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HIGHLY TRANSITIVE SUBGROUPS OF THE SYMMETRIC GROUP ON THE NATURAL NUMBERS

BY

U. B. DARJI (Louisville, KY) and J. D. MITCHELL (St Andrews)

Abstract. Highly transitive subgroups of the symmetric group on the natural numbers are studied using combinatorics and the Baire category method. In particular, elementary combinatorial arguments are used to prove that given any nonidentity permutation α on $\mathbb{N}$ there is another permutation β on $\mathbb{N}$ such that the subgroup generated by α and β is highly transitive. The Baire category method is used to prove that for certain types of permutation α there are many such possibilities for β. As a simple corollary, if $2 \leq \kappa \leq 2^{\aleph_0}$, then the free group of rank κ has a highly transitive faithful representation as permutations on the natural numbers.

1. Introduction. In 1977 McDonough [10] proved that a free group of countable rank at least two has a highly transitive faithful permutation representation in $\mathcal{S}_{\mathbb{N}}$, the group of all permutations on the natural numbers. Since then many results concerning highly transitive free subgroups of $\mathcal{S}_{\mathbb{N}}$ have been published. For example, see Adeleke [1], Glass and McCleary [8], Neumann [12] and Truss [15]. Cameron [3] contains the most comprehensive survey of results in this area.

Of particular interest here is [5], where Dixon uses the Baire category method to show that in a certain sense most highly transitive groups are free. The approach in this paper is in the same spirit as in Dixon's paper. In other words, combinatorial and topological methods are used to obtain algebraic results. The main algebraic result of the paper states that given any nonidentity element $\alpha \in \mathcal{S}_{\mathbb{N}}$, there exists a cycle $\beta \in \mathcal{S}_{\mathbb{N}}$ with support $\mathbb{N}$ such that $\langle \alpha, \beta \rangle$ is highly transitive. The proof is combinatorial in nature. Furthermore, it is shown that if α has infinite support, then a generic β in the set $\mathcal{I}$, of permutations that have no finite cycles, has the property that $\langle \alpha, \beta \rangle$ is highly transitive. That is, there are "many" $\beta \in \mathcal{I}$ that have this property. If α has even less finite structure, that is, $\alpha \in \mathcal{I}$, then a generic $\beta \in \mathcal{S}_{\mathbb{N}}$ has the property that $\langle \alpha, \beta \rangle$ is highly transitive. Dixon's result that

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a generic element in $\mathcal{I} \times \mathcal{I}$ generates a highly transitive subgroup of $\mathcal{S}_{\mathbb{N}}$ is obtained as a corollary. We also deduce as a corollary that if κ is a cardinal with $2 \leq \kappa \leq 2^{\aleph_0}$, then there exists a highly transitive free subgroup of $\mathcal{S}_{\mathbb{N}}$ of rank κ. In [3] Neumann is credited with constructing a highly transitive free subgroup of $\mathcal{S}_{\mathbb{N}}$ of rank $2^{\aleph_0}$ with the additional property that every nonidentity element is *cofinitary*, that is, has at most finitely many fixed points.

The theorems in this paper were originally motivated by similar results for finite symmetric groups. In particular, if $\sigma \in \mathcal{S}_n$, $n \neq 4$, is a nonidentity permutation, then there exists $\tau \in \mathcal{S}_n$ such that $\mathcal{S}_n = \langle \sigma, \tau \rangle$ (see [14]). Furthermore, the probability that a randomly chosen pair of permutations generates $\mathcal{S}_n$ or the alternating group $\mathcal{A}_n$ tends to 1 as n tends to ∞. This result was originally known as Netto's conjecture [11], and was first proved by Dixon in [4]. The results presented here might be considered as infinite analogues of these results.

2. Preliminaries. A topological space is called *Polish* if it is separable and admits a complete metric which generates the given topology. The Baire Category Theorem holds in a Polish space. That is, no nonempty open set is *meagre*, i.e. a countable union of nowhere dense sets. Hence sets that are complements of meagre sets, so-called *comeagre* sets, are "large". To say that *a generic x in a Polish space X has property P* means that the set $\{x \in X : x \text{ does not satisfy } P\}$ is meagre in X. A set which is the countable intersection of open sets is called a *G_δ set*. A theorem of Aleksandrov states that every G_δ subset M of a Polish space X is Polish. Hence, it is meaningful to say that a generic $x \in M$ has property P.

A subset Y of a topological space X has the *Baire property* if

$$Y = (U \setminus N) \cup M$$

where U is open and M, N are meagre. The Baire property is a regularity property analogous to Lebesgue measurability. The following theorem is required. It is an analogue, in the context of category, of Fubini's Theorem.

THEOREM 2.1 (Kuratowski–Ulam Theorem [9, Theorem 8.41]). *Suppose X, Y are Polish spaces and $M \subseteq X \times Y$ has the Baire property. Then*

(i) *if $M_x = \{y \in Y : (x, y) \in M\}$ is comeagre for a comeagre set of $x \in X$, then M is a comeagre subset of $X \times Y$;*
(ii) *if M is comeagre in $X \times Y$, then for a generic $x \in X$, M_x is comeagre in Y.*

For a Polish space X, denote the compact subsets of X equipped with the Hausdorff metric by $\mathcal{K}(X)$. Recall that if X is Polish, then so is $\mathcal{K}(X)$.

The reader is referred to [9] or [13] for further information on the theory of Polish spaces.

