary 1’.** If \( S \in \mathcal{S} \) and \( \mathfrak{A} \) is a generic of \( \mathcal{V}(S) \), then \(|A_b| = 1\) iff \( \mathfrak{A}_1 \not\in \mathcal{S} \).
Proof. \(\Leftarrow\) Follows from Corollary 1.
\[\Rightarrow\] If \(\mathfrak{A}_1 \in S\), then \(\mathcal{V}(S)\) does not satisfy (1) since \(\mathfrak{A}_1\) does not. So \(\mathfrak{A}\) does not satisfy (1). Now by Lemma 1 we get \(|A_b| > 1\).

**Corollary 2.** If \(S \in S, \mathfrak{A} \in \mathcal{V}(S)\) and \(\mathfrak{A}_2 \not\in S\), then \(A_{(+)\mathfrak{A}} = A_b\).

**Proof.** If \(k \neq 2\) and \(A_k \in \text{Ir}(\mathcal{B}^c)\), then \(A_k\) satisfies (2). By (1.ii) we have \(\mathcal{V}(S) = \text{HSP}(S)\), so \(\mathcal{V}(S)\) satisfies (2) and consequently \(\mathfrak{A}\) does. Now by Lemma 2 we have \(A_{(+)\mathfrak{A}} = A_b\).

**Corollary 2'.** If \(S \in S\) and \(\mathfrak{A}\) is a generic of \(\mathcal{V}(S)\), then \(A_{(+)\mathfrak{A}} = A_b\) iff \(\mathfrak{A}_2 \not\in S\).

**Proof.** \(\Leftarrow\) Follows from Corollary 2.
\[\Rightarrow\] If \(\mathfrak{A}_2 \in S\) then \(\mathcal{V}(S)\) does not satisfy (2) since \(\mathfrak{A}_2\) does not. So \(\mathfrak{A}\) does not satisfy (2) and by Lemma 2 we get \(A_{(+)\mathfrak{A}} \neq A_b\).

**Corollary 3.** If \(S \in S, \mathfrak{A} \in \mathcal{V}(S)\) and \(\mathfrak{A}_3 \not\in S\), then \(A_{(+)\mathfrak{A}} = A_b\).

The proof is analogous to that of Corollary 2. It is enough to replace (2) by (3) and (+) by (\(\cdot\)).

**Corollary 3'.** If \(S \in S\) and \(\mathfrak{A}\) is a generic of \(\mathcal{V}(S)\), then \(A_{(+)\mathfrak{A}} = A_b\) iff \(\mathfrak{A}_3 \not\in S\).

The proof is analogous to that of Corollary 2'. It is enough to replace (2) by (3) and (+) by (\(\cdot\)).

**Corollary 4.** If \(S \in S, \mathfrak{A} \in \mathcal{V}(S)\) and \(\mathfrak{A}_5 \not\in S\), then \(A_{(+)\mathfrak{A}} = A_b\).

**Proof.** If \(\mathfrak{A}_5 \not\in S\) then by (1.ii), \(\mathfrak{A}_6 \not\in S\). If \(k \notin \{5, 6\}\) and \(A_k \in \text{Ir}(\mathcal{B}^c)\), then \(A_k\) satisfies (4). So \(\mathcal{V}(S)\) satisfies (4) and \(\mathfrak{A}\) satisfies (4). Now by Lemma 4 we get \(A_{(+)\mathfrak{A}} = A_b\).

**Corollary 4'.** If \(S \in S\) and \(\mathfrak{A}\) is a generic of \(\mathcal{V}(S)\), then \(A_{(+)\mathfrak{A}} = A_b\) iff \(\mathfrak{A}_5 \not\in S\).

**Proof.** \(\Leftarrow\) Follows from Corollary 4.
\[\Rightarrow\] If \(\mathfrak{A}_5 \in S\) then \(\mathcal{V}(S)\) does not satisfy (4) since \(\mathfrak{A}_5\) does not. So \(\mathfrak{A}\) does not satisfy (4) and by Lemma 4 we get \(A_{(+)\mathfrak{A}} \neq A_b\).

**Lemma 5.** \(\mathfrak{A}\) satisfies
\[q_{(+)\mathfrak{A}}(x) \approx q_{(+)\mathfrak{A}}(x)\]
iff for every \(a \in A_{(+)\mathfrak{A}}\) we have \(a' = a\).

