COLLOQUIUM MATHEMATICUM

VOL. 123 2011 NO. 2

ESTIMATES ON INNER AND OUTER RADII OF UNIT BALLS IN NORMED SPACES

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Abstract. The purpose of this paper is to continue the investigations on extremal values for inner and outer radii of the unit ball of a finite-dimensional real Banach space for the Holmes–Thompson and Busemann measures. Furthermore, we give a related new characterization of ellipsoids in $\mathbb{R}^d$ via codimensional cross-section measures.

0. Introduction. Continuing [8], we will establish sharp lower and upper bounds on inner and outer radii of unit balls of finite-dimensional real Banach spaces which are defined with the help of (maximally contained or minimally containing) homothets of isoperimetrices. More precisely, we obtain a sharp lower bound on the inner radius for the Holmes–Thompson measure and a sharp upper bound on the outer radius for the Busemann measure. Also we answer a related question on cross-section measures posed in [8] and [9], getting a new characterization of ellipsoids in $\mathbb{R}^d$ (in the spirit of [10]) and a sharp upper bound on the inner radius for the Busemann measure.

1. Definitions and preliminaries. Recall that a *convex body* K is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$. As usual, S^{d-1} denotes the standard Euclidean unit sphere in $\mathbb{R}^d$. We write λ_i for the *i-dimensional Lebesgue measure* in $\mathbb{R}^d$, where $1 \le i \le d$, and instead of λ_d we simply write λ. We denote by $u^\perp$ the $(d-1)$-dimensional subspace orthogonal to $u \in S^{d-1}$, and by l_u the 1-subspace parallel to u. For a convex body K in $\mathbb{R}^d$, we define the *polar body* K° of K by $K^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \le 1, x \in K\}$ and identify $\mathbb{R}^d$ and its *dual space* $\mathbb{R}^{d*}$ by using the standard basis. In that case, λ_i and λ_i^* coincide in $\mathbb{R}^d$. For a centered convex body K in $\mathbb{R}^d$ we have the *Blaschke–Santaló inequality*

$$\lambda(K)\lambda(K^\circ) \le \epsilon_d^2,$$

2010 *Mathematics Subject Classification*: 46B20, 52A40.
Key words and phrases: Busemann volume, cross-section measures, ellipsoids, Holmes–Thompson volume, inner radius, intersection body, isoperimetrix, Minkowski space, mixed volumes, outer radius, projection body.

DOI: 10.4064/cm123-2-5

with equality exactly for ellipsoids (see [4]). Here ϵ_d stands for the volume of the Euclidean unit ball in $\mathbb{R}^d$. For K a convex body in $\mathbb{R}^d$ and $u \in S^{d-1}$, the *support function* is defined by $h_K(u) = \sup\{\langle u, y\rangle : y \in K\}$, and for $o \in K$ its *radial function* $\rho_K(u)$ by $\rho_K(u) = \max\{\alpha \geq 0 : \alpha u \in K\}$. We always have $h_{\alpha K} = \alpha h_K$ and $\rho_{\alpha K} = \alpha\rho_K$; only these are needed here. We also mention the relation

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \qquad u \in S^{d-1}. \tag{1}$$

The *projection body* ΠK of a convex body K in $\mathbb{R}^d$ is defined by $h_{\Pi K}(u) = \lambda_{d-1}(K|u^\perp)$ for each $u \in S^{d-1}$, where $K|u^\perp$ is the orthogonal projection of K onto $u^\perp$, and $\lambda_{d-1}(K|u^\perp)$ is called the $(d-1)$-*dimensional outer cross-section measure* of K at u. The *intersection body* IK of a convex body $K \subset \mathbb{R}^d$ is defined by $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^\perp)$ for each $u \in S^{d-1}$. If K is a convex body in $\mathbb{R}^d$ containing o, and S is a subspace, then we also have

$$K^\circ \cap S = (K|S)^\circ. \tag{2}$$

Further on, for $K \subset \mathbb{R}^d$ a convex body we denote by $\lambda_{d-1}(K, u^\perp)$ and $\lambda_1(K, u)$ the *inner cross-section measures* of K (i.e., the maximal measure of a hyperplane section of K normal to u and the *maximal chord length* of K at u, respectively). Note that for centered convex bodies maximal chords pass through the origin. By definition $\lambda_1(K|l_u)$ is the *width* of K at u. All the notions given above can be found in the monographs [3], [12], and [14]; see also [6]. And we refer to [5] for a Fourier-analytic characterization of intersection bodies. In [7] the following results for cross-section measures were derived (see also [11] and [13] for generalizations).

