A NOTE ON PRODUCT STRUCTURES ON HOCHSCHILD HOMOLOGY OF SCHEMES

BY

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Abstract. We extend the definition of Hochschild and cyclic homologies of a scheme over a commutative ring \( k \) to define the Hochschild homologies \( \text{HH}_*(X/S) \) and cyclic homologies \( \text{HC}_*(X/S) \) of a scheme \( X \) with respect to an arbitrary base scheme \( S \). Our main purpose is to study product structures on the Hochschild homology groups \( \text{HH}_*(X/S) \). In particular, we show that \( \text{HH}_*(X/S) = \bigoplus_{n \in \mathbb{Z}} \text{HH}_n(X/S) \) carries the structure of a graded algebra.

1. Introduction. Let \( k \) be a commutative ring and let \( A \) be a \( k \)-algebra. The theory of Hochschild and cyclic homologies of \( A \) has been developed extensively in the literature (see [5] for an exposition). If \( X \) is a scheme over \( \text{Spec}(k) \), the Hochschild homology of the scheme \( X \) can be defined by sheafifying the Hochschild complex of \( X \) and taking hypercohomology (see [1, §4]). Following Loday [4, 3.4], the cyclic homology of \( X \) is defined similarly (see also [1], [7]). For approaches to the Hochschild cohomology of schemes, see [2] or [6].

The purpose of this note is to describe a product on the Hochschild homology groups of a scheme \( X \). In fact, we consider the more general situation in which the commutative ground ring \( k \) is replaced by a given scheme \( S \). Then, using hypercohomology as in [1], we introduce the Hochschild homologies \( \text{HH}_*(X/S) \) and cyclic homologies \( \text{HC}_*(X/S) \) of a scheme \( X \) with respect to the base scheme \( S \). When the base scheme \( S \) is \( \text{Spec}(k) \), \( \text{HH}_*(X/S) \) and \( \text{HC}_*(X/S) \) become identical to the Hochschild and cyclic homologies respectively of the scheme \( X \) over \( k \) as in [1], [7].

In particular, when \( X = \text{Spec}(A) \) is an affine scheme and the base scheme \( S = \text{Spec}(k) \), we recover the usual Hochschild and cyclic homologies of the \( k \)-algebra \( A \).

Our main result is the following: given schemes \( X \) and \( Y \) over a base scheme \( S \), there exists a product structure

\[
\text{HH}_q(X/S) \otimes \text{HH}_r(Y/S) \to \text{HH}_{q+r}(X \times_S Y/S) \quad \forall q, r \in \mathbb{Z}.
\]

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From this, we deduce that, for a given scheme $X$ over $S$, there exists a product structure $\text{HH}_q(X/S) \otimes \text{HH}_r(X/S) \to \text{HH}_{q+r}(X/S)$ that makes $\bigoplus_{r \in \mathbb{Z}} \text{HH}_r(X/S)$ into a graded algebra. Moreover, if we change base with respect to a morphism $t : S' \to S$, setting $X_{S'} := X \times_S S'$, we have a product

\[(1.2) \quad \text{HH}_q(X/S) \otimes \text{HH}_r(S'/S) \to \text{HH}_{q+r}(X_{S'}/S') \quad \forall q, r \in \mathbb{Z}.\]

2. Products on Hochschild homology. Throughout this paper, we let $f : X \to S$ be a morphism of schemes over a commutative ring $k$ and let $\mathcal{O}_X$ denote the structure sheaf of the scheme $X$. For any open set $U \subseteq X$ and for any $n \geq 0$, we define

\[(2.1) \quad C_n(X/S)(U) := \mathcal{O}_X(U) \otimes_{\Gamma(S, \mathcal{O}_S)} \cdots \otimes_{\Gamma(S, \mathcal{O}_S)} \mathcal{O}_X(U)\]

(tensor product taken $n + 1$ times) where $\Gamma(S, \mathcal{O}_S)$ denotes the global sections of the structure sheaf $\mathcal{O}_S$ of $S$.