**Proof.** \(\Rightarrow\) Let \(a \in A_{(+)\mathfrak{A}}\). Then by (2.ii) and (5) we have \(a = q_{(+)\mathfrak{A}}^{a\mathfrak{A}}(a) = q_{(+)\mathfrak{A}}^{a\mathfrak{A}}(a)\).
\[\Leftarrow\] Let \(a \in A\). Then \(q_{(+)\mathfrak{A}}^{a\mathfrak{A}}(a) \in A_{(+)\mathfrak{A}}\) by (2.ii). So by (2.ii) and the assumption we have \(q_{(+)\mathfrak{A}}^{a\mathfrak{A}}(a) = (q_{(+)\mathfrak{A}}^{a\mathfrak{A}}(a))' = q_{(+)\mathfrak{A}}^{a\mathfrak{A}}(a)\).
Lemma 6. If \( \mathfrak{A} \) does not satisfy (5) and \( |A_b| = 1 \), then \( |A_{(v)} \setminus A_b| \geq 2 \).

Proof. If \( \mathfrak{A} \) does not satisfy (5) then by Lemma 5 there exists \( a \in A_{(v)} \) with \( a \neq a' \). It cannot be the case that \( a \in A_b \) since by assumption \( \mathfrak{A}_b \) is a 1-element algebra. Consequently, \( a \in A_{(v)} \setminus A_b \). By (2.v) we have \( a' \in A_{(v)} \). We cannot have \( a' \in A_b \) since then, by (2.v), \( a = (a')' \in A_b \), which contradicts the assumption that \( |A_b| = 1 \). Thus \( a' \in A_{(v)} \setminus A_b \). \( \blacksquare \)

Lemma 7. If \( |A_b| \geq 2 \) and \( a \in A_{(v)} \setminus A_b \), then \( a' \neq a \) and \( a' \in A_{(v)} \setminus A_b \). So \( |A_{(v)} \setminus A_b| \geq 2 \).

Proof. By (2.vi) and (2.v) we have \( a' \in A_{(v)} \). Since \( \mathfrak{A}_b \) is a nontrivial Boolean algebra (see (2.v)), for \( b \in A_b \) we must have \( b' \neq b \). Therefore since \( q^a_b \) is an endomorphism of \( \mathfrak{A} \) onto \( \mathfrak{A}_b \), we have \( a' \neq a \). Moreover, \( a' \notin A_b \) since otherwise \( a = (a')' \in A_b \) contrary to the assumptions. Thus \( a' \in A_{(v)} \setminus A_b \). \( \blacksquare \)

Lemma 8. If \( \mathfrak{A}_6 \in S, S \in S \) and \( \mathfrak{A} \) is a generic of \( \mathcal{V}(S) \), then \( |A_{(v)} \setminus A_b| \geq 2 \).

Proof. \( \mathfrak{A} \) does not satisfy (5) since \( \mathfrak{A}_6 \) does not. So if \( |A_b| = 1 \) we get the statement by Lemma 6. Since \( \mathfrak{A}_6 \in S \) and \( \mathfrak{A}_6 \) does not satisfy (4), it follows that \( \mathfrak{A} \) does not satisfy (4) and by Lemma 4 we get \( A_{(v)} \setminus A_b \neq \emptyset \). Hence, if \( |A_b| > 1 \), we get the statement by Lemma 7. \( \blacksquare \)

If a set \( S \) belongs to \( S \) (see (1.iii)), then we shall write \( \mathcal{V}(i_1, \ldots, i_k) \) instead of \( \mathcal{V}(S) \), where \( i_1, \ldots, i_k \) is the sequence of different indices of all algebras from \( S \) written in increasing order. For example \( \mathcal{V}(2, 4) \) stands for \( \mathcal{V}(\{\mathfrak{A}_2, \mathfrak{A}_4\}) \).

Theorem 2. We have

(2.1) If \( \mathfrak{A} \in \mathcal{V}(2, \ldots, 6) \), then \( |A_b| = 1 \).

(2.2) If \( \mathfrak{A} \) is a generic of \( \mathcal{V}(2, \ldots, 6) \), then \( A_{(v)} \setminus A_b \neq \emptyset \neq A_{(v)} \setminus A_b \) and \( |A_{(v)} \setminus A_b| \geq 2 \).

(2.3) The subdirect product

\[
\mathfrak{A}(2, 3, 6) = (\langle a_2, b_3, b_6 \rangle, \langle b_2, a_3, b_6 \rangle, \\
\langle a_2, b_3, a_6 \rangle, \langle b_2, b_3, c_6 \rangle, \langle b_2, b_3, b_6 \rangle); +, \cdot')
\]

of the direct product \( \bigtimes_{2} \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_6 \) is a minimal generic of \( \mathcal{V}(2, \ldots, 6) \), i.e. \( g(\mathcal{V}(2, \ldots, 6)) = 5 \).

