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# A STURM–LIOUVILLE PROBLEM WITH SPECTRAL AND LARGE PARAMETERS IN BOUNDARY CONDITIONS AND THE ASSOCIATED CAUCHY PROBLEM

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**Abstract.** We study a Sturm–Liouville problem containing a spectral parameter in the boundary conditions. We associate to this problem a self-adjoint operator in a Pontryagin space $\Pi_1$. Using this operator-theoretic formulation and analytic methods, we study the asymptotic behavior of the eigenvalues under the variation of a large physical parameter in the boundary conditions. The spectral analysis is applied to investigate the well-posedness and stability of the wave equation of a string.

**1. Introduction.** This paper is devoted to the study of the following Sturm–Liouville problem containing a spectral parameter both in the equation and in the boundary conditions:

$$l(u) = -u''(x) + q(x)u(x) = \lambda u(x), \quad 0 < x < \pi, \tag{1.1}$$

$$u'(0) = 0, \quad u'(\pi) = m\lambda u(\pi); \tag{1.2}$$

here $q \in \mathrm{L}_1[0,\pi]$ and has real values, $\lambda$ is a spectral parameter and $m \in \mathbb{R}$. Physically, such a problem may be derived from the wave equation (1.4)–(1.5) for longitudinal displacements of a homogeneous string whose left end is fixed and on whose right end a servocontrol force is acting. In particular, this situation occurs if there is a massive load at the right end (see for example [7], [21]), and in this case we have $m > 0$.

The sign of the parameter $m$ plays a fundamental role in the study of this problem. In the case $m > 0$, Problem (1.1)–(1.2) is interpreted as a spectral problem for a self-adjoint operator in the Hilbert space $\mathrm{L}_2[0,\pi] \times \mathbb{C}$ (see e.g. [22], [10] and [6]), and hence all the spectral properties of this problem can be obtained by using the well known theory of self-adjoint linear operators in Hilbert space.

It is easily seen that if $m < 0$ then Problem (1.1)–(1.2) is not self-adjoint in $\mathrm{L}_2[0,\pi] \times \mathbb{C}$. This case has been investigated in [5], [9] and [19]. In [3, 4] the authors were mainly interested in the case when $m < 0$ is a small physical

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parameter. In this case Problem (1.1)–(1.2) can be rewritten in the abstract form

$$Au = \lambda G u, \tag{1.3}$$

where $A = A^*$ in the Hilbert space $\mathrm{L}_2[0,\pi] \times \mathbb{C}$ and $G$ is a linear operator which generates a Pontryagin space $\varPi_1$. Using this and developing an analytic approach, it was shown that there exists a value $m_0 < 0$ such that for $m \in [m_0, 0)$, the spectrum of Problem (1.1)–(1.2) consists of a sequence of real eigenvalues tending to $+\infty$. In particular, for $m = m_0$, the smallest eigenvalue has geometric multiplicity 1 and algebraic multiplicity 2 (see Th. 2.3). Furthermore, the asymptotics of the eigenvalues and the corresponding eigenfunctions was established as $m \to 0-$.

The main goal of the present paper is to study the spectral properties of Problem (1.1)–(1.2) for large parameter $m < 0$, and essentially to give the asymptotic estimates of the eigenvalues as $m \to -\infty$. More precisely, we show that if $m < 0$ is sufficiently large, then all the eigenvalues are real, simple and of a definite type (see Definition 1.3). Furthermore, one of the eigenvalues which is of negative type tends to 0 as $m \to -\infty$, and the others tend to the eigenvalues of the standard problem determined by equation (1.1) and the boundary conditions $u'(0) = 0$, $u(\pi) = 0$. Krylov [16] and Timoshenko [21] considered Problem (1.1)–(1.2) for $q(x) \equiv 0$, studying the behavior of the eigenvalues for small and large loads determined by the parameter $m$. These asymptotic estimates can also be found in the book of Tikhonov and Samarskiĭ [20].

Our method is essentially analytic and based on the properties of the meromorphic function $F(\lambda) = u_x(\pi,\lambda)/u(\pi,\lambda)$, where $u(x,\lambda)$ is a solution of equation (1.1) with the initial condition $u(0) = 1$, $u'(0) = 0$. In [4], it was proved that $F(\lambda)$ is concave on the interval $(-\infty, \mu_1')$, where $\mu_1'$ is the first eigenvalue of the boundary value problem mentioned above. Here, we get more information on $F(\lambda)$ for $\lambda > \mu_1'$; namely, we prove that $F''(\lambda)$ decreases from $-\infty$ to $+\infty$ if $\lambda$ increases from one pole to the adjacent one on the right.

As an application of the spectral properties of Problem (1.1)–(1.2), we study the dynamic boundary value problem

$$u_{xx}(x,t) - q(x)u(x,t) = u_{tt}(x,t), \tag{1.4}$$

$$u_x(0,t) = 0, \qquad u_x(\pi,t) + m u_{tt}(\pi,t) = 0. \tag{1.5}$$

We give sufficient conditions on the potential $q(x)$ and on the parameter $m$ for the associated Cauchy problem to be stable (related to boundedness properties of the solutions) and have a unique solution. In particular, we show that if

$$Q = \inf_{\|u\|_{W_2^1[0,\pi]}=1} \left( \int_0^\pi |u'(x)|^2\, dx + \int_0^\pi q(x)|u(x)|^2\, dx \right) > 0,$$

then this problem is not stable for any $m < 0$. While, in the case $Q < 0$, we give a sufficient condition on $q(x)$ such that, for large enough $m < 0$, the Cauchy problem is stable.