Then the objects $C_n(X/S)(U)$ carry a well known Hochschild differential $b(U)_n : C_n(X/S)(U) \to C_{n-1}(X/S)(U)$ for $n \geq 1$ (see [5, §1.1.1]) defining a Hochschild complex $(C^*_h(X/S)(U), b(U))$ given by $C^*_h(X/S)(U) := C_{-n}(X)(U), n \leq 0$. Further, we have Connes’ operator $B(U)_n : C_n(X/S)(U) \to C_{n+1}(X/S)(U)$ for $n \geq 0$ (see [5, §2.1.7]) and the differentials $b(U)_*$ and $B(U)_*$ together form a “mixed (bi)complex” $(BC^{**}(X/S)(U), B(U), b(U))$ with (for $p, q \leq 0$):

\begin{align*}
BC^{p,q}(X/S)(U) &:= C_{p-q}(X/S)(U), \\
b(U) : BC^{p,q}(X/S)(U) &\to BC^{p,q+1}(X/S)(U), \\
B(U) : BC^{p,q}(X/S)(U) &\to BC^{p+1,q}(X/S)(U).
\end{align*}

**Definition 2.1.** Let $f : X \to S$ be a morphism of schemes over a commutative ring $k$. Let $(\tilde{C}^*_h(X/S), b)$ (resp. $(BC^{**}(X/S), B, b)$) denote the sheafification of the complex $U \mapsto (C^*_h(X/S)(U), b(U))$ (resp. of $U \mapsto (BC^{**}(X/S)(U), B(U), b(U))$) to a complex of sheaves of abelian groups on $X$.

Then, for each $q \in \mathbb{Z}$, we refer to the $(-q)$th total hypercohomology of the complex $(\tilde{C}^*_h(X/S)(U), b)$ (resp. $(\text{Tot}(BC^{**}(X/S)), B, b)$) as the $q$th Hochschild (resp. cyclic) homology $\text{HH}_q(X/S)$ (resp. $\text{HC}_q(X/S)$) of $X$ with respect to $S$.

From Definition 2.1, it is clear that the Hochschild homologies $\text{HH}_*(X/S)$ and cyclic homologies $\text{HC}_*(X/S)$ are natural and contravariant on the category $\text{Sch}/S$ of schemes over $S$. In particular, suppose that $S$ is affine, i.e., $S = \text{Spec}(A)$ for some commutative $k$-algebra $A$. Then, following Loday [4, 3.4], we have cyclic homology groups of the scheme $X$ over $\text{Spec}(A)$, which we
denote by $HC_q(X_A)$, $q \in \mathbb{Z}$. Similarly, we have (see [1, 7]) Hochschild homology groups of $X$ over $\text{Spec}(A)$, which we denote by $HH_q(X_A)$, $q \in \mathbb{Z}$.

**Theorem 2.2.** Let $A$ be a commutative $k$-algebra and suppose that $S = \text{Spec}(A)$. Let $f : X \to S$ be a morphism of schemes. Then, for any $q \in \mathbb{Z}$, we have natural isomorphisms

\[(2.3)\quad HH_q(X/S) \cong HH_q(X_A), \quad HC_q(X/S) \cong HC_q(X_A).\]

In particular, when $X = \text{Spec}(B)$ is also affine, we have natural isomorphisms

\[(2.4)\quad HH_q(X/S) \cong HH_q(B/A), \quad HC_q(X/S) \cong HC_q(B/A).\]

for any $q \in \mathbb{Z}$, where $HH_q(B/A)$ and $HC_q(B/A)$ denote respectively the Hochschild and cyclic homologies of $B$ considered as an $A$-algebra.

**Proof.** When $S = \text{Spec}(A)$ is affine, we have $\Gamma(S, \mathcal{O}_S) = A$. Then, for any open set $U \subseteq X$ and any $n \geq 0$, we have $C_n(X/S)(U) = \mathcal{O}_X(U) \otimes_A \cdots \otimes_A \mathcal{O}_X(U)$ (tensor product taken $n + 1$ times). It follows that the sheafified complex $(\tilde{C}_n^*(X/S), b)$ (resp. $(\text{Tot}(\tilde{C}^{**}(X/S)), B, b)$) computing the Hochschild homology (resp. the cyclic homology) of $X$ is identical to the Hochschild complex (resp. cyclic complex) of $X$ considered in [1, §4]. It follows that we have natural isomorphisms

\[(2.5)\quad HH_q(X/S) \cong HH_q(X_A), \quad HC_q(X/S) \cong HC_q(X_A).\]

In particular, suppose that $X = \text{Spec}(B)$, i.e., $X$ corresponds to an affine scheme corresponding to an $A$-algebra $B$. It then follows (see [1, Theorem 4.1] and [7, Theorem 2.5]) that we have natural isomorphisms

\[(2.6)\quad HH_q(X_A) = \mathbb{H}^{-q}(\tilde{C}_n^*(X/S)) \cong HH_q(B/A), \quad HC_q(X_A) = \mathbb{H}^{-q}(\text{Tot}(\tilde{C}^{**}(X/S))) \cong HC_q(B/A).\]

Combining the isomorphisms in (2.5) and (2.6) yields (2.4). \[\blacksquare\]