Proof. (2.1) holds by Corollary 1; (2.2) holds by Corollaries 2', 3' and Lemma 8. It remains to prove (2.3). By (1.v) and (1.iv) we get \( \mathfrak{A}_2, \ldots, \mathfrak{A}_6 \in \text{HSP}(\mathfrak{A}(2, 3, 6)) \). Therefore \( \mathcal{V}(2, \ldots, 6) \subseteq \text{HSP}(\mathfrak{A}(2, 3, 6)) \) by (1.ii). Since \( \mathfrak{A}(2, 3, 6) \in \mathcal{V}(2, \ldots, 6) \), it follows that \( \mathcal{V}(2, \ldots, 6) = \text{HSP}(\mathfrak{A}(2, 3, 6)) \). Thus \( \mathfrak{A}(2, 3, 6) \) is a generic of \( \mathcal{V}(2, \ldots, 6) \) and by (2.2) and (2.iv) it is a minimal generic of \( \mathcal{V}(2, \ldots, 6) \) since it contains five elements. \( \blacksquare \)
Theorem 3. We have

(3.1) If $\mathfrak{A} \in \mathcal{V}(2, \ldots, 5)$, then $|A_b| = 1$.
(3.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(2, \ldots, 5)$, then $A_{(+) \setminus A_b \neq \emptyset}$ and $A_{(-)} \setminus A_b \neq \emptyset$.
(3.3) The subdirect product

$$\mathfrak{A}(2, 3, 5) = (\{(a_2, b_3, b_5), (b_2, a_3, b_5), (b_2, b_3, a_5), (b_2, b_3, b_5); +, \cdot, ^\prime\})$$

of $\mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_5$ is a minimal generic of $\mathcal{V}(2, \ldots, 5)$, and consequently, $g(\mathcal{V}(2, \ldots, 5)) = 4$.

Proof. (3.1) holds by Corollary 1; (3.2) holds by Corollaries 2', 3' and 4'; (3.3) holds by (1.v) for $\{i, j\} = \{3, 5\}$. Thus $\mathcal{V}(2, \ldots, 5) = \text{HSP}(\mathfrak{A}(2, 3, 5))$ and we use the statement of (3.2).

Theorem 4. We have

(4.1) If $\mathfrak{A} \in \mathcal{V}(3, \ldots, 6)$, then $|A_b| = 1$ and $A_{(+) \setminus A_b \neq \emptyset}$.
(4.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(3, \ldots, 6)$, then $A_{(-)} \setminus A_b \neq \emptyset$ and $|A_{(-)} \setminus A_b| \geq 2$.
(4.3) The subdirect product

$$\mathfrak{A}(3, 6) = (\{(a_3, b_6), (b_3, a_6), (b_3, c_6), (b_3, b_6); +, \cdot, ^\prime\})$$

of $\mathfrak{A}_3 \times \mathfrak{A}_6$ is a minimal generic of $\mathcal{V}(3, \ldots, 6)$, and consequently, $g(\mathcal{V}(3, \ldots, 6)) = 4$.

Proof. (4.1) holds by Corollaries 1 and 2; (4.2) holds by Corollary 3' and Lemma 8; (4.3) holds by (1.iv) and (1.v).

Theorem 5. We have

(5.1) If $\mathfrak{A} \in \mathcal{V}(1, 3, 4, 5, 6)$, then $A_{(+) \setminus A_b = \emptyset}$.
(5.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(1, 3, 4, 5, 6)$, then $|A_b| \geq 2$, $A_{(-)} \setminus A_b \neq \emptyset$ and $|A_{(-)} \setminus A_b| \geq 2$.
(5.3) The subdirect product

$$\mathfrak{A}(1, 3, 5) = (\{(a_1, b_3, b_5), (b_1, b_3, b_5), (a_1, a_3, b_5), (a_1, b_3, a_5), (b_1, b_3, a_5); +, \cdot, ^\prime\})$$

of $\mathfrak{A}_1 \times \mathfrak{A}_3 \times \mathfrak{A}_5$ is a minimal generic of $\mathcal{V}(1, 3, 4, 5, 6)$. Consequently, $g(\mathcal{V}(1, 3, 4, 5, 6)) = 5$.

Proof. (5.1) holds by Corollary 2; (5.2) holds by Corollaries 1', 3' and Lemma 8; (5.3) holds by (1.v) and (1.vi).