For a convex body K in $\mathbb{R}^d$, $d \geq 2$, and every direction $u \in S^{d-1}$,

$$\lambda(K) \leq \lambda_{d-1}(K|u^\perp)\lambda_1(K, u) \leq d\lambda(K), \tag{3}$$

with equality on the left if and only if K is a cylinder with u as generator direction, and on the right precisely for K an oblique double cone with respect to u. A convex body K is called an *oblique double cone* with respect to the direction $p - q$ if each boundary point of K can be connected to the boundary points p or q of K by a boundary segment. In other words, any 2-dimensional half-plane with bounding line through the maximal chord pq of K intersects K in a (possibly degenerate) triangle. And for each $u \in S^{d-1}$, a convex body K in $\mathbb{R}^d$, $d \geq 2$, satisfies

$$\lambda(K) \leq \lambda_{d-1}(K, u^\perp)\lambda_1(K|l_u) \leq d\lambda(K), \tag{4}$$

with equality on the left if and only if K is a cylinder whose generators are parallel to u and whose basis is normal to u, and on the right exactly for K a (double) cone whose basis is normal to u.

We write $(\mathbb{R}^d, \|\cdot\|) = \mathbb{M}^d$ for a *d-dimensional real Banach space*, i.e., a *Minkowski space* with *unit ball* B which is a centered convex body; see [14]. The *unit sphere* of $\mathbb{M}^d$ is the boundary ∂B of the unit ball.

2. Surface areas, volumes, and isoperimetrices in Minkowski spaces. A Minkowski space $\mathbb{M}^d$ possesses a Haar measure μ, and this measure is unique up to multiplying the Lebesgue measure by a constant, i.e., $\mu = \sigma_B \lambda$.

The following notions are well known; see Chapter 5 of [14]. The *d-dimensional Holmes–Thompson volume* of a convex body K in $\mathbb{M}^d$ is defined by

$$\mu_B^{\mathrm{HT}}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\epsilon_d}, \quad \text{i.e.,} \quad \sigma_B = \frac{\lambda(B^\circ)}{\epsilon_d},$$

and the *d-dimensional Busemann volume* of K is defined by

$$\mu_B^{\mathrm{Bus}}(K) = \frac{\epsilon_d}{\lambda(B)}\lambda(K), \quad \text{i.e.,} \quad \sigma_B = \frac{\epsilon_d}{\lambda(B)} \ (\text{and } \mu_B^{\mathrm{Bus}}(B) = \epsilon_d).$$

To define the Minkowski surface area of a convex body, one needs similarly to define σ_B in $\mathbb{M}^{d-1}$. That is, for the Holmes–Thompson measure we have $\sigma_B(u) = \lambda_{d-1}((B \cap u^\perp)^\circ)/\epsilon_{d-1}$, and for the Busemann measure $\sigma_B(u) = \epsilon_{d-1}/\lambda(B \cap u^\perp)$ (see [14, pp. 150–151]). The *Minkowski surface area* of K can also be defined in terms of mixed volumes (see [12] for notation and more about mixed volumes) by

$$\mu_B(\partial K) = dV(K[d-1], I_B)\,, \tag{5}$$

where I_B is the convex body whose support function is $\sigma_B(u)$. For the Holmes–Thompson measure, I_B is defined by $I_B^{\mathrm{HT}} = \Pi(B^\circ)/\epsilon_{d-1}$, and therefore it is a centered zonoid. For the Busemann measure we have $I_B^{\mathrm{Bus}} = \epsilon_{d-1}(IB)^\circ$. Among the homothetic images of I_B, one is singled out; it is called the *isoperimetrix* $\hat{I}_B$ and determined by $\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$. The *isoperimetrix for the Holmes–Thompson measure* is defined by

$$\hat{I}_B^{\mathrm{HT}} = \frac{\epsilon_d}{\lambda(B^\circ)} I_B^{\mathrm{HT}}\,, \tag{6}$$

and the *isoperimetrix for the Busemann measure* by

$$\hat{I}_B^{\mathrm{Bus}} = \frac{\lambda(B)}{\epsilon_d} I_B^{\mathrm{Bus}}; \tag{7}$$

see again Chapter 5 of [14] and [9].

Definition 1. If K is a convex body in $\mathbb{M}^d$, the *inner radius* of K is defined by $r(K) := \max\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha\hat{I}_B \subseteq K + x\}$, and the *outer radius* of K is $R(K) := \min\{\alpha : \exists x \in \mathbb{M}^d \text{with } \alpha\hat{I}_B \supseteq K + x\}$.

