COLLOQUIUM MATHEMATICUM

VOL. 112 2008 NO. 1

CONFORMAL GRADIENT VECTOR FIELDS ON A COMPACT RIEMANNIAN MANIFOLD

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Abstract. It is proved that if an n-dimensional compact connected Riemannian manifold (M,g) with Ricci curvature Ric satisfying

$$0 < \mathrm{Ric} \le (n-1)\left(2 - \frac{nc}{\lambda_1}\right)c$$

for a constant c admits a nonzero conformal gradient vector field, then it is isometric to $S^n(c)$, where λ_1 is the first nonzero eigenvalue of the Laplacian operator on M. Also, it is observed that existence of a nonzero conformal gradient vector field on an n-dimensional compact connected Einstein manifold forces it to have positive scalar curvature and ultimately to be isometric to $S^n(c)$, where $n(n-1)c$ is the scalar curvature of the manifold.

1. Introduction. One of the interesting questions in the geometry of Riemannian manifolds is to characterize spheres among the class of compact connected Riemannian manifolds. Also, one of the interesting properties of a sphere $S^n(c)$ is that there exist nonconstant functions f on $S^n(c)$ which satisfy $\nabla_X \nabla f = -cfX$ where ∇f is the gradient of f and ∇_X is the covariant derivative operator with respect to the smooth vector field X. In fact Obata [4] has proved that a compact connected Riemannian manifold that admits a nonconstant solution of the above differential equation is necessarily isometric to $S^n(c)$. Indeed there are nonzero conformal gradient vector fields on $S^n(c)$ (vector fields of type $u = \nabla f$ with a smooth function φ satisfying $\nabla_X u = \varphi X$), and we can interpret Obata's differential equation as defining a specific conformal gradient vector field $u = \nabla f$ with $\varphi = -cf$ on $S^n(c)$. This naturally raises the question: Under what conditions does an n-dimensional compact and connected Riemannian manifold that admits a nonzero conformal gradient vector field have to be isometric to a sphere $S^n(c)$? In this paper we answer this question by proving the following:

2000 *Mathematics Subject Classification*: 53C20, 53A50.

Key words and phrases: Ricci curvature, conformal gradient vector field, eigenvalue of the Laplacian operator.

This work is supported by the Institute of Research and Consultation, King Abdul Aziz University Project No. 155/427.

THEOREM 1. *Let (M,g) be an n-dimensional compact connected Riemannian manifold whose Ricci curvature satisfies*

$$0<\mathrm{Ric}\le(n-1)\left(2-\frac{nc}{\lambda_1}\right)c$$

for a constant c, where λ_1 is the first nonzero eigenvalue of the Laplace operator. If M admits a nonzero conformal gradient vector field, then M is isometric to $S^n(c)$.

THEOREM 2. *Let (M,g) be an n-dimensional compact connected Einstein manifold with Einstein constant $\lambda=(n-1)c$. If M admits a nonzero conformal gradient vector field, then $c>0$ and M is isometric to $S^n(c)$.*

We express our sincere thanks to Professor Derdziński for his kind help in improving this paper.

2. Preliminaries. Let (M,g) be a Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. A vector field $X\in\mathfrak{X}(M)$ is said to be *conformal* if

$$\mathcal{L}_X g=2\varphi g \tag{2.1}$$

for a smooth function $\varphi: M\to R$, where $\mathcal{L}_X$ is the Lie derivative with respect to X. If $u=\nabla f$ is the gradient of a smooth function f on M and u is a conformal vector field, then u is said to be a *conformal gradient vector field*. Since u is then also closed, it follows from (2.1) that a conformal gradient vector field u satisfies

$$\nabla_X u=\varphi X,\quad X\in\mathfrak{X}(M), \tag{2.2}$$

where ∇_X is the covariant derivative operator with respect to X, corresponding to the Riemannian connection on M.

The following lemma is a direct consequence of (2.2).

LEMMA 2.1. *Let u be a conformal gradient vector field on a compact Riemannian manifold (M,g). Then, for $\varphi=n^{-1}\operatorname{div}u$,*

$$\int_M \varphi\,dv=0.$$

For a smooth function f on M, we define an operator $A:\mathfrak{X}(M)\to\mathfrak{X}(M)$ by $A(X)=\nabla_X\nabla f$, ∇f being the gradient of f. The Ricci operator Q is a symmetric $(1,1)$-tensor field defined by $g(QX,Y)=\mathrm{Ric}(X,Y)$, $X,Y\in\mathfrak{X}(M)$, where Ric is the Ricci tensor of the Riemannian manifold.

LEMMA 2.2. *Let (M,g) be a Riemannian manifold and f be a smooth function on M. Then the operator A corresponding to the function f satisfies*

$$\sum_i(\nabla A)(e_i,e_i)=\nabla(\Delta f)+Q(\nabla f)$$

where $\{e_1, \dots, e_n\}$ *is a local orthonormal frame*, Δ *is the Laplace operator on* M *and* $(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$.

Proof. It follows from the definition of A that $(\nabla A)(X,Y) - (\nabla A)(Y,X) = R(X,Y)\nabla f$, where R is the curvature tensor of M. Thus using the fact that $\Delta f = \sum_i g(Ae_i, e_i)$ and that the operator A is symmetric, we arrive at

$$X(\Delta f) = g(X, \sum_i (\nabla A)(e_i, e_i)) - \mathrm{Ric}(X, \nabla f),$$

which proves the lemma.