The topology used on $\mathcal{S}_{\mathbb{N}}$ is that inherited from the product topology on $\mathbb{N}^{\mathbb{N}}$. Recall that $\mathbb{N}^{\mathbb{N}}$ is a Polish space. The group $\mathcal{S}_{\mathbb{N}}$ is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$ and hence is Polish as well. In fact, a specific complete metric on $\mathcal{S}_{\mathbb{N}}$ which generates the aforementioned topology can be defined. For $\sigma, \tau \in \mathcal{S}_{\mathbb{N}}$, define

$$d(\sigma,\tau) = \begin{cases} 0 & \sigma = \tau \\ 1/i + 1/j & \sigma \neq \tau \end{cases}$$

where i is the least natural number where σ and τ differ and j is the least natural number where σ^{-1} and τ^{-1} differ. Moreover, $\mathcal{S}_{\mathbb{N}}$ is a *Polish group*, meaning that the group operation and the unary operation of taking inverses are continuous.

Define $\mathbb{N}^{<\mathbb{N}}$ to be the set of all injections with domain a finite subset of $\mathbb{N}$ and image contained in $\mathbb{N}$. If $\sigma \in \mathbb{N}^{<\mathbb{N}}$, then $[\sigma]$ denotes the clopen (i.e. simultaneously closed and open) set of all $\tau \in \mathcal{S}_{\mathbb{N}}$ such that τ agrees with σ on the domain of σ. Note that the collection of all such sets forms a basis for the topology on $\mathcal{S}_{\mathbb{N}}$. Recall that a subset $X \subseteq \mathcal{S}_{\mathbb{N}}$ is *nowhere dense* if and only if for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$ there exists an extension $\tau \in \mathbb{N}^{<\mathbb{N}}$ of σ such that $[\tau] \cap X = \emptyset$. For any $\alpha \in \mathbb{N}^{<\mathbb{N}}$ we denote by $\mathrm{dom}(\alpha)$ the domain of α and by $\mathrm{im}(\alpha)$ the image (or range) of α.

The order of $\alpha \in \mathcal{S}_{\mathbb{N}}$ is denoted by $|\alpha|$. Denote by 1_X the identity permutation on $X \subseteq \mathbb{N}$. If $\sigma \in \mathcal{S}_{\mathbb{N}}$, then the *support* of σ, denoted by $\mathrm{supp}(\sigma)$, is the set of all $i \in \mathbb{N}$ such that $i\sigma \neq i$. The *fix* of σ, denoted by $\mathrm{fix}(\sigma)$, is the set of all $i \in \mathbb{N}$ such that $i\sigma = i$. The disjoint cycle structure of $\alpha \in \mathcal{S}_{\mathbb{N}}$ is denoted by $(\alpha_1)(\alpha_2)\cdots$. A cycle $\sigma \in \mathcal{S}_{\mathbb{N}}$ is called a *shift* if $\mathrm{supp}(\sigma) = \mathbb{N}$.

The collection $\mathcal{I}$ of all permutations with no finite cycles and the collection $\mathcal{S} \subseteq \mathcal{I}$ of all shifts are central to our investigation. Both of these collections are nowhere dense in $\mathcal{S}_{\mathbb{N}}$ and hence "small". However, both $\mathcal{I}$ and $\mathcal{S}$ are G_δ subsets of $\mathcal{S}_{\mathbb{N}}$ and hence are Polish themselves. Therefore we can meaningfully talk about generic elements of either collection. Neither $\mathcal{I}$ nor $\mathcal{S}$ is a subgroup of $\mathcal{S}_{\mathbb{N}}$ since if σ and ϱ are shifts such that $1\sigma = 2$ and $2\varrho = 1$, then $\sigma\varrho$ fixes 1. We also note that $\mathcal{S}$ is a dense subset of $\mathcal{I}$. Therefore if we show that a generic $\sigma \in \mathcal{S}$ has property P, then so does a generic $\tau \in \mathcal{I}$.

For a subset X of $\mathcal{S}_{\mathbb{N}}$ we denote by $\langle X \rangle$ the subgroup formed in the usual way, by taking all finite products of elements in $X \cup X^{-1}$. It is well-known that two permutations are conjugate in $\mathcal{S}_{\mathbb{N}}$ if and only if they have the same number of disjoint cycles of each length (see, for example, [2, Theorem 2.9]). The following fact will be used frequently.

LEMMA 2.2. *Let G be a permutation group. If $g, h, k \in G$ are such that $\langle g, k^{-1}hk\rangle = G$, then $\langle kgk^{-1}, h\rangle = G$.*

A subgroup G of $\mathcal{S}_{\mathbb{N}}$ is *highly transitive* if the natural action of G on $\mathbb{N}$ is k-transitive for all $k \in \mathbb{N}$. Recall that $G \leq \mathcal{S}_{\mathbb{N}}$ is highly transitive if and only if G is (topologically) dense in $\mathcal{S}_{\mathbb{N}}$. Also bear in mind that if $G \leq \mathcal{S}_{\mathbb{N}}$ is a group, then so is $\mathrm{cl}(G)$, the topological closure of G. If $\alpha \in \mathcal{S}_{\mathbb{N}}$, then define

$$H_\alpha = \{\beta \in \mathcal{S}_{\mathbb{N}} : \langle \alpha, \beta\rangle \text{ is highly transitive}\}. \tag{1}$$

There is a *faithful representation* of a group G in $\mathcal{S}_{\mathbb{N}}$ if there is a monomorphism $\psi : G \to \mathcal{S}_{\mathbb{N}}$. A *highly transitive faithful representation* is a faithful representation in which $\psi(G)$ is a highly transitive subgroup of $\mathcal{S}_{\mathbb{N}}$. The reader is referred to [2] or [6] for more information on permutation groups and to [9] for standard facts about topology and Baire category.