Definitions and notation

1.1. Let $K$ be a linear space equipped with a sesquilinear hermitian form $[\ ,\ ]$ which is nondegenerate in the sense that

$$[f, g] = 0 \text{ for all } g \in K \ \Rightarrow\ f = 0.$$

Then $(K, [\ ,\ ])$ is called a *Pontryagin space* if

$$K = K_+ + K_-,$$

where $K_\pm \subset K$ are linear manifolds such that $(K_\pm, \pm[\ ,\ ])$ are Hilbert spaces, $[K_+, K_-] = 0$ and $\dim K_+ < \infty$ or $\dim K_- < \infty$. We set $\kappa = \dim K_-$ and denote such spaces $K$ by $\Pi_\kappa$.

1.2. Let $A$ and $G$ be linear operators on a linear space $H$, and consider a linear operator pencil of the form $P(\lambda) = A - \lambda G$. The point $\lambda \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$ if there exists an element $u \in H$ such that $P(\lambda)u = 0$; in this case $u$ is called the corresponding eigenfunction.

1.3. Let $P(\lambda) = A - \lambda G$ be a linear pencil defined on a Hilbert space $H$ equipped with an inner product $(\ ,\ )$. We say that an eigenvalue $\lambda$ of the pencil $P(\lambda)$ is of *positive* (resp. *negative*) *type* if $(Gu, u) > 0$ (resp. $< 0$) for all nonzero $u \in \mathrm{Ker}(A - \lambda G)$. If $\lambda$ is not one of these types we say that $\lambda$ is an eigenvalue of *neutral type*.

1.4. Let $A$ be a linear operator on a linear space $H$. An eigenvalue $\lambda$ of $A$ is called *semisimple* if its algebraic multiplicity is equal to its geometric multiplicity.

1.5. A system $\{u_k\}_{k=1}^\infty$ of vectors in a Hilbert space H is said to be a *Riesz basis* if it is equivalent to an orthonormal basis $\{e_k\}_{k=1}^\infty$ in H in the sense that $B(u_k) = e_k$ for all $k \geq 1$ and some bounded operator $B$ with a bounded inverse.

**2. Operator framework and previous results.** It is convenient to represent the spectral problem (1.1)–(1.2) as an eigenvalue problem for a self-adjoint operator in a Pontryagin space $\Pi_1$. We equip the space $\mathrm{H} = \mathrm{L}_2[0,\pi] \times \mathbb{C}$ with the inner product

$$(\widehat{u}_1, \widehat{u}_2)_{\mathrm{H}} = \int_0^\pi u_1(t)\overline{u_2(t)}\, dt + |m| a_1 \overline{a_2}, \tag{2.1}$$

where

$$\widehat{u}_j = \begin{pmatrix} u_j(\cdot) \\ a_j \end{pmatrix} \in \mathrm{H}, \qquad j = 1, 2.$$

We define the operator $\widehat{L}_m$ in H by setting

$$\widehat{L}_m \widehat{u} = \begin{pmatrix} -u'' + q(x)u(x) \\ u'(\pi)m^{-1} \end{pmatrix} \tag{2.2}$$

on the domain

$$\mathcal{D}(\widehat{L}_m) = \left\{ \widehat{u} \;\middle|\; \widehat{u} = \begin{pmatrix} u(x) \\ u(\pi) \end{pmatrix},\ u \in W_2^2[0,\pi],\ l(u) \in \mathrm{L}_2[0,\pi],\ u'(0) = 0 \right\},$$

which is dense in H. We consider the operator

$$\widehat{G} = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix},$$

where $I$ is the identity operator on the space $\mathrm{L}_2[0,\pi]$. The operator $\widehat{G}$ is unitary and symmetric on H, its spectrum consists of two eigenvalues: $-1$ with multiplicity 1, and $+1$ with infinite multiplicity. Hence, it generates a Pontryagin space $\Pi_1$.

Proposition 2.1 (see [4]). *The operator $\widehat{A} = \widehat{G}\widehat{L}_m$ is self-adjoint in the Hilbert space* $\mathrm{H} = \mathrm{L}_2[0,\pi] \times \mathbb{C}$ *and it has a discrete spectrum.*

Obviously, Problem (1.1)–(1.2) is equivalent to the eigenvalue problem

$$\widehat{A}\widehat{u} = \lambda \widehat{G}\widehat{u}, \qquad \widehat{u} \in \mathcal{D}(\widehat{L}_m), \tag{2.3}$$

i.e., the eigenvalues $\lambda_k$ of Problem (1.1)–(1.2) and those of Problem (2.3) coincide; moreover there exists a correspondence between eigenfunctions and associated functions of the two problems. Define

$$Q = \inf_{u \in E} \left( \int_0^\pi |u'(x)|^2 \, dx + \int_0^\pi q(x)|u(x)|^2 \, dx \right), \tag{2.4}$$

where $E = \{u \in W_2^1[0,\pi] \mid \|u\|_{W_2^1[0,\pi]} = 1\}$. The sign of $Q$ plays a fundamental role in the study of the problem. It is clear from the quadratic form

$$(\widehat{A}\widehat{u}, \widehat{u})_H = \int_0^\pi |u'(x)|^2 \, dx + \int_0^\pi q(x)|u(x)|^2 \, dx, \tag{2.5}$$

where $\widehat{u} = \binom{u(x)}{u(\pi)} \in \mathcal{D}(\widehat{A}) = \mathcal{D}(\widehat{L}_m)$, that $\widehat{A} > 0$ if and only if $Q > 0$.

Let us mention the main results on Problem (1.1)–(1.2) established in [4]. These results will be useful in Section 5.