LEMMA 2.3. *Let* u *be a conformal gradient vector field on an* n*-dimensional Riemannian manifold* (M, g). *Then* $Q(u) = -(n-1)\nabla\varphi$, *where* $\nabla\varphi$ *is the gradient of the smooth function* $\varphi = n^{-1}\operatorname{div} u$.

Proof. Since $u = \nabla f$ is a conformal vector field, equation (2.2) gives $\Delta f = n\varphi$ and consequently $A(X) = (n^{-1}\Delta f)X$, $X \in \mathfrak{X}(M)$. Thus for a local orthonormal frame $\{e_1, \dots, e_n\}$ we get $\sum_i (\nabla A)(e_i, e_i) = n^{-1}\nabla(\Delta f)$, and consequently Lemma 2.2 gives $Q(u) = -(n-1)\nabla\varphi$.

LEMMA 2.4. *Let* (M, g) *be an* n*-dimensional compact Riemannian manifold and* u *be a conformal gradient vector field on* M. *Then, for* $\varphi = n^{-1}\operatorname{div} u$,

$$\int_M \{\mathrm{Ric}(u,u) - n(n-1)\varphi^2\}\, dv = 0.$$

Proof. Note that Lemma 2.3 implies that $\mathrm{Ric}(u,u) = -(n-1)g(\nabla\varphi, u) = -(n-1)u(\varphi)$. However, $\operatorname{div}(\varphi u) = u(\varphi) + \varphi \operatorname{div} u = u(\varphi) + n\varphi^2$; using this in the previous equation and integrating we get the result.

Also, as a direct consequence of Lemma 2.3, we have

LEMMA 2.5. *Let* (M, g) *be an* n*-dimensional compact Riemannian manifold and* u *be a conformal gradient vector field on* M. *Then, for* $\varphi = n^{-1}\operatorname{div} u$,

$$\int_M \{\mathrm{Ric}(\nabla\varphi, u) + (n-1)\|\nabla\varphi\|^2\}\, dv = 0.$$

3. Proof of Theorem 1. Let (M, g) be an n-dimensional compact connected Riemannian manifold and u be a nonzero conformal gradient vector field on M. For $\varphi = n^{-1}\operatorname{div} u$, we have

$$\mathrm{Ric}(\nabla\varphi + cu, \nabla\varphi + cu) = \mathrm{Ric}(\nabla\varphi, \nabla\varphi) + c^2\mathrm{Ric}(u,u) + 2c\mathrm{Ric}(\nabla\varphi, u).$$

Integration of the above equation using Lemmas 2.3–2.5 leads to

$$\int_M \mathrm{Ric}(\nabla\varphi + cu, \nabla\varphi + cu)\, dv = \int_M \{\mathrm{Ric}(\nabla\varphi, \nabla\varphi) + n(n-1)c^2\varphi^2 - 2(n-1)c\|\nabla\varphi\|^2\}\, dv. \tag{3.1}$$

Lemma 2.1 together with the minimum principle gives

$$\int_M \|\nabla\varphi\|^2 dv \geq \lambda_1 \int_M \varphi^2\, dv, \tag{3.2}$$

where λ_1 is the first nonzero eigenvalue of the Laplacian on M. Using (3.2) in (3.1) we get

$$\int_M \operatorname{Ric}(\nabla\varphi + cu, \nabla\varphi + cu)\, dv \leq \int_M \left\{ \operatorname{Ric}(\nabla\varphi, \nabla\varphi) - (n-1)\left(2 - \frac{nc}{\lambda_1}\right) c\|\nabla\varphi\|^2 \right\} dv.$$

Since M has positive Ricci curvature, the condition in the statement of the theorem together with the above inequality implies that

$$\nabla\varphi = -cu, \tag{3.3}$$

which together with (2.2) gives $\nabla_X \nabla\varphi = -c\varphi X$, $X \in \mathfrak{X}(M)$. If φ is a constant, Lemma 2.1 will imply $\varphi = 0$, and consequently $\Delta f = 0$ on M compact, so that f is a constant, and that in turn will lead to $u = 0$, a contradiction. Hence φ is a nonconstant function satisfying Obata's differential equation and hence M is isometric to $S^n(c)$.

4. Proof of Theorem 2. Since (M, g) is an Einstein manifold with Einstein constant $\lambda = (n-1)c$, Lemma 2.3 gives

$$\nabla\varphi = -cu;$$

this is just equation (3.3) except that here we need to check that $c > 0$. To this end, from the above equation we get

$$\Delta\varphi = -nc\varphi.$$

As in the proof of Theorem 1, φ is a nonconstant function and thus nc is a nonzero eigenvalue of Δ, which guarantees $c > 0$. Thus, as in Theorem 1, we conclude that M is isometric to $S^n(c)$.

Remark. For a compact Riemannian manifold (M, g) of constant scalar curvature S admitting a nonzero conformal gradient vector field u, using Lemma 2.3 we arrive at $(1-n)\nabla\varphi = Q(\nabla f)$. Taking divergence in this equation we get

$$n(n-1)\Delta\varphi = -S\varphi;$$

this confirms that $S > 0$ as φ is a nonconstant eigenfunction of Δ. It is interesting to note that a compact connected Riemannian manifold (M, g) of nonpositive constant scalar curvature does not admit a nonzero conformal gradient vector field.

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Received 9 July 2007;
revised 31 August 2007 (4956)