3. The main results. In this section we state the main results of this paper. Some discussion is devoted to showing that these results are, in some sense, the best possible. Most of the proofs of these results are left to the next section. We begin by showing that no single permutation generates a highly transitive subgroup of $\mathcal{S}_{\mathbb{N}}$.

LEMMA 3.1. *Let $\alpha = (\alpha_1)(\alpha_2)\cdots \in \mathcal{S}_{\mathbb{N}}$. Then $\langle\alpha\rangle$ is a nowhere dense subgroup of $\mathcal{S}_{\mathbb{N}}$ and so it is not highly transitive.*

Proof. If α has an infinite cycle, then no sequence $\alpha^{n_1}, \alpha^{n_2}, \ldots$ converges and so $\mathrm{cl}(\langle\alpha\rangle) = \langle\alpha\rangle$, which is countable. Therefore $\langle\alpha\rangle$ is nowhere dense in $\mathcal{S}_{\mathbb{N}}$.

Alternatively, each α_i is a finite cycle. Let $\phi \in \mathbb{N}^{<\mathbb{N}}$ and choose numbers $k, l \notin \mathrm{dom}(\phi) \cup \mathrm{im}(\phi)$ that belong to different cycles of α. If $\tau \in \mathbb{N}^{<\mathbb{N}}$ is any extension of ϕ satisfying $k\tau = l$, then $[\tau] \cap \mathrm{cl}(\langle\alpha\rangle) = \emptyset$, and so $\langle\alpha\rangle$ is nowhere dense. ■

The next example shows it is possible for $\mathrm{cl}(\langle\alpha\rangle)$ to be uncountable.

EXAMPLE 3.2. Let $p_1, p_2, \ldots$ be distinct primes and let $\alpha = (\alpha_1)(\alpha_2)\cdots \in \mathcal{S}_{\mathbb{N}}$ be any element such that for each i, $|\alpha_i| = p_i$. Take any sequence $\{r_i\}_{i\in\mathbb{N}}$ satisfying $0 \leq r_i < p_i$. By the Chinese Remainder Theorem, there exists a sequence of positive integers $\{x_i\}_{i\in\mathbb{N}}$ such that for each $1 \leq j \leq i$, we have

$$x_i \equiv r_j \bmod p_j.$$

The sequence $\{\alpha^{x_i}\}_{i\in\mathbb{N}}$ converges to $(\alpha_1^{r_1})(\alpha_2^{r_2})\cdots$. Hence, for each sequence $\{r_i\}_{i\in\mathbb{N}}$ with $0 \leq r_i < p_i$ for all i, we have a distinct element of $\mathcal{S}_{\mathbb{N}}$. Therefore $\mathrm{cl}(\langle\alpha\rangle)$ is uncountable.

Next, we state our main theorem.

THEOREM 3.3. *Let α be an arbitrary nonidentity permutation on $\mathbb{N}$. Then*

(i) *if α has finite support, then there exists a shift σ such that $\langle\alpha,\sigma\rangle$ is highly transitive, that is, $\sigma \in H_\alpha \cap \mathcal{S}$;*
(ii) *if α has infinite support, then $H_\alpha \cap \mathcal{S}$ is comeagre in $\mathcal{S}$ and $H_\alpha \cap \mathcal{I}$ is comeagre in $\mathcal{I}$;*
(iii) *if $\alpha \in \mathcal{I}$, then H_α is comeagre in $\mathcal{S}_{\mathbb{N}}$.*

The proof of this theorem is given in the next section. Let us first consider some corollaries.

COROLLARY 3.4. *Let $\alpha \in \mathcal{S}_{\mathbb{N}}$ be a nonidentity permutation. Then there is a shift σ such that $\langle\alpha,\sigma\rangle$ is highly transitive.*

Proof. This simply follows from Theorems 3.3(i) and (ii). ■

COROLLARY 3.5 (Dixon, [5]). *A generic element of $\mathcal{S}^2$, respectively a generic element of $\mathcal{I}^2$, generates a highly transitive subgroup of $\mathcal{S}_{\mathbb{N}}$.*

Proof. This simply follows from the Kuratowski–Ulam Theorem (Theorem 2.1) and Theorem 3.3(ii). ■

COROLLARY 3.6. *A generic compact set K in $\mathcal{K}(\mathcal{S})$ has the property that $\langle K\rangle$ is highly transitive. Moreover, there exist $\alpha,\beta \in K$ such that $\langle\alpha,\beta\rangle$ is highly transitive.*

Proof. By Corollary 3.5, the set $T = \{(\alpha,\beta) \in \mathcal{S}^2 : \langle\alpha,\beta\rangle$ is highly transitive$\}$ is comeagre in $\mathcal{S}^2$. It follows that $\{K \in \mathcal{K}(\mathcal{S}) : \exists\alpha,\beta \in K$ such that $(\alpha,\beta) \in T\}$ is comeagre in $\mathcal{K}(\mathcal{S})$. ■