Theorem 6. We have

(6.1) If $\mathfrak{A} \in \mathcal{V}(3, 4, 5)$, then $|A_b| = 1$ and $A_{(+) \setminus A_b = \emptyset}$.
(6.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(3, 4, 5)$, then $A_{(-)} \setminus A_b \neq \emptyset \neq A_{(v)} \setminus A_b$. 
(6.3) The subdirect product

\[ \mathfrak{A}(3, 5) = \{ \{a_3, b_5\}, \{b_3, b_5\}, \{b_3, a_5\}\} ; +, \cdot, ' \] of \( \mathfrak{A}_3 \times \mathfrak{A}_5 \) is a minimal generic of \( \mathcal{V}(3, 4, 5) \). So \( g(\mathcal{V}(3, 4, 5)) = 3 \).

Proof. (6.1) holds by Corollaries 1 and 2; (6.2) holds by Corollaries 3' and 4'; (6.3) holds by (1.v). ■

The proofs of the next three theorems are analogous to those of Theorems 4–6.

Theorem 7. We have

(7.1) If \( \mathfrak{A} \in \mathcal{V}(2, 4, 5, 6) \), then \( |A_b| = 1 \) and \( A(\cdot) = A_b \).

(7.2) If \( \mathfrak{A} \) is a generic of \( \mathcal{V}(2, 4, 5, 6) \), then \( A(\cdot) \setminus A_b \neq \emptyset \) and \( |A(\nu) \setminus A_b| \geq 2 \).

(7.3) The subdirect product

\[ \mathfrak{A}(2, 6) = \{ \{a_2, b_6\}, \{b_2, a_6\}, \{b_2, c_6\}, \{b_2, b_6\}\} ; +, \cdot, ' \] of \( \mathfrak{A}_2 \times \mathfrak{A}_6 \) is a minimal generic of \( \mathcal{V}(2, 4, 5, 6) \). Consequently, \( g(\mathcal{V}(2, 4, 5, 6)) = 4 \).

Theorem 8. We have

(8.1) If \( \mathfrak{A} \in \mathcal{V}(1, 2, 4, 5, 6) \), then \( A(\cdot) \setminus A_b = \emptyset \).

(8.2) If \( \mathfrak{A} \) is a generic of \( \mathcal{V}(1, 2, 4, 5, 6) \), then \( |A_b| \geq 2 \), \( A(\cdot) \setminus A_b \neq \emptyset \) and \( |A(\nu) \setminus A_b| \geq 2 \).

(8.3) The subdirect product

\[ \mathfrak{A}(1, 2, 5) = \{ \{a_1, b_2, b_5\}, \{b_1, b_2, b_5\}, \{a_1, a_2, b_5\}, \{a_1, b_2, a_5\}, \{b_1, b_2, a_5\}\} ; +, \cdot, ' \] of \( \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_5 \) is a minimal generic of \( \mathcal{V}(1, 2, 4, 5, 6) \). Consequently, \( g(\mathcal{V}(1, 2, 4, 5, 6)) = 5 \).

Theorem 9. We have

(9.1) If \( \mathfrak{A} \in \mathcal{V}(2, 4, 5) \), then \( |A_b| = 1 \) and \( A(\cdot) \setminus A_b = \emptyset \).

(9.2) If \( \mathfrak{A} \) is a generic of \( \mathcal{V}(2, 4, 5) \), then \( A(\cdot) \setminus A_b \neq \emptyset \neq A(\nu) \setminus A_b \).

(9.3) The subdirect product

\[ \mathfrak{A}(2, 5) = \{ \{a_2, b_5\}, \{b_2, b_5\}, \{b_2, a_5\}\} ; +, \cdot, ' \] of \( \mathfrak{A}_2 \times \mathfrak{A}_5 \) is a minimal generic of \( \mathcal{V}(2, 4, 5) \). So \( g(\mathcal{V}(2, 4, 5)) = 3 \).

Theorem 10. We have

(10.1) If \( \mathfrak{A} \in \mathcal{V}(2, 3, 4) \), then \( |A_b| = 1 \) and \( A(\nu) \setminus A_b = \emptyset \).

(10.2) If \( \mathfrak{A} \) is a generic of \( \mathcal{V}(2, 3, 4) \), then \( A(\cdot) \setminus A_b \neq \emptyset \neq A(\cdot) \setminus A_b \).
(10.3) The subdirect product
\[ \mathfrak{A}(2, 3) = \langle (a_2, b_3), (b_2, a_3), (b_2, b_3); +, ', \rangle \]
of \( \mathfrak{A}_2 \times \mathfrak{A}_3 \) is a minimal generic of \( V(2, 3, 4) \). So \( g(V(2, 3, 4)) = 3 \).

Proof. (10.1) holds by Corollaries 1 and 4; (10.2) holds by Corollaries 2' and 3'; (10.3) holds by (1,v) and (10.2).