3. Estimates for inner and outer radii of the unit ball. Notice that when K is a centered convex body, $r(K)$ and $R(K)$ can also be defined in terms of the support functions of K and $\hat{I}_B$. Namely, $r(K)$ is the maximum value of α such that $\alpha \le h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$. Similarly, $R(K)$ is the minimal α such that $\alpha \ge h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$.

Setting $K = B^\circ$ in (3), we obtain $\lambda(B^\circ) \le 2\rho_{B^\circ}(u)h_{\Pi B^\circ}(u) \le d\lambda(B^\circ)$ for each $u \in S^{d-1}$. This gives

$$\frac{\lambda(B^\circ)}{2\epsilon_{d-1}}h_B(u) \le h_{I_B^{\mathrm{HT}}}(u) \le \frac{d\lambda(B^\circ)}{2\epsilon_{d-1}}h_B(u)$$

for each $u \in S^{d-1}$. Since the last inequality does not change under dilations, we may assume that $\lambda(B^\circ) = \epsilon_d$. This yields

$$\frac{\epsilon_d}{2\epsilon_{d-1}} \le \frac{h_{I_B}(u)}{h_B(u)} \le \frac{d\epsilon_d}{2\epsilon_{d-1}}$$

(cf. [2]). Since in that case $\hat{I}_B^{\mathrm{HT}} = I_B$, the following result is established.

THEOREM 2. *Let B be the unit ball of a Minkowski space. Then for the Holmes–Thompson measure we have the estimate*

$$R(B) \le \frac{2\epsilon_{d-1}}{\epsilon_d}.$$

This estimate is sharp. From [7] it follows that equality holds when B a centered cylinder with u as generator direction.

REMARK. As in [8], we also find that the Holmes–Thompson measure satisfies the inequality

$$r(B) \ge \frac{2\epsilon_{d-1}}{d\epsilon_d},$$

with equality when B is an oblique double cone with respect to u; this also follows from [7].

For $K = B$, (4) yields $\lambda(B) \le 2\rho_{IB}(u)h_B(u) \le d\lambda(B)$ for any direction u, implying

$$\frac{\lambda(B)}{2}h_{I_B^{\mathrm{Bus}}}(u) \le \epsilon_{d-1}h_B(u) \le \frac{d\lambda(B)}{2}h_{I_B^{\mathrm{Bus}}}(u)$$

for each $u \in S^{d-1}$. Since the last inequality will not change under dilations, we may assume that $\lambda(B) = \epsilon_d$. Therefore we have

$$\frac{2\epsilon_{d-1}}{d\epsilon_d} \le \frac{h_{I_B}(u)}{h_B(u)} \le \frac{2\epsilon_{d-1}}{\epsilon_d}$$

(cf. [2]). Since in that case $\hat{I}_B^{\mathrm{Bus}} = I_B$, we have established the following result.

THEOREM 3. *Let B be the unit ball of a Minkowski space. Then the Busemann measure satisfies the estimate*

$$r(B) \geq \frac{\epsilon_d}{2\epsilon_{d-1}}.$$

Again, this estimate is sharp. From [7] it follows that equality holds when B is a centered cylinder with basis direction u.

REMARK. As in [8], we also infer that for the Busemann measure we have

$$R(B) \leq \frac{d\epsilon_d}{2\epsilon_{d-1}},$$

with equality when B is a (double) cone whose basis is normal to u; this follows from [7]. Clearly, both measures satisfy $R(B)/r(B) \leq d$.

THEOREM 4. *Let B be the unit ball of a Minkowski space with $d \geq 3$. Then there exists a direction $u \in S^{d-1}$ such that*

$$\frac{\lambda_{d-1}(B \cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\epsilon_{d-1}}{\epsilon_d}.$$

Furthermore, equality holds for each $u \in S^{d-1}$ if and only if B is an ellipsoid.