COROLLARY 3.7. *Let $2 \le \kappa \le 2^{\aleph_0}$. Then there is a highly transitive free subgroup of $\mathcal{S}_{\mathbb{N}}$ with rank κ.*

Proof. In [7] Gartside and Knight prove that a generic compact set $K \in \mathcal{K}(\mathcal{S}_{\mathbb{N}})$ freely generates a free subgroup of $\mathcal{S}_{\mathbb{N}}$. It is straightforward to modify their proof to show that a generic $K \in \mathcal{K}(\mathcal{S})$ freely generates a free subgroup of $\mathcal{S}_{\mathbb{N}}$. Moreover, a generic compact subset of S is perfect. Together with Corollary 3.6 these facts imply that there exists a compact subset K of $\mathcal{S}$ such that

- $|K| = 2^{\aleph_0}$;
- there exist $\alpha,\beta \in K$ such that $\langle\alpha,\beta\rangle$ is highly transitive;
- K freely generates a free subgroup of $\mathcal{S}_{\mathbb{N}}$.

(In fact, a generic compact set in $\mathcal{K}(\mathcal{S})$ has these three properties.)

Let A be a subset of K with $\alpha,\beta \in A$ and $|A| = \kappa$. Then $\langle A\rangle$ is the required group. ■

The sharpness of Theorem 3.3 is now discussed. In other words, we ask: is it possible to obtain the conclusion of Theorem 3.3(ii) or (iii) under the hypothesis of Theorem 3.3(i) or (ii)? More precisely, if $\alpha \in \mathcal{S}_{\mathbb{N}} \setminus \{1_{\mathbb{N}}\}$ has finite support, is $H_\alpha \cap \mathcal{S}$ comeagre in $\mathcal{S}$ or H_α comeagre in $\mathcal{S}_{\mathbb{N}}$? If α has infinite support, but contains a finite cycle, then is H_α comeagre in $\mathcal{S}_{\mathbb{N}}$? The following examples show that the answers to these questions are, in general, no.

Example 3.8. Let $\alpha = (1\ 2)$. Then we will show that $H_\alpha \cap \mathcal{S}$ is not dense in $\mathcal{S}$, and so it is certainly not comeagre in $\mathcal{S}$. To this end, define $\phi, \eta \in \mathbb{N}^{<\mathbb{N}}$ such that $1\phi = 3$ & $3\phi = 2$ and $1\eta = 3,\ 3\eta = 1,\ \&\ 4\eta = 4$. We prove that $\langle \alpha, \sigma\rangle \cap [\eta] = \emptyset$ for all σ in the open set $[\phi] \cap \mathcal{S}$. From this we deduce that $\langle \alpha, \sigma\rangle$ is not dense in $\mathcal{S}_{\mathbb{N}}$ and thus not highly transitive. Consequently, the intersection of $[\phi] \cap \mathcal{S}$ and H_α is empty.

To obtain a contradiction assume there is $\sigma \in [\phi] \cap \mathcal{S}$ and $\beta \in \langle \alpha, \sigma\rangle \cap [\eta]$. We may write $\beta = \sigma^{k_1}\alpha\sigma^{k_2}\cdots\alpha\sigma^{k_n}$ where $k_i \in \mathbb{Z}$. If $i, j \in \mathbb{Z}$ and $\tau = (1\sigma^{-i}\ 2\sigma^{-i})$, then $\sigma^i\alpha\sigma^j = \tau\sigma^{i+j}$. Applying this recursively to β, we obtain

$$\beta = \tau_1 \cdots \tau_{n-1}\sigma^{k_1+\cdots+k_n},$$

where each τ_i is a transposition of the form $(j\, j\sigma^2)$. This implies that $i\tau_1\tau_2\cdots\tau_{n-1} = i\sigma^{2k}$ for all $i \in \mathbb{N}$ and some $k \in \mathbb{Z}$ that depends on i. Moreover, since $4\eta = 4$, $k_1 + \cdots + k_n$ is divisible by 2. Thus $1\beta = 1\tau_1\cdots\tau_{n-1}\sigma^{k_1+\cdots+k_n} = 1\sigma^m$ for some number m that is divisible by 2. In particular, $1\beta \neq 3$, a contradiction.

Since $\mathcal{S}$ is dense in $\mathcal{I}$, Example 3.8 also demonstrates that $H_\alpha \cap \mathcal{I}$ is not, in general, dense in $\mathcal{I}$. The set $\mathcal{S}$ is a nowhere dense in $\mathcal{S}_{\mathbb{N}}$ and so Example 3.8 does not rule out the possibility that H_α may be comeagre in $\mathcal{S}_{\mathbb{N}}$. However, the following easy example shows that H_α may not be comeagre in $\mathcal{S}_{\mathbb{N}}$.

Example 3.9. If α is any permutation with a finite cycle ϕ, then for any $\beta \in [\phi]$ the subgroup $\langle \alpha, \beta\rangle$ is not highly transitive.

Not only is H_α not comeagre (in $\mathcal{S}_{\mathbb{N}}$) when α contains a finite cycle, but something much stronger is true.