Theorem 11. We have

(11.1) If \( \mathfrak{A} \in V(1, 2, 3, 4) \), then \( A^{(\nu)} \setminus A_b = \emptyset \).

(11.2) If \( \mathfrak{A} \) is a generic of \( V(1, 2, 3, 4) \), then \( |A_b| \geq 2 \) and \( A^{(+)} \setminus A_b \neq \emptyset \neq A^{(\cdot)} \setminus A_b \).

(11.3) The subdirect product
\[ \mathfrak{A}(1, 2, 3) = \langle (a_1, b_2, b_3), (b_1, b_2, b_3), (a_1, a_2, b_3), (a_1, b_2, a_3); +, ', \rangle \]
of \( \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \) is a minimal generic of \( V(1, 2, 3, 4) \). Consequently, 
\[ g(V(1, 2, 3, 4)) = 4. \]

Proof. (11.1) holds by Corollary 4; (11.2) holds by Corollaries 1', 2' and 3'; (11.3) holds by (1,v).

Lemma 9. If \( S \in \mathcal{S} \), \( \mathfrak{A} \) is a generic of \( V(S) \) and \( \mathfrak{A}_4 \in V(S) \), then \( \mathfrak{A} \) satisfies none of the identities \( q^{(+)}(x) \approx x \), \( q^{(\cdot)}(x) \approx x \), \( q^{(\nu)}(x) \approx x \), \( q^{(\cdot)}(x) \approx x \).

In fact, \( \mathfrak{A}_4 \) satisfies none of these identities, so neither does \( \mathfrak{A} \).

Lemma 10. \( A = A_b \) iff \( \mathfrak{A} \) satisfies

\[ q_b(x) \approx x. \]

Proof. \( \Rightarrow \) If \( a \in A \), then \( a \in A_b \), so \( q_b(a) = a \) by (2.ii).

\( \Leftarrow \) If (6) holds, then for every \( a \in A \) we have \( a \in A_b \) by (2.ii), so \( A \subseteq A_b \)
and \( A = A_b \).

Similarly, we prove that

Lemma 11. \( A = A^{(+)} \) iff \( \mathfrak{A} \) satisfies

\[ q^{(+)}(x) \approx x. \]

Lemma 12. \( A = A^{(\cdot)} \) iff \( \mathfrak{A} \) satisfies

\[ q^{(\cdot)}(x) \approx x. \]

Lemma 13. \( A = A^{(\nu)} \) iff \( \mathfrak{A} \) satisfies

\[ q^{(\nu)}(x) \approx x. \]

Lemma 14. If \( \mathfrak{A} \) satisfies (5), then \( A = A^{(\nu)} \) iff \( \mathfrak{A} \) satisfies

\[ q^{(\cdot)}(x) \approx x. \]

This follows at once from Lemma 13.
Theorem 12. We have

(12.1) If $\mathfrak{A} \in \mathcal{V}(4)$, then $|A_b| = 1$ and $A_{(+)} = A_{(\cdot)} = A_{(\cdot')} = A_b$.
(12.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(4)$, then $A \setminus A_b \neq \emptyset$.
(12.3) The algebra $\mathfrak{A}_4$ is a minimal generic of $\mathcal{V}(4)$. So $g(\mathcal{V}(4)) = 2$.

Proof. (12.1) holds by Corollaries 1–4; (12.2) holds by Lemmas 10 and 9. In fact, $\mathfrak{A}$ does not satisfy (6) since $\mathfrak{A}_4$ does not. (12.3) holds by (1.ii). ■

Theorem 13. We have

(13.1) If $\mathfrak{A} \in \mathcal{V}(1, 4)$, then $A_{(+)} = A_{(\cdot)} = A_{(\cdot')} = A_b$.
(13.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(1, 4)$, then $|A_b| \geq 2$ and $A \setminus A_b \neq \emptyset$.
(13.3) The subdirect product

$$\mathfrak{A}(1, 4) = (\langle a_1, b_4 \rangle, \langle b_1, a_4 \rangle, \langle b_1, b_4 \rangle); +, \cdot, \cdot)$$

of $\mathfrak{A}_1 \times \mathfrak{A}_4$ is a minimal generic of $\mathcal{V}(1, 4)$. So $g(\mathcal{V}(1, 4)) = 3$.

Proof. (13.1) holds by Corollaries 2–4; (13.2) holds by Corollary 1', Lemmas 10 and 9; (13.3) holds by (1.ii). ■

Theorem 14. We have

(14.1) If $\mathfrak{A} \in \mathcal{V}(2)$, then $|A_b| = 1$ and $A_{(\cdot)} = A_{(\cdot')} = A_b$ and $A = A_{(+)}$.
(14.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(2)$, then $A_{(+)} \setminus A_b \neq \emptyset$.
(14.3) The algebra $\mathfrak{A}_2$ is a minimal generic of $\mathcal{V}(2)$. So $g(\mathcal{V}(2)) = 2$.