THEOREM 2.2 (see [4]). *Let $m < 0$, $Q > 0$, where $Q$ is defined by* (2.4). *Then $\widehat{A} > 0$ and the spectrum of Problem* (1.1)–(1.2) *consists of a sequence of real and simple eigenvalues tending to $+\infty$. Moreover, all the other eigenvalues have a definite type* (*see Definition* 1.3): *there exists a unique eigenvalue $\lambda_0(m) < 0$ which is of negative type, and all the other eigenvalues $\lambda_1(m), \lambda_2(m), \dots$ are positive and of positive type. The negative eigenvalue has the asymptotics*

$$\lambda_0 = -\frac{1}{m^2} + \mathrm{O}(1) \qquad \textit{as } m \to -0, \tag{2.6}$$

*and the corresponding eigenfunction is*

$$y_0(x) = \left(\cosh\frac{x}{|m|}\right)(1 + \mathrm{O}(m)) \qquad \textit{as } m \to -0. \tag{2.7}$$

*For positive eigenvalues and the corresponding eigenfunctions, we have*

$$\lambda_k = \mu_k - \mu_k\alpha_k m + \mathrm{O}(m), \qquad k = 1, 2, \dots, \tag{2.8}$$

$$y_k(x) = y_k^0 + \mathrm{O}(m), \quad k = 1, 2, \dots, \quad \textit{as } m \to -0, \tag{2.9}$$

*where $\mu_k$ and $y_k^0$ are respectively the eigenvalues and the corresponding eigenfunctions of Problem* (1.1)–(1.2) *for $m = 0$ and*

$$\alpha_k = \frac{|y_k(\pi)|^2}{\left(y_k^0, y_k^0\right)_{\mathrm{L}_2[0,\pi]}}. \tag{2.10}$$

*Proof.* We only have to prove that all the eigenvalues are algebraically simple. In fact, it is known that an eigenvalue of a definite type is necessarily real and semisimple (see Definition 1.4). In our case, all the eigenvalues have a definite type and geometric multiplicity 1. Therefore, they are algebraically simple. ∎

THEOREM 2.3 (see [4]). *Let $m < 0$ and $Q < 0$. Then there exists a number $m_0 < 0$ such that, for all $m \in (m_0, 0)$, Problem* (1.1)–(1.2) *has only real and simple eigenvalues; moreover, the smallest eigenvalue $\lambda_0(m)$ is of negative type and tends to $-\infty$ as $m \to -0$, in accordance with the asymptotics* (2.6). *An eigenfunction of the form* (2.7) *corresponds to this eigenvalue. All the other eigenvalues $\lambda_k(m)$, $k = 1, 2, \dots,$ are of positive type. For $m = m_0$, Problem* (1.1)–(1.2) *has only real eigenvalues such that the smallest one is a multiple eigenvalue with geometric multiplicity* 1 *and algebraic multiplicity* 2. *For $m \in (m_0, m_0 + \varepsilon)$, where $\varepsilon$ is sufficiently small, the problem has one pair of complex eigenvalues, and all the other eigenvalues are real simple and of positive type.*

**3. Spectral properties for $m \in (-\infty, m_0)$.** In this section we investigate the spectral properties of Problem (1.1)–(1.2) for $Q < 0$ and

$m \in (-\infty, m_0)$, where $Q$ is defined by (2.4) and $m_0$ is introduced in Theorem 2.3. The spectral properties in the case $Q > 0$ and $m < m_0$ are completely described by Theorem 2.2.

We introduce the function

$$F(\lambda) = \frac{u_x(\pi, \lambda)}{u(\pi, \lambda)}, \tag{3.1}$$

where $u(x, \lambda)$ is a solution of equation (1.1) satisfying the initial conditions

$$u'(0) = 0, \qquad u(0) = 1. \tag{3.2}$$

Let $\mu_k$, $k = 1, 2, \dots$, be the eigenvalues of the problem

$$\begin{cases} -u'' + qu = \lambda u, \\ u'(0) = u'(\pi) = 0, \end{cases} \tag{3.3}$$

and let $\mu'_k$, $k = 1, 2, \dots$, be the eigenvalues of the problem

$$\begin{cases} -u'' + qu = \lambda u, \\ u'(0) = u(\pi) = 0. \end{cases} \tag{3.4}$$

It is known [16] that the eigenvalues $\mu_k$, $\mu'_k$ are real and simple. Moreover, they satisfy

$$\mu_1 < \mu'_1 < \mu_2 < \mu'_2 < \cdots .$$

Obviously, $\mu_k$ and $\mu'_k$, $k = 1, 2, \dots$, coincide respectively with the zeros and poles of the function $F(\lambda)$. Moreover, the eigenvalues of Problem (1.1)–(1.2) coincide with the zeros of the equation

$$F(\lambda) = m\lambda. \tag{3.5}$$

It is known ([1, Chap. 8]) that $F(\lambda)$ decreases as $\lambda$ varies from $-\infty$ to $\mu'_1$, and from $\mu'_k$ to $\mu'_{k+1}$, $k = 1, 2, \dots$. In the following we obtain more information on this function, partially stated in [4].

Proposition 3.1. *The function $F(\lambda)$ defined by* (3.1) *is concave on the interval $(-\infty, \mu'_1)$, where $\mu'_1$ is the first eigenvalue of Problem* (3.4). *Moreover, each interval $(\mu'_k, \mu'_{k+1})$, $k = 1, 2, \dots$, contains exactly one point $\rho_k$ such that $F''(\rho_k) = 0$, $F''(\lambda) > 0$ for $\lambda \in (\mu'_k, \rho_k)$ and $F''(\lambda) < 0$ for $\lambda \in (\rho_k, \mu'_{k+1})$.*

