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A NOTE ON PRODUCT STRUCTURES ON HOCHSCHILD HOMOLOGY OF SCHEMES

BY

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Abstract. We extend the definition of Hochschild and cyclic homologies of a scheme over a commutative ring k to define the Hochschild homologies $\mathrm{HH}_*(X/S)$ and cyclic homologies $\mathrm{HC}_*(X/S)$ of a scheme X with respect to an arbitrary base scheme S. Our main purpose is to study product structures on the Hochschild homology groups $\mathrm{HH}_*(X/S)$. In particular, we show that $\mathrm{HH}_*(X/S) = \bigoplus_{n\in\mathbb{Z}} \mathrm{HH}_n(X/S)$ carries the structure of a graded algebra.

1. Introduction. Let k be a commutative ring and let A be a k-algebra. The theory of Hochschild and cyclic homologies of A has been developed extensively in the literature (see [5] for an exposition). If X is a scheme over $\mathrm{Spec}(k)$, the Hochschild homology of the scheme X can be defined by sheafifying the Hochschild complex of X and taking hypercohomology (see [1, §4]). Following Loday [4, 3.4], the cyclic homology of X is defined similarly (see also [1], [7]). For approaches to the Hochschild cohomology of schemes, see [2] or [6].

The purpose of this note is to describe a product on the Hochschild homology groups of a scheme X. In fact, we consider the more general situation in which the commutative ground ring k is replaced by a given scheme S over k. Then, using hypercohomology as in [1], we introduce the Hochschild homologies $\mathrm{HH}_*(X/S)$ and cyclic homologies $\mathrm{HC}_*(X/S)$ of a scheme X with respect to the base scheme S. When the base scheme S is $\mathrm{Spec}(k)$, $\mathrm{HH}_*(X/S)$ and $\mathrm{HC}_*(X/S)$ become identical to the Hochschild and cyclic homologies respectively of the scheme X over k as in [1], [7]. In particular, when $X = \mathrm{Spec}(A)$ is an affine scheme and the base scheme $S = \mathrm{Spec}(k)$, we recover the usual Hochschild and cyclic homologies of the k-algebra A.

Our main result is the following: given schemes X and Y over a base scheme S, there exists a product structure

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(Y/S) \to \mathrm{HH}_{q+r}(X \times_S Y/S) \quad \forall q, r \in \mathbb{Z}. \tag{1.1}$$

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From this, we deduce that, for a given scheme X over S, there exists a product structure $\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(X/S) \to \mathrm{HH}_{q+r}(X/S)$ that makes $\bigoplus_{r\in\mathbb{Z}} \mathrm{HH}_r(X/S)$ into a graded algebra. Moreover, if we change base with respect to a morphism $t : S' \to S$, setting $X_{S'} := X \times_S S'$, we have a product

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(S'/S) \to \mathrm{HH}_{q+r}(X_{S'}/S') \quad \forall q, r \in \mathbb{Z}. \tag{1.2}$$

2. Products on Hochschild homology. Throughout this paper, we let $f : X \to S$ be a morphism of schemes over a commutative ring k and let $\mathcal{O}_X$ denote the structure sheaf of the scheme X. For any open set $U \subseteq X$ and for any $n \geq 0$, we define

$$C_n(X/S)(U) := \mathcal{O}_X(U) \otimes_{\Gamma(S,\mathcal{O}_S)} \cdots \otimes_{\Gamma(S,\mathcal{O}_S)} \mathcal{O}_X(U) \tag{2.1}$$

(tensor product taken $n + 1$ times) where $\Gamma(S, \mathcal{O}_S)$ denotes the global sections of the structure sheaf $\mathcal{O}_S$ of S.