Proof. We set B to be B° with $\lambda(B^\circ) = \epsilon_d$, since the inequality does not change under dilations. From the Blaschke–Santaló inequality and (2) we obtain

$$\begin{aligned}\lambda_{d-1}(B^\circ \cap u^{\perp})\lambda_1(B^\circ|l_u) &\leq \frac{\epsilon_{d-1}^2\lambda_1(B^\circ|l_u)}{\lambda_{d-1}((B^\circ \cap u^{\perp})^\circ)} \\ &= \frac{\epsilon_{d-1}^2\lambda_1(B^\circ|l_u)}{\lambda_{d-1}(B|u^{\perp})} = \frac{2\epsilon_{d-1}^2 h_{B^\circ}(u)}{h_{\Pi B}(u)} = \frac{2\epsilon_{d-1}h_{B^\circ}(u)}{h_{I_{B^\circ}^{\mathrm{HT}}}(u)}.\end{aligned}$$

If $h_{B^\circ}(u)/h_{I_{B^\circ}^{\mathrm{HT}}}(u) > 1$ for all $u \in S^{d-1}$, then $\hat{I}_{B^\circ}^{\mathrm{HT}} \subset B^\circ$. But this contradicts the fact that $\hat{I}_B^{\mathrm{HT}} \subset B$ if and only if B is an ellipsoid (see [14, p. 216]). Note that in our setting $\hat{I}_B^{\mathrm{HT}} = I_B^{\mathrm{HT}}$ (for $d = 2$, $I_B^{\mathrm{HT}} = B$ holds also for Radon curves). Hence there is a direction $u \in S^{d-1}$ such that $h_{B^\circ}(u)/h_{I_{B^\circ}^{\mathrm{HT}}}(u) \leq 1$. And, clearly, equality holds for each $u \in S^{d-1}$ if and only if B is an ellipsoid. ∎

COROLLARY 5. *Let B be the unit ball of a Minkowski space with $d \geq 3$. Then for the Busemann measure we have the sharp estimate $r(B) \leq 1$.*

Proof. By the theorem above there is a direction $u \in S^{d-1}$ such that $\rho_{IB}(u)h_B(u) \leq \lambda(B)\epsilon_{d-1}/\epsilon_d$. Applying (1) and (7), we obtain $h_B(u)/h_{\hat{I}_B^{\mathrm{Bus}}}(u) \leq 1$ for some $u \in S^{d-1}$, establishing the result. ∎

REMARK. For the Busemann measure the equality $r(B) = 1$ holds not only for ellipsoids. For example, if B is an affine image of the dual-Archimedean rhombic dodecahedron in $\mathbb{M}^3$, then $r(B) = 1$; see [9].

Finding sharp bounds on $\mu_B(\partial B)$ for both measures in $\mathbb{M}^d$, $d \geq 3$, is a challenging problem. It is known that for the Busemann measure we have $\mu_B(\partial B) \leq 2d\epsilon_{d-1}$, with equality if and only if B is a parallelotope. It has been conjectured that the Busemann measure satisfies $\mu_B(\partial B) \geq d\epsilon_d$. From properties of mixed volumes it follows that for both measures we have

$$\lambda(B) = V(B[d-1], B) \geq r(B)V(B[d-1], \hat{I}_B),$$
$$\lambda(B) = V(B[d-1], B) \leq R(B)V(B[d-1], \hat{I}_B).$$

Thus we obtain $r(B)\mu_B(\partial B) \leq d\epsilon_d$ and $R(B)\mu_B(\partial B) \geq d\epsilon_d$ for the Busemann measure, and $r(B)\mu_B(\partial B) \leq d\epsilon_d$ for the Holmes–Thompson measure.

An important open problem is whether B has to be an ellipsoid if B is a solution of the isoperimetric problem in $\mathbb{M}^d$, $d \geq 3$ (see [1]). For the Holmes–Thompson measure, this would mean that B has to be an ellipsoid if B and ΠB° are homothetic (see [3, p. 180], [6], [12, p. 416], and [14, Problem 6.5.4]). And for the Busemann measure, it would mean that B has to be an ellipsoid if B and $(IB)^\circ$ are homothetic (see [3, p. 336], [6], [12, p. 416], and [14, Problem 7.4.4]). These problems are equivalent to the following two questions in $\mathbb{M}^d$, $d \geq 3$, the first meant for the Holmes–Thompson measure, and the second for the Busemann measure: Is there a constant c such that, for all $u \in S^{d-1}$,

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} = c \quad \text{or} \quad \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} = c?$$

From our Theorem 4, and Theorem 9 of [8], we see that $c = 2\epsilon_{d-1}/\epsilon_d$ if and only if B is an ellipsoid. Can c be equal to another constant? One should also notice that if such a constant c not equal to $2\epsilon_{d-1}/\epsilon_d$ exists, then for the Holmes–Thompson measure $c > 2\epsilon_{d-1}/\epsilon_d$, and for the Busemann measure $c < 2\epsilon_d/\epsilon_d$, since $h_{\hat{I}_B}$ cannot be strictly smaller than h_B.

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Received 14 January 2011;
revised 1 April 2011 (5462)