Let $A$ and $B$ be two given $k$-algebras. For any $p \geq 0$ and given $(a_0, a_1, \ldots, a_p) \in C_p(A)$, a permutation $\sigma \in S_p$ acts on $(a_0, a_1, \ldots, a_p)$ as follows (see [5, 4.2.1.1]):

$$\sigma \cdot (a_0, a_1, \ldots, a_p) := (a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(p)}).$$

Then, if $p, q \geq 0$ are integers and $S_{p,q}$ denotes the set of $(p, q)$ shuffles, i.e., of all $\sigma \in S_{p+q}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \cdots < \sigma(p + q)$, we know that the shuffle product (see [5, 4.2.1.2])

$$\text{sh}_{p,q}((a_0, \ldots, a_p) \otimes (b_0, \ldots, b_q))$$

$$= \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) \sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \ldots, a_p \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_q)$$
induces a product $\sh_{p,q} : \HH_p(A) \otimes_k \HH_q(B) \to \HH_{p+q}(A \otimes B)$ on Hochschild homology groups. Our purpose is to extend this to the Hochschild homology groups of schemes with respect to the base scheme $S$.

**Theorem 2.3.** Suppose that $f : X \to S$ and $g : Y \to S$ are morphisms of schemes over a commutative ring $k$. Then there is a multiplication on Hochschild homology groups with respect to $S$:

\begin{equation}
\HH_q(X/S) \otimes \HH_r(Y/S) \to \HH_{q+r}(X \times_S Y/S) \quad \forall q, r \in \mathbb{Z}.
\end{equation}

**Proof.** We consider the fibre product $X \times_S Y$ along with the projections $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$. Then there is a multiplication on $\HH_{q+r}(X \times_S Y/S)$.

Further, we consider the complex $(C^*_h(X/S), b)$ (resp. $C^*_h(Y/S), b)$) of presheaves of abelian groups on $X$ (resp. $Y$). For any given $i \in I$, $m \in M_i$ and $n \in M_i$, we note that, for all $q, r \geq 0$, the inverse image presheaves can be described as

\begin{equation}
p_X^{-1}(C^{-q}_h(X/S))(X_{im} \times_{S_i} Y_{in}) = \mathcal{O}_X(X_{im})^\otimes q+1,
p_Y^{-1}(C^{-r}_h(Y/S))(X_{im} \times_{S_i} Y_{in}) = \mathcal{O}_Y(Y_{in})^\otimes r+1,
\end{equation}

where all the tensor products are taken over $\Gamma(S, \mathcal{O}_S)$. Then, for all $q, r \geq 0$, we have shuffle maps

\begin{equation}
\sh_{q,r} : \mathcal{O}_X(X_{im})^\otimes q+1 \otimes \Gamma(S, \mathcal{O}_S) \mathcal{O}_Y(Y_{in})^\otimes r+1
\to \mathcal{O}_X(X_{im}) \otimes \Gamma(S, \mathcal{O}_S) \mathcal{O}_Y(Y_{in})^\otimes q+r+1.
\end{equation}

We also consider the morphisms

\begin{equation}
(\mathcal{O}_X(X_{im}) \otimes \Gamma(S, \mathcal{O}_S) \mathcal{O}_Y(Y_{in}))^\otimes q+r+1
\to (\mathcal{O}_X(X_{im}) \otimes \mathcal{O}_S(S_i) \mathcal{O}_Y(Y_{in}))^\otimes q+r+1
\end{equation}

induced by the natural morphism $(\mathcal{O}_X(X_{im}) \otimes \Gamma(S, \mathcal{O}_S) \mathcal{O}_Y(Y_{in})) \to (\mathcal{O}_X(X_{im}) \otimes \mathcal{O}_S(S_i) \mathcal{O}_Y(Y_{in}))$ of algebras. Composing the morphisms in (2.9) and (2.10) and using (2.8), we have natural morphisms

\begin{equation}
p_X^{-1}(C^{-q}_h(X/S))(X_{im} \times_{S_i} Y_{in}) \otimes p_Y^{-1}(C^{-r}_h(Y/S))(X_{im} \times_{S_i} Y_{in})
\end{equation}

\begin{equation}
\downarrow
\end{equation}

\begin{equation}
p_X^{-1}(C^{-q}_h(X/S))(X_{im} \times_{S_i} Y_{in}) \otimes \Gamma(S, \mathcal{O}_S) p_Y^{-1}(C^{-r}_h(Y/S))(X_{im} \times_{S_i} Y_{in})
\end{equation}