Proof. (14.1) holds by Corollaries 1, 3, 4 and Lemma 11. In fact, $\mathcal{V}(2)$ satisfies (7) since $\mathfrak{A}_2$ does. (14.2) holds by Corollary 2', and (14.3) is obvious. ■

Theorem 15. We have

(15.1) If $\mathfrak{A} \in \mathcal{V}(1, 2)$, then $A_{(\cdot)} = A_{(\cdot')} = A_b$ and $A = A_{(+)}$.
(15.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(1, 2)$, then $|A_b| \geq 2$ and $A_{(+)} \setminus A_b \neq \emptyset$.
(15.3) The subdirect product

$$\mathfrak{A}(1, 2) = (\langle a_1, a_2 \rangle, \langle a_1, b_2 \rangle, \langle b_1, b_2 \rangle); +, \cdot, \cdot)$$

of $\mathfrak{A}_1 \times \mathfrak{A}_2$ is a minimal generic of $\mathcal{V}(1, 2)$. So $g(\mathcal{V}(1, 2)) = 3$.

Proof. (15.1) holds by Corollaries 3, 4 and Lemma 11; (15.2) holds by Corollaries 1' and 2'; (15.3) is obvious. ■

Theorem 16. We have

(16.1) If $\mathfrak{A} \in \mathcal{V}(2, 4)$, then $|A_b| = 1$ and $A_{(\cdot)} = A_{(\cdot')} = A_b$.
(16.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(2, 4)$, then $A_{(+)} \setminus A_b \neq \emptyset \neq A \setminus A_{(+)}$.
(16.3) The subdirect product

$$\mathfrak{A}(2, 4) = (\langle a_2, b_4 \rangle, \langle b_2, a_4 \rangle, \langle b_2, b_4 \rangle); +, \cdot, \cdot)$$

of $\mathfrak{A}_2 \times \mathfrak{A}_4$ is a minimal generic of $\mathcal{V}(2, 4)$. So $g(\mathcal{V}(2, 4)) = 3$. 


Proof. (16.1) holds by Corollaries 1, 3 and 4; (16.2) holds by Corollary 2′ and Lemmas 11 and 9; (16.3) holds by (1.ii). ■

**Theorem 17.** We have

(17.1) If $\mathfrak{A} \in \mathcal{V}(1, 2, 4)$, then $A_{(\cup)} = A_{(\cap)} = A_b$.
(17.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(1, 2, 4)$, then $|A_b| \geq 2$ and $A_{(\cup)} \setminus A_b \neq \emptyset \neq A \setminus A_{(\cap)}$.
(17.3) The subdirect product

$\mathfrak{A}(1, 2, 4) = \{\langle a_1, a_2, b_4\rangle, \langle a_1, b_2, b_4\rangle, \langle a_1, b_2, a_4\rangle, \langle b_1, b_2, b_4\rangle\}; +, \cdot, \prime\}$

of $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_4$ is a minimal generic of $\mathcal{V}(1, 2, 4)$. Consequently, $g(\mathcal{V}(1, 2, 4)) = 4$.

Proof. (17.1) holds by Corollaries 3 and 4; (17.2) holds by Corollaries 1′, 2′, Lemmas 11 and 9; (17.3) is obvious. ■

The proofs of Theorems 18–21 are analogous to those of Theorems 14–17, respectively. However, we must replace Lemma 11 by Lemma 12.

**Theorem 18.** We have

(18.1) If $\mathfrak{A} \in \mathcal{V}(3)$, then $|A_b| = 1$ and $A_{(\cup)} = A_{(\cap)} = A_b$ and $A = A_{(\cdot)}$.
(18.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(3)$, then $A_{(\cup)} \setminus A_b \neq \emptyset$.
(18.3) The algebra $\mathfrak{A}_3$ is a minimal generic of $\mathcal{V}(3)$. So $g(\mathcal{V}(3)) = 2$.

**Theorem 19.** We have

(19.1) If $\mathfrak{A} \in \mathcal{V}(1, 3)$, then $A_{(\cup)} = A_{(\cap)} = A_b$ and $A = A_{(\cdot)}$.
(19.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(1, 3)$, then $|A_b| \geq 2$ and $(A_{(\cup)} \setminus A_b) \neq \emptyset$.
(19.3) The subdirect product

$\mathfrak{A}(1, 3) = \{\langle a_1, a_3\rangle, \langle a_1, b_3\rangle, \langle b_1, b_3\rangle\}; +, \cdot, \prime\}$

of $\mathfrak{A}_1 \times \mathfrak{A}_3$ is a minimal generic of $\mathcal{V}(1, 3)$. So $g(\mathcal{V}(1, 3)) = 3$.