Then the objects $C_n(X/S)(U)$ carry a well known Hochschild differential $b(U)_n : C_n(X/S)(U) \to C_{n-1}(X/S)(U)$ for $n \geq 1$ (see [5, §1.1.1]) defining a Hochschild complex $(C_h^*(X/S)(U), b(U))$ given by $C_h^n(X/S)(U) := C_{-n}(X)(U)$, $n \leq 0$. Further, we have Connes' operator $B(U)_n : C_n(X/S)(U) \to C_{n+1}(X/S)(U)$ for $n \geq 0$ (see [5, §2.1.7]) and the differentials $b(U)_*$ and $B(U)_*$ together form a "mixed (bi)complex" $(BC^{**}(X/S)(U), B(U), b(U))$ with (for $p, q \leq 0$):

$$\begin{aligned} BC^{p,q}(X/S)(U) &:= C_{p-q}(X/S)(U), \\ b(U) : BC^{p,q}(X/S)(U) &\to BC^{p,q+1}(X/S)(U), \\ B(U) : BC^{p,q}(X/S)(U) &\to BC^{p+1,q}(X/S)(U). \end{aligned} \tag{2.2}$$

DEFINITION 2.1. Let $f : X \to S$ be a morphism of schemes over a commutative ring k. Let $(\tilde{C}_h^*(X/S), b)$ (resp. $(\tilde{BC}^{**}(X/S), B, b)$) denote the sheafification of the complex $U \mapsto (C_h^*(X/S)(U), b(U))$ (resp. of $U \mapsto (BC^{**}(X/S)(U), B(U), b(U))$) to a complex of sheaves of abelian groups on X.

Then, for each $q \in \mathbb{Z}$, we refer to the $(-q)$th total hypercohomology of the complex $(\tilde{C}_h^*(X/S)(U), b)$ (resp. $(\mathrm{Tot}(\tilde{BC}^{**}(X/S)), B, b)$) as the qth *Hochschild* (resp. *cyclic*) *homology* $\mathrm{HH}_q(X/S)$ (resp. $\mathrm{HC}_q(X/S)$) of X with respect to S.

From Definition 2.1, it is clear that the Hochschild homologies $\mathrm{HH}_*(X/S)$ and cyclic homologies $\mathrm{HC}_*(X/S)$ are natural and contravariant on the category Sch/S of schemes over S. In particular, suppose that S is affine, i.e., $S = \mathrm{Spec}(A)$ for some commutative k-algebra A. Then, following Loday [4, 3.4], we have cyclic homology groups of the scheme X over $\mathrm{Spec}(A)$, which we denote by $\mathrm{HC}_q(X_A)$, $q \in \mathbb{Z}$. Similarly, we have (see [1], [7]) Hochschild homology groups of X over $\mathrm{Spec}(A)$, which we denote by $\mathrm{HH}_q(X_A)$, $q \in \mathbb{Z}$.

THEOREM 2.2. *Let A be a commutative k-algebra and suppose that $S = \mathrm{Spec}(A)$. Let $f : X \to S$ be a morphism of schemes. Then, for any $q \in \mathbb{Z}$, we have natural isomorphisms*

$$\mathrm{HH}_q(X/S) \cong \mathrm{HH}_q(X_A), \quad \mathrm{HC}_q(X/S) \cong \mathrm{HC}_q(X_A). \tag{2.3}$$

In particular, when $X = \mathrm{Spec}(B)$ is also affine, we have natural isomorphisms

$$\mathrm{HH}_q(X/S) \cong \mathrm{HH}_q(B|A), \quad \mathrm{HC}_q(X/S) \cong \mathrm{HC}_q(B|A). \tag{2.4}$$

for any $q \in \mathbb{Z}$, where $\mathrm{HH}_q(B|A)$ and $\mathrm{HC}_q(B|A)$ denote respectively the Hochschild and cyclic homologies of B considered as an A-algebra.

