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TWO COMMUTING MAPS WITHOUT COMMON MINIMAL POINTS

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Abstract. We construct an example of two commuting homeomorphisms S, T of a compact metric space X such that the union of all minimal sets for S is disjoint from the union of all minimal sets for T. In other words, there are no common minimal points. This answers negatively a question posed in [C-L]. We remark that Furstenberg proved the existence of "doubly recurrent" points (see [F]). Not only are these points recurrent under both S and T, but they recur along the same sequence of powers. Our example shows that nothing similar holds if recurrence is replaced by the stronger notion of uniform recurrence.

The purpose of this note is to prove the following

THEOREM. *There exists a compact metric space X and two commuting transformations (homeomorphisms) $S : X \to X$ and $T : X \to X$ such that no point belongs simultaneously to a set minimal under S and a set minimal under T.*

Recall that a set A is called *minimal* under a continuous map T if it is closed, T-invariant ($T(A) \subset A$) and contains no proper subset with these two properties. It is worth mentioning (although we will not use this characterization) that a point belongs to a minimal set if and only if it is *uniformly recurrent*, i.e., returns to each of its neighborhoods along a syndetic (having bounded gaps) sequence of iterates. Such points are often called *minimal.* In the title, by *common minimal* points we mean points which are minimal simultaneously for two maps.

The rest of this note contains the construction of a relevant example.

Let $Y = \{0,1\}^{\mathbb{Z}^2}$ be the set of all 0-1 valued arrays $x = [x(n,m)]_{n\in\mathbb{Z},m\in\mathbb{Z}}$. We will interpret the coordinates as in the Cartesian system rather than in the matrix labeling, i.e., the first coordinate n is the column number (and grows to the right) while the second coordinate m represents the row

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number (and grows upward). On this space we have two natural commuting homeomorphisms: the horizontal shift (to the left) and the vertical shift (downward), given by the formulae

$$S(x)(n,m) = x(n+1,m), \qquad T(x)(n,m) = x(n,m+1).$$

In Y we will select a closed subspace X invariant under both S and T. We will do that by defining a "language", a sequence of families of square blocks (matrices) of which the elements of X will be built. We start the induction by defining the family $\mathcal{B}_1$ containing two 1×1 blocks, called the 1-*blocks*: [0] and [1]. Next we define four 2-blocks of dimensions 2×2:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note that the first two of them have constant rows. They will be called *horizontal* 2-*blocks*. The other two have constant columns and will be called *vertical* 2-*blocks*.

Suppose we have defined the family $\mathcal{B}_k$ consisting of 2^k square k-blocks of some dimensions $p_k \times p_k$. Say $\mathcal{B}_k = \{a, b, c, \ldots, y, z\}$ (we are not assuming that there are 26 of them; this labeling allows us to avoid another indexation). Here is how we define $\mathcal{B}_{k+1}$: it consists of 2^k *horizontal* $(k+1)$-*blocks* and 2^k *vertical* $(k+1)$-*blocks* (in total, 2^{k+1} blocks). The horizontal blocks are the following:

$$\begin{bmatrix} a & a & \cdots & a \\ b & b & \cdots & b \\ c & c & \cdots & c \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ y & y & \cdots & y \\ z & z & \cdots & z \end{bmatrix}, \begin{bmatrix} b & b & \cdots & b \\ c & c & \cdots & c \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ y & y & \cdots & y \\ z & z & \cdots & z \\ a & a & \cdots & a \end{bmatrix}, \begin{bmatrix} c & c & \cdots & c \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ y & y & \cdots & y \\ z & z & \cdots & z \\ a & a & \cdots & a \\ b & b & \cdots & b \end{bmatrix}, \ldots, \begin{bmatrix} z & z & \cdots & z \\ a & a & \cdots & a \\ b & b & \cdots & b \\ c & c & \cdots & c \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ y & y & \cdots & y \end{bmatrix},$$

while the vertical $(k+1)$-blocks are:

$$\begin{bmatrix} a & b & c & \cdots & y & z \\ a & b & c & \cdots & y & z \\ a & b & c & \cdots & y & z \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ a & b & c & \cdots & y & z \end{bmatrix}, \begin{bmatrix} b & c & \cdots & y & z & a \\ b & c & \cdots & y & z & a \\ b & c & \cdots & y & z & a \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\ b & c & \cdots & y & z & a \end{bmatrix}, \begin{bmatrix} c & \cdots & y & z & a & b \\ c & \cdots & y & z & a & b \\ c & \cdots & y & z & a & b \\ \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots \\ c & \cdots & y & z & a & b \end{bmatrix}, \ldots, \begin{bmatrix} z & a & b & c & \cdots & y \\ z & a & b & c & \cdots & y \\ z & a & b & c & \cdots & y \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ z & a & b & c & \cdots & y \end{bmatrix}.$$

The dimensions of these blocks (in both directions) are $p_{k+1} = 2^k p_k$. We skip the explicit formula for p_{k+1}.

Note that the horizontal blocks no longer have all rows constant, for instance, if q is a vertical block then $q, q, \ldots, q$ represents p_k nonconstant rows. Nevertheless, as we will show, some of the rows are indeed constant.