Theorem 3.10. *If $k \in \mathbb{N}$, then the set $\{(\alpha_1, \ldots, \alpha_k) \in \mathcal{S}_{\mathbb{N}}^k : \langle \alpha_1, \ldots, \alpha_k\rangle$ is nowhere dense in $\mathcal{S}_{\mathbb{N}}\}$ is comeagre in $\mathcal{S}_{\mathbb{N}}^k$.*

Corollary 3.11. *The set H_α is meagre in $\mathcal{S}_{\mathbb{N}}$ for a generic $\alpha \in \mathcal{S}_{\mathbb{N}}$.*

Proof. Theorem 3.10 shows that a generic pair $(\alpha, \beta) \in \mathcal{S}_{\mathbb{N}}^2$ generates a nowhere dense subgroup of $\mathcal{S}_{\mathbb{N}}$. By the Kuratowski–Ulam Theorem, there exists a comeagre set A in $\mathcal{S}_{\mathbb{N}}$ such that for all $\alpha \in A$ the set

$$M_\alpha = \{\beta \in \mathcal{S}_{\mathbb{N}} : \langle \alpha, \beta\rangle \text{ is nowhere dense (in } \mathcal{S}_{\mathbb{N}})\}$$

is comeagre in $\mathcal{S}_{\mathbb{N}}$. Thus $H_\alpha = \{\beta \in \mathcal{S}_{\mathbb{N}} : \langle \alpha, \beta\rangle \text{ is dense in } \mathcal{S}_{\mathbb{N}}\} \subseteq \mathcal{S}_{\mathbb{N}} \setminus M_\alpha$ is meagre for all $\alpha \in A$. ∎

Since the set of all permutations with finite support is meagre in $\mathcal{S}_{\mathbb{N}}$, a generic permutation α in $\mathcal{S}_{\mathbb{N}}$ has infinite support and H_α meagre in $\mathcal{S}_{\mathbb{N}}$. This demonstrates that Theorem 3.3(ii) is sharp.

If X is any Polish space, then

$$\{(x_1, x_2, \ldots) \in X^\omega : \{x_1, x_2, \ldots\} \text{ is dense in } X\}$$

is comeagre in X^ω. Hence the analogue of Theorem 3.10 does not hold when $k = \omega$.

Recall that $\mathcal{S}_{\mathbb{N}}$ is nowhere locally compact and hence every compact subset of $\mathcal{S}_{\mathbb{N}}$ is nowhere dense in $\mathcal{S}_{\mathbb{N}}$. Of course, it follows from Corollary 3.4 that there are compact sets K such that $\langle K\rangle$ is highly transitive, but the following theorem shows that the contrary is true for generic compact sets in $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$.

THEOREM 3.12. *A generic $K \in \mathcal{K}(\mathcal{S}_{\mathbb{N}})$ has the property that $\langle K\rangle$ is nowhere dense in $\mathcal{S}_{\mathbb{N}}$.*

4. Proofs of Theorems 3.3, 3.10 and 3.12. The first thing to prove is Theorem 3.3(i). The proof is started by the next four lemmas.

LEMMA 4.1. *Let σ_t be a cycle in the disjoint cycle decomposition of $\sigma \in \mathcal{S}_{\mathbb{N}}$ and let $\tau = (i\ i\sigma_t) = (i\ i\sigma)$ for some $i \in \operatorname{supp}(\sigma_t)$. Then $\langle\sigma, \tau\rangle$ contains every transposition τ' with $\operatorname{supp}(\tau') \subseteq \operatorname{supp}(\sigma_t)$.*

Proof. For any $j \in \operatorname{supp}(\sigma_t)$ there exists $r \in \mathbb{Z}$ such that $j = i\sigma_t^r = i\sigma^r$. Note that $(i\sigma_t^r\ i\sigma_t^{r+1}) = \sigma_t^{-r}(i\ i\sigma_t)\sigma_t^r = \sigma^{-r}\tau\sigma^r$. Let $k, l \in \operatorname{supp}(\sigma_t)$ be arbitrary. Then there exists $n \in \mathbb{Z}$ such that $l = k\sigma_t^n = k\sigma^n$. Therefore

$$\begin{aligned}(k\ l) = (k\ k\sigma_t^n) &= (k\sigma_t^{n-1}\ k\sigma_t^n)\cdots(k\sigma_t^2\ k\sigma_t^3)(k\sigma_t\ k\sigma_t^2)(k\ k\sigma_t)\\ &\circ (k\sigma_t\ k\sigma_t^2)(k\sigma_t^2\ k\sigma_t^3)\cdots(k\sigma_t^{n-1}\ k\sigma_t^n) \in \langle\sigma, \tau\rangle,\end{aligned}$$

and the proof is complete. ∎

LEMMA 4.2. *Let σ be a shift and $\tau = (i\ i\sigma)$ for some $i \in \mathbb{N}$. Then $\langle\sigma, \tau\rangle$ is highly transitive.*

Proof. Every transposition $(k\ l) \in \langle\sigma, \tau\rangle$ by Lemma 4.1. Since the set of all permutations with finite support set is dense in $\mathcal{S}_{\mathbb{N}}$ and every permutation with finite support set is a product of transpositions, we have $\operatorname{cl}(\langle\sigma, \tau\rangle) = \mathcal{S}_{\mathbb{N}}$. ∎

LEMMA 4.3. *Let σ be a shift and $\tau = (i\ i\sigma)(i\sigma^n\ i\sigma^{n+1})$, $n \geq 2$. Then $\langle\sigma, \tau\rangle$ is highly transitive.*