Proof. When $S = \mathrm{Spec}(A)$ is affine, we have $\Gamma(S, \mathcal{O}_S) = A$. Then, for any open set $U \subseteq X$ and any $n \geq 0$, we have $C_n(X/S)(U) = \mathcal{O}_X(U) \otimes_A \cdots \otimes_A \mathcal{O}_X(U)$ (tensor product taken $n+1$ times). It follows that the sheafified complex $(\tilde{C}_h^*(X/S), b)$ (resp. $(\mathrm{Tot}(\tilde{BC}^{**}(X/S)), B, b)$) computing the Hochschild homology (resp. the cyclic homology) of X is identical to the Hochschild complex (resp. cyclic complex) of X considered in [1, §4]. It follows that we have natural isomorphisms

$$\mathrm{HH}_q(X/S) \cong \mathrm{HH}_q(X_A), \quad \mathrm{HC}_q(X/S) \cong \mathrm{HC}_q(X_A). \tag{2.5}$$

In particular, suppose that $X = \mathrm{Spec}(B)$, i.e., X corresponds to an affine scheme corresponding to an A-algebra B. It then follows (see [1, Theorem 4.1] and [7, Theorem 2.5]) that we have natural isomorphisms

$$\begin{aligned} \mathrm{HH}_q(X_A) &= \mathbb{H}^{-q}(\tilde{C}_h^*(X/S)) \cong \mathrm{HH}_q(B|A), \\ \mathrm{HC}_q(X_A) &= \mathbb{H}^{-q}(\mathrm{Tot}(\tilde{BC}^{**}(X/S))) \cong \mathrm{HC}_q(B|A). \end{aligned} \tag{2.6}$$

Combining the isomorphisms in (2.5) and (2.6) yields (2.4). ∎

Let A and B be two given k-algebras. For any $p \geq 0$ and given $(a_0, a_1, \ldots, a_p) \in C_p(A)$, a permutation $\sigma \in S_p$ acts on $(a_0, a_1, \ldots, a_p)$ as follows (see [5, 4.2.1.1]):

$$\sigma \cdot (a_0, a_1, \ldots, a_p) := (a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(p)}).$$

Then, if $p, q \geq 0$ are integers and $S_{p,q}$ denotes the set of (p,q) *shuffles*, i.e., of all $\sigma \in S_{p+q}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$, we know that the shuffle product (see [5, 4.2.1.2])

$$\begin{aligned} &\mathrm{sh}_{p,q}((a_0, \ldots, a_p) \otimes (b_0, \ldots, b_q)) \\ &\quad = \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma)\sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \ldots, a_p \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_q) \end{aligned}$$

induces a product $\mathrm{sh}_{p,q} : \mathrm{HH}_p(A)\otimes_k \mathrm{HH}_q(B) \to \mathrm{HH}_{p+q}(A\otimes B)$ on Hochschild homology groups. Our purpose is to extend this to the Hochschild homology groups of schemes with respect to the base scheme S.

THEOREM 2.3. *Suppose that $f : X \to S$ and $g : Y \to S$ are morphisms of schemes over a commutative ring k. Then there is a multiplication on Hochschild homology groups with respect to S:*

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(Y/S) \to \mathrm{HH}_{q+r}(X \times_S Y/S) \quad \forall q, r \in \mathbb{Z}. \tag{2.7}$$

Proof. We consider the fibre product $X \times_S Y$ along with the projections $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$. We then choose an affine cover $\{S_i\}_{i\in I}$ of S and set $X_i := f^{-1}(S_i)$ and $Y_i = g^{-1}(S_i)$, $i \in I$. For each $i \in I$, let $\{X_{im}\}_{m\in M_i}$ (resp. $\{Y_{in}\}_{n\in N_i}$) be a basis of X_i (resp. Y_i) consisting of affine open subsets. Then the fibre product $X \times_S Y$ has a basis consisting of open sets of the form $\{X_{im} \times_{S_i} Y_{in}\}_{i\in I,\, m\in M_i,\, n\in N_i}$.