*Proof.* The concavity of $F(\lambda)$ on $(-\infty, \mu'_1)$ is proved in [4]. By the Mittag-Leffler theorem ([11, Chap. 4]), $F(\lambda)$ admits a decomposition

$$F(\lambda) = G(\lambda) + \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu'_k}\right)^s \frac{C_k}{\lambda - \mu'_k}, \tag{3.6}$$

where $G(\lambda)$ is an entire function,

$$C_k = \operatorname{res}_{\lambda=\mu'_k} F(\lambda) = \frac{u_x(\pi, \mu'_k)}{u_\lambda(\pi, \mu'_k)},$$

and $s$ is chosen so that the series (3.6) is convergent. In [4] it was proved that $C_k > 0$, $k = 1, 2, \dots$, $G(\lambda) \equiv F(0)$ and $s = 1$. Therefore, (3.6) takes the form

$$F(\lambda) = F(0) + \sum_{k=0}^{\infty} \frac{\lambda C_k}{\mu'_k(\lambda - \mu'_k)}.$$

By differentiating this series twice and three times, we obtain

$$F''(\lambda) = 2\sum_{k=0}^{\infty} \frac{C_k}{(\lambda - \mu'_k)^3}, \qquad F'''(\lambda) = -6\sum_{k=0}^{\infty} \frac{C_k}{(\lambda - \mu'_k)^4}.$$

Thus, $F''(\lambda)$ is a decreasing function on each interval $(\mu'_k, \mu'_{k+1})$, $k=1,2,\dots$. On the other hand we have

$$\lim_{\lambda \to \mu'_k + 0} F''(\lambda) = +\infty, \qquad \lim_{\lambda \to \mu'_{k+1} - 0} F''(\lambda) = -\infty.$$

Hence, the function $F''(\lambda)$ has only one zero in each interval $(\mu'_k, \mu'_{k+1})$ and so the claim follows. ∎

From the graph of the function $F(\lambda)$ and the straight line $g(\lambda) := m\lambda$, one can deduce

COROLLARY 3.2. *Let $Q < 0$ and $m < 0$.*

1. *If $0 \in (\mu_1, \mu'_1)$, then for large enough $m < 0$, equation* (3.5) *has two positive roots in the interval $(\mu_1, \mu'_1)$.*
2. *If $0 \in \big(\mu'_M, \mu'_{M+1}\big)$ for some integer $M \geq 1$, then for large enough $m<0$, equation* (3.5) *has exactly three roots in the interval $\big(\mu'_M, \mu'_{M+1}\big)$ and one root in each interval $(\mu'_n, \mu'_{n+1})$, $n \geq 1$ and $n \neq M$.*

We now state results on the location of the eigenvalues for $Q < 0$ and $m < m_0$, where $m_0$ is introduced in Theorem 2.3.

THEOREM 3.3. *Let $m < 0$, $Q < 0$ and assume that $0 \in (\mu_1, \mu'_1)$. Then there exists a number $m_1 \in (-\infty, m_0)$ such that for $m \in (-\infty, m_1)$, Problem* (1.1)–(1.2) *has only positive and simple eigenvalues $\{\lambda_k(m)\}_{k=1}^{\infty}$. In particular, the smallest eigenvalue $\lambda_1(m)$ is of negative type and all the other $\lambda_k(m)$, $k \geq 2$, are of positive type. Furthermore, the eigenvalues $\lambda_k(m)$ and $\mu'_k$ interlace in the following sense:*

$$\begin{aligned}(3.7) \qquad 0 < \lambda_1(m) < \lambda_2(m) < \mu'_1 < \lambda_3(m) < \mu'_2 < \cdots \\ < \mu'_{n-2} < \lambda_n(m) < \mu'_{n-1}.\end{aligned}$$

*For $m = m_1$, there exists one multiple eigenvalue $\lambda_1(m) = \lambda_2(m) = \mu \in (\mu_1, \mu'_1)$ with algebraic multiplicity* 2 *and geometric multiplicity* 1, *and of neutral type, while all the other eigenvalues are positive simple and of positive type. For $m \in (m_1 - \delta, m_1)$, where $\delta > 0$ is sufficiently small, the problem* *has one pair of complex eigenvalues, and a sequence of positive and simple eigenvalues of positive type tending to* $+\infty$*.*

THEOREM 3.4. *Let* $m < 0$*,* $Q < 0$ *and* $0 \in (\mu'_M, \mu'_{M+1})$ *for some integer* $M \geq 1$*. Then there exists a number* $m_1 \in (-\infty, m_0)$ *such that for* $m \in (-\infty, m_1)$*, Problem* (1.1)–(1.2) *has only real and simple eigenvalues* $\{\lambda_k(m)\}_{k=1}^{\infty}$*. In particular, the eigenvalue* $\lambda_{M+1}(m)$ *is of negative type, and all the other* $\lambda_k(m)$*,* $k \neq M+1$*, are of positive type. The eigenvalues* $\lambda_k(m)$ *and* $\mu'_k$ *interlace in the following sense:*

$$\begin{aligned} (3.8)\qquad \mu'_1 < \lambda_1(m) < \mu'_2 < \cdots < \mu'_M < \lambda_M(m) \\ < \lambda_{M+1}(m) < \lambda_{M+2}(m) < \mu'_{M+1}. \end{aligned}$$

*For* $m = m_1$*, there exists one multiple eigenvalue* $\mu \in (\mu'_M, \mu'_{M+1})$ *with algebraic multiplicity* 2 *or* 3 *and geometric multiplicity* 1*, and of neutral type, while all the other eigenvalues are real simple and of positive type. For* $m \in (m_1 - \delta, m_1)$*, where* $\delta > 0$ *is sufficiently small, the problem has one pair of complex eigenvalues.*

We now prove Theorem 3.4. The proof of Theorem 3.3 is the same.