Proof. Without loss of generality we assume that $n = 2$. Let

$$\alpha_1 = \sigma^{-2}\tau\sigma^2\tau = (i\ i\sigma)(i\sigma^4\ i\sigma^5),$$

$$\alpha_2 = \sigma^{-4}\alpha_1\sigma^4\alpha_1 = (i\ i\sigma)(i\sigma^8\ i\sigma^9),\ \ldots,\ \alpha_{i+1} = \sigma^{-2^{(i+1)}}\alpha_i\sigma^{2^{(i+1)}}\alpha_i,\ \ldots.$$

Then the limit of the sequence $(\alpha_i)_{i\in\mathbb{N}}$ is the transposition $(i\ i\sigma)$. Therefore $(i\ i\sigma) \in \mathrm{cl}(\langle \sigma, \tau\rangle)$ and so $\mathcal{S}_{\mathbb{N}} = \mathrm{cl}(\langle \sigma, (i\ i\sigma)\rangle) \subseteq \mathrm{cl}(\langle \sigma, \tau\rangle)$. ∎

LEMMA 4.4. *Let σ be a shift and let $\tau = (i\ i\sigma\ i\sigma^2\ \ldots\ i\sigma^n)$ for some $n \geq 1$. Then $\langle \sigma, \tau\rangle$ is highly transitive.*

Proof. If $n = 1$, then we are finished by Lemma 4.2. Suppose that $n \geq 2$ and define

$$\alpha = \sigma^{-n}\tau\sigma^n = (i\sigma^n\ i\sigma^{n+1}\ \ldots\ i\sigma^{2n}).$$

Hence

$$\beta = \tau\alpha\tau^{-1} = (i\sigma^{n-1}\ i\sigma^{n+1}\ i\sigma^{n+2}\ \ldots\ i\sigma^{2n}),$$

and so

$$\delta = \sigma^{n-1}\beta\sigma^{-(n-1)} = (i\ i\sigma^2\ i\sigma^3\ \ldots\ i\sigma^{n+1}).$$

Finally,

$$\gamma = \tau\delta^{-1} = (i\ i\sigma)(i\sigma^n\ i\sigma^{n+1}).$$

Now, by Lemma 4.3, it follows that $\mathcal{S}_{\mathbb{N}} = \mathrm{cl}(\langle \sigma, \gamma\rangle) \subseteq \mathrm{cl}(\langle \sigma, \tau\rangle) \subseteq \mathcal{S}_{\mathbb{N}}$, giving equality throughout. ∎

It is now possible to complete the proof of Theorem 3.3(i).

Proof of Theorem 3.3(i). Recall that $\alpha \in \mathcal{S}_{\mathbb{N}}$ is a nonidentity with finite support. Let p be a prime which divides $|\alpha|$. Then $\alpha^{|\alpha|/p}$ is a permutation with order p and finite support. Hence we may assume that α is a permutation with finite support which has order p. Let $\varrho \in \mathcal{S}_{\mathbb{N}}$ be any shift. There exists $\beta \in \mathcal{S}_{\mathbb{N}}$ such that

$$\delta = \beta\alpha\beta^{-1} = (1\varrho\ \ldots\ 1\varrho^p)(1\varrho^{p+1}\ \ldots\ 1\varrho^{2p})\cdots(1\varrho^{kp+1}\ \ldots\ 1\varrho^{(k+1)p}).$$

By an argument similar to that used in the proof of Lemma 4.3 we can obtain a sequence in $\langle \delta, \varrho\rangle$ whose limit is $\gamma = (1\varrho\ \ldots\ 1\varrho^p)$. Therefore, by Lemma 4.4, we have $\mathcal{S}_{\mathbb{N}} = \mathrm{cl}(\langle \varrho, \gamma\rangle) \subseteq \mathrm{cl}(\langle \varrho, \delta\rangle) = \mathrm{cl}(\langle \varrho, \beta\alpha\beta^{-1}\rangle)$. Hence, by Lemma 2.2, we have $\mathrm{cl}(\langle \alpha, \beta^{-1}\varrho\beta\rangle) = \mathcal{S}_{\mathbb{N}}$. It follows that $\sigma = \beta^{-1}\varrho\beta$ is the desired shift. ∎

Next is the proof of part (ii) of the main theorem in the previous section.

Proof of Theorem 3.3(ii). It suffices to prove the case for $\mathcal{S}$. If k is a positive integer, then denote by $\mathbb{N}^k$ the set of all injections from $\{1, \ldots, k\}$ into $\mathbb{N}$. It is enough to show for every $k \in \mathbb{N}$ and every $\eta \in \mathbb{N}^k$ that the intersection of $\mathcal{S}$ and the set

$$H_{\alpha,\eta} = \{\sigma \in \mathcal{S}_{\mathbb{N}} : [\eta] \cap \langle \alpha, \sigma\rangle \neq \emptyset\}$$

is comeagre in $\mathcal{S}$. This is so because

$$H_\alpha \cap \mathcal{S} = \bigcap_{k=1}^{\infty} \bigcap_{\eta\in\mathbb{N}^k} (H_{\alpha,\eta} \cap \mathcal{S}).$$

Each of the sets $H_{\alpha,\eta}$ is open. Hence we need only show that $H_{\alpha,\eta} \cap \mathcal{S}$ is dense in $\mathcal{S}$. To this end, let ϕ be any element in $\mathbb{N}^{<\mathbb{N}}$ which contains no cycle. It remains to show that there is $\sigma \in [\phi] \cap \mathcal{S}$ such that $[\eta] \cap \langle\alpha, \sigma\rangle \neq \emptyset$.