**Theorem 20.** We have

(20.1) If $\mathfrak{A} \in \mathcal{V}(3, 4)$, then $|A_b| = 1$ and $A_{(\cup)} = A_{(\cap)} = A_b$.
(20.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(3, 4)$, then $A_{(\cup)} \setminus A_b \neq \emptyset \neq A \setminus A_{(\cap)}$.
(20.3) The subdirect product

$\mathfrak{A}(3, 4) = \{\langle a_3, b_4\rangle, \langle b_3, a_4\rangle, \langle b_3, b_4\rangle\}; +, \cdot, \prime\}$

of $\mathfrak{A}_3 \times \mathfrak{A}_4$ is a minimal generic of $\mathcal{V}(3, 4)$. So $g(\mathcal{V}(3, 4)) = 3$.

**Theorem 21.** We have

(21.1) If $\mathfrak{A} \in \mathcal{V}(1, 3, 4)$, then $A_{(\cup)} = A_{(\cap)} = A_b$.
(21.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(1, 3, 4)$, then $|A_b| \geq 2$ and $A_{(\cup)} \setminus A_b \neq \emptyset \neq A \setminus A_{(\cap)}$. 
(21.3) The subdirect product
\[ \mathfrak{A}(1, 3, 4) = \{ \langle a_1, a_3, b_4 \rangle, \langle a_1, b_3, b_4 \rangle, \langle a_1, b_3, a_4 \rangle, \langle b_1, b_3, b_4 \rangle \}; +, ' \} \]
of \( \mathfrak{A}_1 \times \mathfrak{A}_3 \times \mathfrak{A}_4 \) is a minimal generic of \( V(1, 3, 4) \). Consequently, 
\[ g(\mathcal{V}(1, 3, 4)) = 4. \]

Theorem 22. We have
\[ \begin{align*}
(22.1) & \quad \text{If } \mathfrak{A} \in V(5, 6), \text{ then } |A_b| = 1 \text{ and } A_{(+)\,+} = A_{(-)\,-} = A_b \text{ and } A = A_{(\nu)\,\nu}. \\
(22.2) & \quad \text{If } \mathfrak{A} \text{ is a generic of } V(5, 6), \text{ then } |A_{(\nu)\,\nu}\,\nu| \setminus A_b \geq 2. \\
(22.3) & \quad \text{The algebra } \mathfrak{A}_6 \text{ is a minimal generic of } V(5, 6). \text{ So } g(\mathcal{V}(5, 6)) = 3.
\end{align*} \]

Proof. (22.1) holds by Corollaries 1–3 and Lemma 13; (22.2) holds by Lemma 8; (22.3) is obvious by (1.iv).

Theorem 23. We have
\[ \begin{align*}
(23.1) & \quad \text{If } \mathfrak{A} \in V(1, 5, 6), \text{ then } A_{(+)\,+} = A_{(-)\,-} = A_b \text{ and } A = A_{(\nu)\,\nu}. \\
(23.2) & \quad \text{If } \mathfrak{A} \text{ is a generic of } V(1, 5, 6), \text{ then } |A_b| \geq 2 \text{ and } |A_{(\nu)\,\nu}\,\nu| \setminus A_b \geq 2. \\
(23.3) & \quad \text{The algebra } \mathfrak{A}_1 \times \mathfrak{A}_5 \text{ is a minimal generic of } V(1, 5, 6). \text{ Consequently, } g(\mathcal{V}(1, 5, 6)) = 4.
\end{align*} \]

Proof. (23.1) holds by Corollaries 2–3 and Lemma 13; (23.2) holds by Corollary 1’ and Lemma 8; (23.3) is obvious, by (1.vi).

Theorem 24. We have
\[ \begin{align*}
(24.1) & \quad \text{If } \mathfrak{A} \in V(4, 5, 6), \text{ then } |A_b| = 1 \text{ and } A_{(+)\,+} = A_{(-)\,-} = A_b. \\
(24.2) & \quad \text{If } \mathfrak{A} \text{ is a generic of } V(4, 5, 6), \text{ then } |A_{(\nu)\,\nu}\,\nu| \setminus A_b \geq 2 \text{ and } A \setminus A_{(\nu)\,\nu} \neq \emptyset. \\
(24.3) & \quad \text{The subdirect product} \\
\mathfrak{A}(4, 6) = \{ \langle a_4, b_6 \rangle, \langle b_4, b_6 \rangle, \langle b_4, a_6 \rangle, \langle b_4, c_6 \rangle \}; +, ' \} \\
of \mathfrak{A}_4 \times \mathfrak{A}_6 \text{ is a minimal generic of } V(4, 5, 6). \text{ So } g(\mathcal{V}(4, 5, 6)) = 4.
\end{align*} \]

Proof. (24.1) holds by Corollaries 1–3; (24.2) holds by Lemmas 8, 13 and 9; (24.3) is obvious, by (1.iv).