*Proof of Theorem 3.4.* If $0 \in (\mu'_M, \mu'_{M+1})$, $M \geq 1$, then by Corollary 3.2, for large enough $m < 0$, equation (3.5) has exactly three roots in the interval $(\mu'_M, \mu'_{M+1})$, denoted by $\lambda_M(m)$, $\lambda_{M+1}(m)$ and $\lambda_{M+2}(m)$. We show that $\lambda_{M+1}(m)$ is of negative type. By (2.1), we have

$$(3.9)\qquad (\widehat{G}\widehat{u}, \widehat{u})_{\mathrm{H}} = (u, u)_{\mathrm{L}_2(0,\pi)} + m|u(\pi)|^2$$

for $\widehat{u} = \binom{u(x)}{u(\pi)} \in \mathcal{D}(\widehat{L}_m)$. On the other hand, it is known (see e.g. [1, Chap. 6]) that

$$(3.10)\qquad F'(\lambda) = -u^{-2}(\pi, \lambda) \int_0^{\pi} u^2(x)\, dx,$$

where $F(\lambda)$ is defined by (3.5)). From (3.9) and (3.10) we obtain

$$(3.11)\qquad (\widehat{G}\widehat{u}, \widehat{u})_{\mathrm{H}} = -|u(\pi)|^2 (F'(\lambda) - m).$$

It is easily seen, from the convexity of $F(\lambda)$ on the interval $(\mu'_M, \rho_M)$ and its concavity on $(\rho_M, \mu'_{M+1})$ ($\rho_M$ is introduced in Proposition 3.1), that there exists $m_1$ ($m_1 < m_0$) such that for $m = m_1$ we have $F'(\lambda_{M+1}) = m_1$, and for $m \in (-\infty, m_1)$, $F'(\lambda_{M+1}) - m > 0$. Hence, from (3.11) we get $(\widehat{G}\widehat{u}_{M+1}, \widehat{u}_{M+1})_{\mathrm{H}} \leq 0$, where $\widehat{u}_{M+1}$ is the eigenfunction of Problem (2.3) corresponding to $\lambda_{M+1}$. Therefore, $\lambda_{M+1}(m)$ is an eigenvalue of neutral or negative type for all $m \in (-\infty, m_1]$. Since the dimension of the maximal $\widehat{G}$-nonpositive subspace does not exceed 1 (recall that the operator $\widehat{G}$ generates a Pontryagin space $\Pi_1$), it follows from the Pontryagin space theory [2] that all the other eigenvalues $\lambda_k(m)$, $k \neq M+1$, of Problem (1.1)–(1.2) are of positive type. If we suppose that there exists an integer $p \neq M$ such that equation (3.5) has for $m \in (-\infty, m_1)$ more than one root in $(\mu'_p, \mu'_{p+1})$, then by the same arguments used above, one of these roots is of negative type, and this contradicts the uniqueness of the eigenvalue of the neutral or negative type. Therefore, the interlacing property (3.8) follows. It is known that an eigenvalue of a definite type is necessarily semisimple (see Definition 1.4). In our case, if $m \in (-\infty, m_1)$ then all the eigenvalues are of a definite type and have geometric multiplicity 1, since all the eigenfunctions are generated by $u(x, \lambda)$ solution of Problem (1.1), (3.2). Therefore, for $m \in (-\infty, m_1)$, they are algebraically simple.

For $m = m_1$, we have either $\lambda_M(m) = \lambda_{M+1}(m)$, $\lambda_{M+1}(m) = \lambda_{M+2}(m)$, or $\lambda_M(m) = \lambda_{M+1}(m) = \lambda_{M+2}(m)$. Therefore, in this case Problem (1.1)–(1.2) has in $(\mu'_M, \mu'_{M+1})$ one multiple eigenvalue with algebraic multiplicity 2 or 3. If we denote this multiple eigenvalue by $\mu$, then $F'(\mu) = m$, and hence, by (3.11), we have $(\widehat{G}\widehat{u}(\mu), \widehat{u}(\mu))_{\mathrm{H}} = 0$, where $\widehat{u}(\mu)$ is the eigenfunction corresponding to $\mu$. Therefore $\mu$ is an eigenvalue of neutral type. The proof of the remaining result, about the existence of one pair of complex eigenvalues, is similar to that of Theorem 2.3. ■

**4. Asymptotics of the eigenvalues.** In this section we study the asymptotic behavior of the eigenvalues as $m \to -\infty$.

LEMMA 4.1. *Let $\lambda(m) \in \mathbb{R}$ and $\widehat{U} = \{u(x), u(\pi)\}$ be an eigenpair of Problem* (2.3). *Then*

$$(\widehat{G}\widehat{u}, \widehat{u})_{\mathrm{H}} = -\frac{\lambda(m)u^2(\pi)}{\frac{d\lambda(m)}{dm}}. \tag{4.1}$$

*Proof.* Let $\lambda(m)$ be a real eigenvalue of Problem (1.1)–(1.2). Then

$$F(\lambda(m)) = m\lambda(m).$$

Differentiating this relation with respect to $m$, we obtain

$$\frac{d\lambda(m)}{dm} = \frac{\lambda(m)}{F'(\lambda) - m}.$$

This and (3.11) yield (4.1). ■

THEOREM 4.2. *Let $\mu'_k$, $k = 1, 2, \ldots,$ be the eigenvalues of Problem* (3.4), *and let $u_k$ be the corresponding eigenfunctions. Then we have the following asymptotics:*

1) *If* $0 \in (\mu_1, \mu'_1)$ *then*

$$\lambda_1(m) = \frac{\alpha}{m} + \mathrm{o}(1/m) \qquad \text{as } m \to -\infty, \tag{4.2}$$

*where* $\alpha = F(0)$*, and for* $k \geq 2$*,*

$$(4.3)\quad \lambda_k(m) = \mu'_{k-1} + \frac{u'^2_{k-1}(\pi)}{m\mu'_{k-1}\,(u_{k-1}, u_{k-1})_{\mathrm{L}_2[0,\pi]}} + \mathrm{o}(1/m) \quad \textit{as } m \to -\infty,$$

*where* $(\cdot,\cdot)_{\mathrm{L}_2[0,\pi]}$ *is the usual scalar product in the space* $\mathrm{L}_2[0,\pi]$*, and* $u'_k := du_k/dx$*.*