By extending the domain and the range, if necessary, assume that both the domain and range of η equal the set $\{1, \ldots, l\}$, and that the domain of ϕ is $\{1, \ldots, l\}$ and the range of ϕ is contained in $\{1, \ldots, l+1\}$. Moreover, order the elements of $\{1, \ldots, l+1\} = \{n_1, \ldots, n_{l+1}\}$ so that $(n_i)\phi = n_{i+1}$ for $1 \leq i \leq l$. Let $n_{l+2}, n_{l+3}, \ldots, n_{2l+2} \in \operatorname{supp}(\alpha)$ be chosen recursively so that

$$n_{l+i+1} \notin \{n_1, \ldots, n_{l+i}\} \cup \{n_1, \ldots, n_{l+i}\}\alpha \cup \{n_1, \ldots, n_{l+i}\}\alpha^{-1}$$

for $1 \leq i \leq l+1$. Then $n_{l+i+1}\alpha \notin \{n_1, \ldots, n_{2l+2}\}$. It is possible to choose such elements because α has infinite support.

Since $\operatorname{dom}(\eta) = \operatorname{im}(\eta) = \{1, \ldots, l\}$, we may find $u_1, \ldots, u_l, v_1, \ldots, v_l \in \{1, \ldots, l\}$ such that $n_{u_i} = i$ and $n_{v_i} = (i)\eta$. Let $n_{-l}, n_{-l+1}, \ldots, n_0$ be points in $\mathbb{N} \setminus \{n_1, \ldots, n_{2l+2}\}$ such that $n_{v_i-l-1} = (n_{u_i+l+1})\alpha$ for $1 \leq i \leq l$. Note that, from the definition, the points n_{l+i+1}, $i \geq 1$, can all be taken to be distinct.

Finally, let σ be any shift which satisfies $(n_i)\sigma = n_{i+1}$ for each $-l \leq i \leq 2l+1$. Clearly, $\sigma \in [\phi] \cap \mathcal{S}$. Next we show that $\sigma^{l+1}\alpha\sigma^{l+1} \in [\eta]$. If $1 \leq i \leq l$, then

$$\begin{aligned}(i)\sigma^{l+1}\alpha\sigma^{l+1} &= (n_{u_i})\sigma^{l+1}\alpha\sigma^{l+1} = (n_{u_i+l+1})\alpha\sigma^{l+1}\\ &= (n_{v_i-l-1})\sigma^{l+1} = n_{v_i} = (i)\eta.\end{aligned}$$

Thus $[\eta] \cap \langle\alpha, \sigma\rangle \neq \emptyset$, completing the proof. ∎

Proof of Theorem 3.3(iii). Fix $\alpha \in \mathcal{I}$. As in the proof of Theorem 3.3(ii), for each $\eta \in \mathbb{N}^k$, let $H_{\alpha,\eta} = \{\sigma \in \mathcal{S}_{\mathbb{N}} : \langle\alpha, \sigma\rangle \cap [\eta] \neq \emptyset\}$. It suffices to show that $H_{\alpha,\eta}$ is comeagre in $\mathcal{S}_{\mathbb{N}}$. Since $H_{\alpha,\eta}$ is clearly open, it only remains to show that it is dense in $\mathcal{S}_{\mathbb{N}}$. To this end, let $\phi \in \mathbb{N}^{<\mathbb{N}}$. We show that there is $\sigma \in [\phi]$ such that $\langle\alpha, \sigma\rangle \cap [\eta] \neq \emptyset$. Let

$$A = \operatorname{dom}(\phi) \cup \operatorname{im}(\phi) \cup \operatorname{dom}(\eta) \cup \operatorname{im}(\eta).$$

Since α contains no finite cycle and A is finite, there is a positive integer l such that for all $i \in A$ and $l' \in \mathbb{Z}$ with $|l'| \geq l$, we have $(i)\alpha^{l'} \cap A = \emptyset$. For $1 \leq i \leq k$, let $\tau_i = ((i)\alpha^l\ (i)\eta\alpha^{-l})$. Note that for all i, $\operatorname{supp}(\tau_i) \cap \operatorname{dom}(\phi) = \emptyset$, and since $(i)\alpha^l \neq (j)\eta\alpha^{-l}$ for all $1 \leq i, j \leq k$ we have $\operatorname{supp}(\tau_i) \cap \operatorname{supp}(\tau_j) = \emptyset$ when $i \neq j$. Now, let $\sigma \in [\phi]$ be such that every τ_i is a cycle of σ. Then, for each $1 \leq i \leq k$, we have

$$(i)\alpha^l\sigma\alpha^l = (i)\alpha^l\tau_i\alpha^l = (i)\eta\alpha^{-l}\alpha^l = (i)\eta,$$

completing the proof. ∎

Proof of Theorem 3.10. The case when $k = 1$ follows from Lemma 3.1.