Theorem 25. We have
\[ \begin{align*}
(25.1) & \quad \text{If } \mathfrak{A} \in V(1, 4, 5, 6), \text{ then } A_{(+)\,+} = A_{(-)\,-} = A_b. \\
(25.2) & \quad \text{If } \mathfrak{A} \text{ is a generic of } V(1, 4, 5, 6), \text{ then } |A_b| \geq 2 \text{ and } |A_{(\nu)\,\nu}\,\nu| \setminus A_b \geq 2 \\
& \quad \text{and } A \setminus A_{(\nu)\,\nu} \neq \emptyset. \\
(25.3) & \quad \text{The subdirect product} \\
\mathfrak{A}(1, 4, 5) = \{ \langle a_1, b_4, b_5 \rangle, \langle b_1, b_4, b_5 \rangle, \\
\langle a_1, a_4, b_5 \rangle, \langle a_1, b_4, a_5 \rangle, \langle b_1, b_4, a_5 \rangle \}; +, ' \} \\
of \mathfrak{A}_1 \times \mathfrak{A}_4 \times \mathfrak{A}_5 \text{ is a minimal generic of } V(1, 4, 5, 6). \text{ Consequently, } 
\[ g(\mathcal{V}(1, 4, 5, 6)) = 5. \]
\]

Proof. (25.1) holds by Corollaries 2 and 3; (25.2) holds by Corollary 1’, Lemmas 8, 13 and 9; (25.3) is obvious, by (1.vi).
Theorem 26. We have

(26.1) If $\mathfrak{A} \in \mathcal{V}(5)$, then $|A_b| = 1$ and $A(+) = A(\cdot) = A_b$ and $A = A(\prime) = \{x \in A : x' = x\}$. 

(26.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(5)$, then $A(\prime) \setminus A_b \neq \emptyset$. 

(26.3) The algebra $\mathfrak{A}_5$ is a minimal generic of $\mathcal{V}(5)$. So $g(\mathcal{V}(5)) = 2$.

Proof. (26.1) holds by Corollaries 1–3 and Lemma 14; (26.2) holds by Corollary 4'; (26.3) is obvious. ■

Theorem 27. We have

(27.1) If $\mathfrak{A} \in \mathcal{V}(4, 5)$, then $|A_b| = 1$ and $A(+) = A(\cdot) = A_b$. 

(27.2) If $\mathfrak{A}$ is a generic of $\mathcal{V}(4, 5)$, then $A(\prime) \setminus A_b \neq \emptyset \neq A \setminus A(\prime)$. 

(27.3) The subdirect product

$$\mathfrak{A}(4, 5) = \{\langle a_4, b_5 \rangle, \langle b_4, b_5 \rangle, \langle b_4, a_5 \rangle\}; +, , , ,$$

of $\mathfrak{A}_4 \times \mathfrak{A}_5$ is a minimal generic of $\mathcal{V}(4, 5)$. So $g(\mathcal{V}(4, 5)) = 3$.

Proof. (27.1) holds by Corollaries 1–3; (27.2) holds by Corollary 4', Lemmas 14 and 9; (27.3) is obvious. ■

Obviously we have:

(2.viii) A 1-element algebra of type $\tau_b$ is a minimal generic of the trivial variety $\mathcal{V}(\emptyset)$ (satisfying $x \approx y$).

It is known that

(2.ix) The algebra $\mathfrak{A}_1$ is a minimal generic of the variety $\mathcal{V}(1) = \mathcal{B}$.

In (2.vii) we noticed that $g(\mathcal{B}'(5)) = 6$, which was proved in [4]. Now having Corollaries 1'–3' and Lemma 8 of the present paper the reader can easily see that $g(\mathcal{B}'(5)) \geq 6$, which together with the algebra $\mathfrak{A}(1, 2, 3, 5)$ gives the statement of (2.vii).

We hope that the observations and methods of our paper will also be useful in other cases of finding minimal generics of varieties.

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Institute of Mathematics
Polish Academy of Sciences
Kopernika 18
51-617 Wroclaw, Poland
E-mail: jersalk@wp.pl

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