2) *If* $0 \in (\mu'_M, \mu'_{M+1})$*, then* $\lambda_{M+1}$ *satisfies the asymptotics* (4.2)*. For* $\lambda_k$*,* $k \leq M$*, we have*

$$(4.4)\qquad \lambda_k(m) = \mu'_K + \frac{u'^2_k(\pi)}{m\mu'_k\,(u_k, u_k)_{\mathrm{L}_2[0,\pi]}} + \mathrm{o}(1/m) \qquad \textit{as } m \to -\infty,$$

*and for* $k \geq M+2$ *we have the asymptotics*

$$(4.5)\quad \lambda_k(m) = \mu'_{k-1} + \frac{u'^2_{k-1}(\pi)}{m\mu'_{k-1}\,(u_{k-1}, u_{k-1})_{\mathrm{L}_2[0,\pi]}} + \mathrm{o}(1/m) \qquad \textit{as } m \to -\infty.$$

*Proof.* 1) Let $0 \in (\mu_1, \mu'_1)$. According to Theorem 3.3, for large enough $m < 0$, all the eigenvalues $\lambda_k$ of Problem (1.1)–(1.2) are positive and simple. We first prove that $\lambda_1(m) = o(1)$ as $m \to -\infty$. In fact, since $(\widehat{G}\widehat{u}_1, \widehat{u}_1)_{\mathrm{H}} < 0$ ($\widehat{u}_1$ is the eigenfunction corresponding to $\lambda_1(m)$), (4.1) implies that $\lambda_1(m)$ moves to the left as $m \to -\infty$. Therefore, equation (3.5) and the analyticity of $F(\lambda)$ on $[0, \mu'_1)$ yield

$$\lambda_1(m) = \frac{F(\lambda_1(m))}{m} \to 0 \qquad \text{as } m \to -\infty.$$

From this and the asymptotic equation

$$F(\lambda) = F(0) + o(1) = m\lambda,$$

we get (4.2).

We now prove that for $k \geq 2$, $\lambda_k(m) - \mu'_{k-1} = o(1)$ as $m \to -\infty$. In view of Theorem 3.3, the eigenfunctions $\widehat{u}_k$ corresponding to $\lambda_k(m)$, $k \geq 2$, satisfy $(\widehat{G}\widehat{u}_k, \widehat{u}_k)_{\mathrm{H}} > 0$. Hence, again by use of (4.1), $\lambda_k(m)$ ($k \geq 2$) move to the right as $m \to -\infty$. Therefore, from (3.7) and the analyticity of $F(\lambda)$ on the intervals $(\mu_1, \mu'_1)$ and $(\mu'_{k-2}, \mu'_{k-1})$, $k \geq 3$, we obtain

$$\lambda_k(m) = \frac{F(\lambda_k(m))}{m} \to \mu'_{k-1}, \qquad k \geq 2, \qquad \text{as } m \to -\infty.$$

Put $\delta = \lambda_k(m) - \mu'_{k-1}$ and $G(\lambda) = 1/F(\lambda)$. Then

$$G(\mu'_{k-1} + \delta) = \frac{1}{m(\mu'_{k-1} + \delta)}.$$

Remarking that $G(\mu'_k) = 0$, in a small enough neighborhood of $\mu'_{k-1}$ the last equality takes the form

$$G'(\mu'_{k-1})\delta + \mathrm{o}(\delta) = \frac{1}{m\mu'_{k-1}}\left(1 - \frac{\delta}{\mu'_{k-1}} + \mathrm{o}(\delta)\right).$$

By (3.10), we have

$$G'(\lambda) = (u')^{-2}(\pi, \lambda) \int_0^\pi u^2(x)\, dx.$$

Thus

$$\delta = \frac{1}{m\mu'_{k-1} G'(\mu'_{k-1})} + \mathrm{o}(1/m) = \frac{u'^2_{k-1}(\pi)}{m\mu'_{k-1}\,(u_{k-1}, u_{k-1})_{\mathrm{L}_2[0,\pi]}} + \mathrm{o}(1/m),$$

from which (4.3) follows.

2) Let now $0 \in (\mu'_M, \mu'_{M+1})$. According to Theorem 3.4, the eigenvalue $\lambda_{M+1}(m)$ is of negative type. Hence, by the same arguments used in the previous case for (4.2), if $\lambda_{M+1}(m) > 0$ then $\lambda_{M+1}(m) \to +0$, and if $\lambda_{M+1}(m) < 0$ then $\lambda_{M+1}(m) \to -0$ as $m \to -\infty$. From this and the analyticity of $F(\lambda)$ at $\lambda = 0$, it follows that $\lambda_{M+1}(m)$ also satisfies (4.2). Since for $m < m_1$, the eigenvalues $\lambda_k(m)$, $k \neq M+1$, are of positive type, (4.1) implies that the eigenvalues $\lambda_k(m)$, $k \leq M$, move to the left, while $\lambda_k(m)$, $k \geq M+2$, move to the right as $m \to -\infty$. Hence, again by use of the analyticity of $F(\lambda)$ on finite intervals, together with (3.7), it follows that for $k \leq M$, $\lambda_k(m) \to \mu'_k + 0$ and $\lambda_k(m) \to \mu'_{k-1} - 0$ for $k \geq M+2$ as $m \to -\infty$. By the same argument used above, we get the asymptotics (4.4) and (4.5). ∎

**5. The associated Cauchy problem and its stability.** In this section we investigate the well-posedness and stability of the wave equation (1.4)–(1.5). We have to impose the initial conditions

$$u(0, x) = \phi(x), \qquad u'_t(0, x) = \Psi(x). \tag{5.1}$$

Obviously, the Cauchy Problem (1.4), (1.5), (5.1) can be rewritten in the abstract form

$$\frac{d^2\widehat{u}}{dt^2} = -\widehat{L}_m \widehat{u}, \tag{5.2}$$

$$\widehat{u}(0) = \widehat{\phi} = \{\phi(x), \phi(\pi)\}, \qquad \widehat{u}_t(0) = \widehat{\Psi} = \{\Psi(x), \Psi(\pi)\}, \tag{5.3}$$

where $\widehat{L}_m$ is the operator defined by (2.2). Here we restrict ourselves to the case $m < 0$. In the case $m > 0$, the operator $\widehat{L}_m$ is self-adjoint in H, and hence we can use the self-adjoint operator theory.