It remains to prove the assertion when $k > 1$. We do this in the case $k = 2$; the general case follows by a similar argument. To start, we show that a generic $(\alpha, \beta) \in \mathcal{S}_{\mathbb{N}}^2$ has the property that there is an infinite set fixed by both α and β. To this end, enumerate the finite subsets of $\mathbb{N}$ of size n as $F_{n,1}, F_{n,2}, \ldots$ and let

$$\mathcal{A}_{n,m} = \{(\alpha_1, \alpha_2) \in \mathcal{S}_{\mathbb{N}}^2 : F_{n,m} \subseteq \operatorname{fix}(\alpha_1) \cap \operatorname{fix}(\alpha_2)\}.$$

Observe that $\mathcal{A}_{n,m}$ is open in $\mathcal{S}_{\mathbb{N}}^2$ and so the countable union

$$\mathcal{A}_n = \bigcup_{m=1}^{\infty} \mathcal{A}_{n,m}$$

is open as well. We show that $\mathcal{A}_n$ is dense in $\mathcal{S}_{\mathbb{N}}^2$. Let $(\beta_1, \beta_2) \in \mathcal{S}_{\mathbb{N}}^2$ and let $\overline{\beta}_1, \overline{\beta}_2 \in \mathbb{N}^{<\mathbb{N}}$ be any initial segments of β_1 and β_2, respectively. There exists m such that $F_{n,m}$ has empty intersection with the domains and the images of $\overline{\beta}_1$ and $\overline{\beta}_2$. Hence there exists $(\gamma_1, \gamma_2) \in \mathcal{A}_{n,m}$ such that γ_1 and γ_2 are extensions of $\overline{\beta}_1$ and $\overline{\beta}_2$, respectively. Therefore $\mathcal{A}_n$ is dense. It follows that the set $\mathcal{A} = \bigcap_{n=1}^{\infty} \mathcal{A}_n$ is a dense G_δ subset of $\mathcal{S}_{\mathbb{N}}^2$.

Note that if $(\alpha_1, \alpha_2) \in \mathcal{A}$, then there is an infinite set B, depending on α_1 and α_2, such that $\alpha_1|_B = \alpha_2|_B = 1_B$. We prove that $\langle \alpha_1, \alpha_2 \rangle$ is nowhere dense in $\mathcal{S}_{\mathbb{N}}$ by showing that given $\phi \in \mathbb{N}^{<\mathbb{N}}$ there exists $\tau \in \mathbb{N}^{<\mathbb{N}}$ such that $\tau|_{\operatorname{dom}(\phi)} = \phi$ and $\langle \alpha_1, \alpha_2 \rangle \cap [\tau] = \emptyset$. To this end, let $i \in B$ such that i is not in the domain of ϕ nor in the image of ϕ. Then any τ which is an extension of ϕ with $i\tau \neq i$ is the desired map. ■

Proof of Theorem 3.12. The proof of this result is analogous to that of Theorem 3.10. Enumerate the finite subsets of $\mathbb{N}$ of size n as $F_{n,1}, F_{n,2}, \ldots$. Let

$$\mathcal{A}_{n,m} = \{K \in \mathcal{K}(\mathcal{S}_{\mathbb{N}}) : \alpha|_{F_{n,m}} = 1_{F_{n,m}} \text{ for all } \alpha \in K\}.$$

We note that $\mathcal{A}_{n,m}$ is open in $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$. Let $\mathcal{A}_n = \bigcup_{m=1}^{\infty} \mathcal{A}_{n,m}$. Clearly, $\mathcal{A}_n$ is open in $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$. We next show that it is dense in $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$. Recall that the set of all finite sets is dense in $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$. It will suffice to show that if $M \in \mathcal{K}(\mathcal{S}_{\mathbb{N}})$ is any finite set and $\mathcal{U}$ is any open set in $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$ with $M \in \mathcal{U}$, then there is $K \in \mathcal{A}_n$ such that $K \in \mathcal{U}$. Enumerate the elements of M as $\beta_1, \ldots, \beta_k$. Let $\overline{\beta}_i$ be a restriction of β_i such that if $N = \{\gamma_1, \ldots, \gamma_k\}$ and $\gamma_i \in [\overline{\beta}_i]$ for each i, then $N \in \mathcal{U}$. Let $F_{n,m}$ have empty intersection with the domain and image of $\overline{\beta}_i$ for all $1 \leq i \leq k$. Now we may choose a γ_i, an extension of $\overline{\beta}_i$, such that $\gamma_i|_{F_{n,m}} = 1|_{F_{n,m}}$. Then $\{\gamma_1, \ldots, \gamma_k\} \in \mathcal{U} \cap \mathcal{A}_{n,m} \subseteq \mathcal{U} \cap \mathcal{A}_n$.

Now consider the dense G_δ subset $\mathcal{A} = \bigcap_{n=1}^{\infty} \mathcal{A}_n$ of $\mathcal{K}(\mathcal{S}_{\mathbb{N}})$. We want to show that for each $K \in \mathcal{A}$, $\langle K \rangle$ is nowhere dense in $\mathcal{S}_{\mathbb{N}}$. To this end, let $\phi \in \mathbb{N}^{<\mathbb{N}}$. Let $n > \max\{\operatorname{dom}(\phi) \cup \operatorname{im}(\phi)\}$ and let m be such that $K \in \mathcal{A}_{n,m}$.

Then there is an $i \in F_{n,m} \setminus (\mathrm{dom}(\phi) \cup \mathrm{im}(\phi))$. Let τ be any extension of ϕ such that $i\tau \neq i$. Then $\langle K \rangle \cap [\tau] = \emptyset$. ∎

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Mathematics Department
University of Louisville
Louisville, KY 40292, U.S.A.
E-mail: ubdarj01@louisville.edu

Mathematics Institute
University of St Andrews
North Haugh
St Andrews, Fife, KY16 9SS, UK
E-mail: jamesm@mcs.st-andrews.ac.uk

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