The space X is defined as the intersection of spaces X_k, where X_k consists of all arrays that can be represented as aligned concatenations of k-blocks (by aligned concatenations we mean those where the division lines between the component blocks form a lattice of straight lines). Notice that the spaces X_k are all closed (hence compact), nonempty, and invariant under both S and T. Since the sequence X_k decreases, X is also closed, nonempty and invariant under both transformations.

We intend to prove that no element of X belongs simultaneously to a minimal set under S and a minimal set under T. First we need a small observation.

LEMMA 1. *Fix some k and recall that p_k denotes the dimension of k-blocks. Next, fix $n \in \{1, \ldots, p_k\}$. Then there exists a (horizontal) k-block whose nth row consists only of zeros and another (horizontal) k-block whose nth row consists only of ones. Similarly, there exists a (vertical) k-block whose nth column consists only of zeros, and another whose nth column consists only of ones.*

Proof. For $k = 1$ and 2 this property is visible with the naked eye. Suppose this is true for some k. We will prove the property for $(k+1)$-blocks with zeros in the horizontal rows. The other cases are completely analogous. Fix some n between 1 and p_{k+1} and let $m = n(\mathrm{mod})p_k$ (we use the convention that $n(\mathrm{mod})p$ assumes the values from 1 to p). By the inductive assumption, there exists a k-block, say q, whose mth row consists exclusively of zeros. By construction, there exists a horizontal $(k+1)$-block in which the nth row passes through $q, q, \ldots, q$. It is clear that this row becomes the mth row in each of the copies of q. So, this row consists entirely of zeros. ∎

We are in a position to prove the main claim.

PROPOSITION 1. *Let $x \in X$. Then at least one of the conditions below holds:*

(A) *The 0th row of x contains arbitrarily long (horizontal) strings of zeros and arbitrarily long (horizontal) strings of ones.*
(B) *The zero column of x contains arbitrarily long (vertical) strings of zeros and arbitrarily long (vertical) strings of ones.*

In particular, there are no common minimal points in X, i.e., points that belong simultaneously to a minimal set under the horizontal shift S and to a minimal set under the vertical shift T.

Proof. Consider the coordinate $(0,0)$ in x. By the definition of X, for every k this coordinate is covered by some $(k+1)$-block. It does not matter whether the covering $(k+1)$-block is uniquely determined or not (in fact, we believe it is, but it is not necessary to verify this claim). For each k we simply pick one possible $(k+1)$-block that we can see around the origin. This block is either horizontal or vertical. Suppose it is vertical. It is then easily seen, from the way the vertical $(k+1)$-blocks are built of k-blocks, that the 0th row of x passes (inside this vertical $(k+1)$-block) through all possible k-blocks, and in each of them this row becomes row n (with some n between 1 and p_k), where n is the same for all the crossed k-blocks. Among them, there will be the (horizontal) k-block whose nth row consists only of zeros as well as the (horizontal) k-block whose nth row consists only of ones (see Lemma 1). We have proved that if $(0,0)$ is covered by a vertical $(k+1)$-block then the 0th row of x contains a (horizontal) string of zeros of length p_k and a similar string of ones. If this happens for infinitely many indices k then x satisfies the condition (A). An analogous argument proves (B) in the symmetric case. Since there are infinitely many indices k, either (A) or (B) must hold for x.

The last statement of the proposition is obvious. A point x which satisfies the condition (A) cannot belong to a minimal set under the horizontal shift S. Its orbit closure under S contains a point y whose entire 0th row consists of zeros, and then x does not belong to the orbit closure of y (because the 0th row of x contains some ones). Analogously, a point that satisfies (B) does not belong to a minimal set under the vertical shift T. A common minimal point would have to fail both (A) and (B), and we have just proved that there are no such points. ∎

For completeness, below we outline (without detailed proofs) some other properties of our system.

1. The above system factors into the direct product $G \times G$ of the odometer G to base (p_k) with itself. The odometer factor is responsible for the positioning of the vertical and horizontal division lines between the k-blocks.

2. The system is NOT transitive under S (and similarly, it is not transitive under T). Here we have a very simple argument. Every point x has a structure of horizontal division lines. This structure will not change under the horizontal shift. So, this point is not transitive under S, as its S-orbit cannot reach points with differently positioned division lines. More generally, any system with two commuting transformations, say S and T, which factors into a direct product of two nondegenerate systems, say $(G_1, R_1) \times (G_2, R_2)$, cannot be transitive for S or T. Namely, every point x factors into a pair (g_1, g_2), and then its orbit under the action of S remains within the preimage of $G_1 \times \{g_2\}$ (this preimage is S-invariant), while under the action of T it remains within the preimage of $\{g_1\} \times G_2$ (invariant under T).

3. For a fixed element g in the odometer G, the preimage of $G \times \{g\}$ is closed and, as we have said, S-invariant. This set is already transitive under S, and within this set, every point x satisfying the condition (A) is transitive (we skip the proof).

4. There are many points which satisfy both (A) and (B). These points are transitive within their corresponding fiber sets corresponding to $G \times \{g\}$ (under S) and to $\{h\} \times G$ (under T).

5. The example can probably be modified to become transitive under both S and T. In that case there would be many common transitive points (the last statement follows from the general fact that in a transitive system the transitive points form a dense G_δ set). The above mentioned modification requires killing the odometer factor. This can be done by replacing the square blocks by some rhombuses, so that the horizontal and vertical shifts do not preserve the structure of the division lines. Other details would become slightly more complicated.

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