Further, we consider the complex $(C^*_h(X/S), b)$ (resp. $C^*_h(Y/S), b)$) of presheaves of abelian groups on X (resp. Y). For any given $i \in I$, $m \in M_i$ and $n \in N_i$, we note that, for all $q, r \geq 0$, the inverse image presheaves can be described as

$$\begin{aligned} p_X^{-1}(C_h^{-q}(X/S))(X_{im} \times_{S_i} Y_{in}) &= \mathcal{O}_X(X_{im})^{\otimes q+1}, \\ p_Y^{-1}(C_h^{-r}(Y/S))(X_{im} \times_{S_i} Y_{in}) &= \mathcal{O}_Y(Y_{in})^{\otimes r+1}, \end{aligned} \tag{2.8}$$

where all the tensor products are taken over $\Gamma(S, \mathcal{O}_S)$. Then, for all $q, r \geq 0$, we have shuffle maps

$$\begin{aligned} \mathrm{sh}_{q,r} : \mathcal{O}_X(X_{im})^{\otimes q+1} \otimes_{\Gamma(S,\mathcal{O}_S)} \mathcal{O}_Y(Y_{in})^{\otimes r+1} \\ \to \mathcal{O}_X(X_{im}) \otimes_{\Gamma(S,\mathcal{O}_S)} \mathcal{O}_Y(Y_{in})^{\otimes q+r+1}. \end{aligned} \tag{2.9}$$

We also consider the morphisms

$$\begin{aligned} &(\mathcal{O}_X(X_{im}) \otimes_{\Gamma(S,\mathcal{O}_S)} \mathcal{O}_Y(Y_{in}))^{\otimes q+r+1} \\ &\to (\mathcal{O}_X(X_{im}) \otimes_{\mathcal{O}_S(S_i)} \mathcal{O}_Y(Y_{in}))^{\otimes q+r+1} = \mathcal{O}_{X\times_S Y}(X_{im} \times_{S_i} Y_{in})^{\otimes q+r+1} \end{aligned} \tag{2.10}$$

induced by the natural morphism $(\mathcal{O}_X(X_{im}) \otimes_{\Gamma(S,\mathcal{O}_S)} \mathcal{O}_Y(Y_{in})) \to (\mathcal{O}_X(X_{im}) \otimes_{\mathcal{O}_S(S_i)} \mathcal{O}_Y(Y_{in}))$ of algebras. Composing the morphisms in (2.9) and (2.10) and using (2.8), we have natural morphisms

$$\begin{array}{c} p_X^{-1}(C_h^{-q}(X/S))(X_{im} \times_{S_i} Y_{in}) \otimes p_Y^{-1}(C_h^{-r}(Y/S))(X_{im} \times_{S_i} Y_{in}) \\ \downarrow \\ p_X^{-1}(C_h^{-q}(X/S))(X_{im} \times_{S_i} Y_{in}) \otimes_{\Gamma(S,\mathcal{O}_S)} p_Y^{-1}(C_h^{-r}(Y/S))(X_{im} \times_{S_i} Y_{in}) \\ \downarrow \\ C_h^{-q-r}(X \times_S Y/S)(X_{im} \times_{S_i} Y_{in}) \end{array} \tag{2.11}$$

Since the open sets $\{X_{im} \times_{S_i} Y_{in}\}_{i\in I,\, m\in M_i,\, n\in N_i}$ form a basis of $X \times_S Y$, it follows from the morphisms in (2.11) that we have a morphism of complexes of sheaves of abelian groups on $X \times_S Y$:

$$p_X^{-1}(\tilde{C}_h^*(X/S)) \otimes p_Y^{-1}(\tilde{C}_h^*(Y/S)) \to \tilde{C}_h^*(X \times_S Y/S). \tag{2.12}$$

Then, for all q, $r \in \mathbb{Z}$, the morphism in (2.12) induces a multiplication on hypercohomology:

(2.13)
$$\mathbb{H}^{-q}(p_X^{-1}(\tilde{C}_h^*(X/S))) \otimes \mathbb{H}^{-r}(p_Y^{-1}(\tilde{C}_h^*(Y/S))) \to \mathbb{H}^{-q-r}(\tilde{C}_h^*(X \times_S Y/S)).$$