Definition 5.1.

(i) We say that $\widehat{u}(t)$ is a *solution of Problem* (5.2)–(5.3) on the semi-axis $\mathbb{R}^+$ if $\widehat{u}(t) \in \mathcal{D}(\widehat{L}_m)$ for all $t > 0$, has two continuous derivatives in H and satisfies (5.3) (see Chapter 3 of [11]).

(ii) We say that Problem (5.2)–(5.3) is *stable* if each solution $\widehat{u}(t)$ is uniformly bounded in H for all $t > 0$.

It is known (see e.g. [8, Chap. 2]) that Problem (5.2)–(5.3) is stable if and only if the operator $\widehat{L}_m$ is similar to a uniformly positive operator. We now establish some preliminary results.

PROPOSITION 5.2. *Assume that the operator $\widehat{A} = \widehat{G}\widehat{L}_m$ (introduced in Proposition 2.1) is positive in* H, *and let* $\mathrm{H}_\theta$ *be a scale of Hilbert spaces generated by* $\widehat{A}$. *Then* $\mathrm{H}_{1/2} = \{\widehat{u} = \{u(x), u(\pi)\} \mid u \in W_2^1[0,\pi]\} = \mathcal{D}(\widehat{A}^{1/2})$. *Moreover, the map* $\widehat{u} \mapsto u$ *is a homeomorphism from* $\mathrm{H}_{1/2}$ *into* $W_2^1[0,\pi]$. *If* $\widehat{A}$ *is not positive, then it can be replaced by* $\widetilde{A} = \widehat{A} + CI$, *where* $C$ *is a large enough positive constant, and* $I$ *is the identity operator defined in* H.

*Proof.* A similar result is proved at the end of [13]. ∎

PROPOSITION 5.3. *Let* $m < 0$. *Then the system of eigenfunctions and associated functions (Jordan chains) of the operator* $\widehat{L}_m$ *forms a Riesz basis in the Hilbert space* H.

*Proof.* It follows from Proposition 2.1 that for $m < 0$, the operator $\widehat{L}_m$ is self-adjoint in the Pontryagin space $\Pi_1$ and has a discrete spectrum. Hence, the system of eigenfunctions forms a Riesz basis in H (see e.g. [2, Chap. 6]). ∎

LEMMA 5.4. *Let* $m < 0$ *and suppose that* $\widehat{L}_m$ *has no multiple eigenvalues. Then* $\widehat{L}_m$ *is similar to a normal operator in* H, *i.e., there exists a bounded operator* $\widehat{B}$ *defined in* H *with bounded inverse such that the operator* $\widehat{S} = \widehat{B}\widehat{L}_m\widehat{B}^{-1}$ *is normal in* H. *In particular, if* $m < 0$ *and* $Q > 0$ *then* $\widehat{S}$ *is self-adjoint in* H.

*Proof.* If $m < 0$ then according to Proposition 5.3, the system of eigenfunctions and associated functions of $\widehat{L}_m$ forms a Riesz basis in H. So, there exists a bounded operator $\widehat{B}$ with bounded inverse such that the system $\{\widehat{B}\widehat{u}_k\}_{k=1}^{\infty}$ forms an orthonormal basis in H. The eigenvalues of the operator $\widehat{S} = \widehat{B}\widehat{L}_m\widehat{B}^{-1}$ coincide with those of $\widehat{L}_m$. If $\widehat{L}_m$ has only algebraically simple eigenvalues, then the system of eigenfunctions of the operators $\widehat{S}$ coincides with the orthonormal basis $\{\widehat{B}\widehat{u}_k\}_{k=1}^{\infty}$. Therefore $\widehat{S}$ is a normal operator in H. In particular, if $Q > 0$ then by Theorem 2.2, all the eigenvalues are real and simple, and hence $\widehat{S} = \widehat{S}^*$ in H. ∎

In view of Theorems 2.3, 3.3 and 3.4, $\widehat{L}_m$ may have two nonreal conjugate eigenvalues. In this case $\widehat{S}$ is a normal operator, and hence it admits the spectral decomposition

$$\widehat{S} = \widehat{S}_+ - \widehat{S}_- + \lambda_0\widehat{S}_0 + \overline{\lambda_0}\,\overline{\widehat{S}_0}, \tag{5.4}$$

where $\widehat{S}_+$, $\widehat{S}_-$ ($\widehat{S}_+ \geq 0$, $\widehat{S}_- \geq 0$) are the orthogonal projectors on the subspaces spanned by the eigenfunctions corresponding respectively to positive and to negative eigenvalues, and $\widehat{S}_0$ (resp. $\overline{\widehat{S}_0}$) is the orthogonal projector on the subspace spanned by the eigenfunctions corresponding to the eigenvalue $\lambda_0$ (resp. $\overline{\lambda_0}$). In particular, if $Q > 0$ then by Theorem 2.2, all the eigenvalues are real and simple. Hence $\widehat{S}_0 = \overline{\widehat{S}_0} \equiv 0$.