It follows from general properties of hypercohomology (see [3]) that we have natural morphisms $\mathbb{H}^{-q}(\tilde{C}_h^*(X/S)) \to \mathbb{H}^{-q}(p_X^{-1}(\tilde{C}_h^*(X/S)))$ and $\mathbb{H}^{-r}(\tilde{C}_h^*(Y/S)) \to \mathbb{H}^{-r}(p_Y^{-1}(\tilde{C}_h^*(Y/S)))$. Combining this with (2.13), we have a multiplication

$$\mathbb{H}^{-q}(\tilde{C}_h^*(X/S)) \otimes \mathbb{H}^{-r}(\tilde{C}_h^*(Y/S)) \to \mathbb{H}^{-q-r}(\tilde{C}_h^*(X \times_S Y)). \tag{2.14}$$

The result of (2.7) now follows directly from the definition of Hochschild homology with respect to S. ∎

THEOREM 2.4. (a) *Let $f : X \to S$ be a morphism of schemes over a commutative ring k. Then $\bigoplus_{r\in\mathbb{Z}} \mathrm{HH}_r(X/S)$ carries the structure of a graded algebra.*

(b) *Let $f : X \to S$ and $t : S' \to S$ be morphisms of schemes over a commutative ring k. Then, setting $X_{S'} = X \times_S S'$, we have a product structure*

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(S'/S) \to \mathrm{HH}_{q+r}(X_{S'}/S') \quad \forall q, r \in \mathbb{Z}. \tag{2.15}$$

Proof. (a) From Proposition 2.3, it follows that we have a multiplication

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(X/S) \to \mathrm{HH}_{q+r}(X \times_S X/S) \tag{2.16}$$

for each $q, r \in \mathbb{Z}$. Further, the diagonal map $\Delta_{X/S} : X \to X \times_S X$ induces a morphism

$$\Delta_{X/S}^* : \mathrm{HH}_{q+r}(X \times_S X/S) \to \mathrm{HH}_{q+r}(X/S). \tag{2.17}$$

Composing the morphisms in (2.16) and (2.17), we have a product structure

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(X/S) \to \mathrm{HH}_{q+r}(X/S) \quad \forall q, r \in \mathbb{Z} \tag{2.18}$$

that makes $\bigoplus_{r\in\mathbb{Z}} \mathrm{HH}_r(X/S)$ into a graded algebra.

(b) Using Proposition 2.3, it follows that we have a product (for all $q, r \in \mathbb{Z}$)

$$\mathrm{HH}_q(X/S) \otimes \mathrm{HH}_r(S'/S) \to \mathrm{HH}_{q+r}(X_{S'}/S) = \mathrm{HH}_{q+r}(X_{S'}/S). \tag{2.19}$$

Further, we note that the natural morphisms

(2.20)
$$C_h^{-q-r}(X_{S'}/S) = \mathcal{O}_{X_{S'}}^{\otimes_{\Gamma(S,\mathcal{O}_S)} q+r+1} \to \mathcal{O}_{X_{S'}}^{\otimes_{\Gamma(S',\mathcal{O}_{S'})} q+r+1} = C_h^{-q-r}(X_{S'}/S')$$

induce morphisms of complexes $\tilde{C}_h^*(X_{S'}/S) \to \tilde{C}_h^*(X_{S'}/S')$ and hence a morphism of hypercohomologies

$$(2.21)\qquad \mathrm{HH}_{q+r}(X_{S'}/S) = \mathbb{H}^{-q-r}(\tilde{C}_h^*(X_{S'}/S)) \\ \to \mathbb{H}^{-q-r}(\tilde{C}_h^*(X_{S'}/S')) = \mathrm{HH}_{q+r}(X_{S'}/S').$$

Composing (2.19) and (2.21) yields (2.15). ■

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