Thus, the square root operator (which we denote by $\widehat{W}$) is defined by the expression

$$\widehat{W} = \widehat{B}^{-1}(\widehat{S}_+^{1/2} + i\widehat{S}_-^{1/2} + \sqrt{\lambda_0}\,\widehat{S}_0^{1/2} + \sqrt{\overline{\lambda_0}}\,\overline{\widehat{S}_0}^{1/2})\widehat{B} \tag{5.5}$$

$$= \widehat{W}_+ + i\widehat{W}_-. \tag{5.6}$$

It is clear that $\mathcal{D}(\widehat{L}_m) = \mathcal{D}(\widetilde{A}) = \mathcal{D}(\widehat{W^2})$ (where $\widetilde{A}$ is defined in Proposition 5.2). Therefore, by Heinz's theorem ([14, Chap. 12]) we have

$$\mathcal{D}(\widehat{W}) = \mathcal{D}(\widetilde{A}^{1/2}) = \mathrm{H}_{1/2},$$

where $\mathrm{H}_{1/2}$ is described in Proposition 5.2. Now we can state the main result of this section.

THEOREM 5.5. *Let $m < 0$ and assume that all the eigenvalues of the operator $\widehat{L}_m$ are algebraically simple. Then, Problem* (5.2)–(5.3) *has a unique classical solution for all $\widehat{\phi} \in \mathcal{D}(\widehat{L}_m)$ and $\widehat{\Psi} \in \mathrm{H}_{1/2} \cap (\operatorname{Ker} \widehat{L}_m^*)^\perp$ (if $Q > 0$ then $\widehat{\Psi} \in \mathrm{H}_{1/2}$), where $\mathrm{H}_{1/2}$ is defined in Proposition* 5.2, *i.e., for all $\phi \in W_2^2[0,\pi]$ such that $\phi'(0) = 0$ and $\psi \in W_2^1[0,\pi]$. This solution is given by the expression*

$$\widehat{u}(t) = \cos(\widehat{W}t)\widehat{\phi} + \sin(\widehat{W}t)\widehat{W}^{-1}\widehat{\Psi}, \tag{5.7}$$

*where $\widehat{W}$ is defined by* (5.5).

*If $m < 0$ and $Q > 0$, then Problem* (5.2)–(5.3) *is not stable, while if*

$$\mu_1 < 0 < \mu_1', \tag{5.8}$$

*then it is stable for $m \in (-\infty, m_1)$ (where $m_1$ is introduced in Theorem* 3.3). *Note that the condition* (5.8) *depends only on the potential $q(x)$.*

*Proof.* If $m < 0$ and $Q < 0$, then $\lambda = 0$ can be an eigenvalue of the operator $\widehat{L}_m$. In this case we have to impose some restrictions on the vector $\widehat{\Psi}$ for the Cauchy problem (5.2)–(5.3) to be well-posed; namely, we have to require $\widehat{\Psi} \in \operatorname{Im} \widehat{W}$ (see [15, Chap. 3.1, p. 95]). The operator $\widehat{L}_m$ has a discrete spectrum. Therefore, the subspace $\operatorname{Im} \widehat{L}_m$ is closed and coincides with the orthogonal complement of $\operatorname{Ker} \widehat{L}_m^*$, i.e.,

$$\operatorname{Im} \widehat{L}_m = \overline{\operatorname{Im} \widehat{L}_m} = (\operatorname{Ker} \widehat{L}_m^*)^\perp,$$

where the orthogonality is in H. Obviously,

$$\operatorname{Ker} \widehat{L}_m^* = \operatorname{Ker} (\widehat{W}^*)^2 = \operatorname{Ker} \widehat{W}^*.$$

Hence,

$$\operatorname{Im} \widehat{W} = \operatorname{Im} \widehat{L}_m = (\operatorname{Ker} \widehat{L}_m^*)^\perp.$$

Thus, $\widehat{\Psi} \in \operatorname{Im}\widehat{W}$ if and only if $\widehat{\Psi} \perp \operatorname{Ker}\widehat{L}_m^*$, and therefore $\widehat{W}^{-1}\widehat{\Psi}$ is well defined. If $m < 0$ and $Q > 0$ then by Theorem 2.2, $\operatorname{Ker}\widehat{L}_m = 0$. Therefore, $\operatorname{Im}\widehat{W} = \mathrm{H}$.

Since $\widehat{S}_+ \geq 0$, $\widehat{S}_- \geq 0$, where $\widehat{S}_+$ is self-adjoint and $\operatorname{rank}\widehat{S}_- < \infty$, then the operator semigroup

$$e^{i\widehat{W}t} = e^{i\widehat{W}_+ t}e^{-\widehat{W}_- t}$$

is correctly defined (see e.g. [12, Chap. 9]). Referring to [15, Chap. 3.1], we can easily verify that for $\widehat{\phi} \in \mathcal{D}(\widehat{L}_m)$ and $\widehat{\Psi} \in \mathrm{H}_{1/2} \cap (\operatorname{Ker}\widehat{L}_m^*)^\perp$, the function $\widehat{u}(t)$ defined by (5.7) is a solution of Problem (5.2)–(5.3). According to Kreĭn's Theorem [15, Chap. 3.1], this solution is unique.

We now establish the stability criterion. Obviously, Problem (5.2)–(5.3) is stable if and only if $\widehat{S}_- = \widehat{S}_0 = \overline{\widehat{S}_0} \equiv 0$ (i.e., if and only if $\widehat{L}_m$ has only positive and semisimple eigenvalues). In the case $m < 0$, $Q > 0$, by Theorem 2.2, $\widehat{L}_m$ has one negative eigenvalue, so that Problem (5.2)–(5.3) is not stable. In the case $m < 0$ and $Q < 0$, if $\mu_1 < 0 < \mu_1'$ and $m \in (-\infty, m_1)$, then according to Theorem 3.3, $\widehat{L}_m$ has only positive and simple eigenvalues. Therefore, in this case Problem (5.2)–(5.3) is stable. The theorem is proved. ∎

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