\begin{equation}
\downarrow
\end{equation}

\begin{equation}
C^{-q-r}_h(X \times_S Y/S)(X_{im} \times_{S_i} Y_{in})
\end{equation}
Since the open sets \( \{X_{im} \times_{S_i} Y_{in}\}_{i \in I, m \in M_i, n \in N_i} \) form a basis of \( X \times_S Y \), it follows from the morphisms in (2.11) that we have a morphism of complexes of sheaves of abelian groups on \( X \times_S Y \):

\[
(2.12) \quad p_X^{-1}(\tilde{C}_h^*(X/S)) \otimes p_Y^{-1}(\tilde{C}_h^*(Y/S)) \rightarrow \tilde{C}_h^*(X \times_S Y/S).
\]

Then, for all \( q, r \in \mathbb{Z} \), the morphism in (2.12) induces a multiplication on hypercohomology:

\[
(2.13) \quad H^{-q}(p_X^{-1}(\tilde{C}_h^*(X/S))) \otimes H^{-r}(p_Y^{-1}(\tilde{C}_h^*(Y/S))) \rightarrow H^{-q-r}(\tilde{C}_h^*(X \times_S Y/S)).
\]

It follows from general properties of hypercohomology (see [3]) that we have natural morphisms \( H^{-q}(\tilde{C}_h^*(X/S)) \rightarrow H^{-q}(p_X^{-1}(\tilde{C}_h^*(X/S))) \) and \( H^{-r}(\tilde{C}_h^*(Y/S)) \rightarrow H^{-r}(p_Y^{-1}(\tilde{C}_h^*(Y/S))) \). Combining this with (2.13), we have a multiplication

\[
(2.14) \quad H^{-q}(\tilde{C}_h^*(X/S)) \otimes H^{-r}(\tilde{C}_h^*(Y/S)) \rightarrow H^{-q-r}(\tilde{C}_h^*(X \times_S Y/S)).
\]

The result of (2.7) now follows directly from the definition of Hochschild homology with respect to \( S \).

**Theorem 2.4.** (a) Let \( f : X \rightarrow S \) be a morphism of schemes over a commutative ring \( k \). Then \( \bigoplus_{r \in \mathbb{Z}} HH_r(X/S) \) carries the structure of a graded algebra.

(b) Let \( f : X \rightarrow S \) and \( t : S' \rightarrow S \) be morphisms of schemes over a commutative ring. Then, setting \( X_{S'} = X \times_S S' \), we have a product structure

\[
(2.15) \quad HH_q(X/S) \otimes HH_r(S'/S) \rightarrow HH_{q+r}(X_{S'/S'}) \quad \forall q, r \in \mathbb{Z}.
\]

**Proof.** (a) From Proposition 2.3 it follows that we have a multiplication

\[
(2.16) \quad HH_q(X/S) \otimes HH_r(X/S) \rightarrow HH_{q+r}(X \times_S X/S)
\]

for each \( q, r \in \mathbb{Z} \). Further, the diagonal map \( \Delta_{X/S} : X \rightarrow X \times_S X \) induces a morphism

\[
(2.17) \quad \Delta_{X/S}^* : HH_{q+r}(X \times_S X/S) \rightarrow HH_{q+r}(X/S).
\]

Composing the morphisms in (2.16) and (2.17), we have a product structure

\[
(2.18) \quad HH_q(X/S) \otimes HH_r(X/S) \rightarrow HH_{q+r}(X/S) \quad \forall q, r \in \mathbb{Z}
\]

that makes \( \bigoplus_{r \in \mathbb{Z}} HH_r(X/S) \) into a graded algebra.

(b) Using Proposition 2.3 it follows that we have a product (for all \( q, r \in \mathbb{Z} \))

\[
(2.19) \quad HH_q(X/S) \otimes HH_r(S'/S) \rightarrow HH_{q+r}(X_{S'/S}) = HH_{q+r}(X_{S'/S}).
\]

Further, we note that the natural morphisms

\[
(2.20) \quad C_{h}^{-q-r}(X_{S'}/S') = C_{X_{S'}}^{\otimes R(s, o_{S})q+r+1} \rightarrow C_{X_{S'}}^{\otimes R(s', o_{S'})q+r+1} = C_{h}^{-q-r}(X_{S'}/S').
\]
induce morphisms of complexes $\tilde{C}_h^*(X_{S'}/S) \to \tilde{C}_h^*(X_{S'}/S')$ and hence a morphism of hypercohomologies

$$\text{(2.21)} \quad \text{HH}_{q+r}(X_{S'}/S) = \mathbb{H}^{-q-r}(\tilde{C}_h^*(X_{S'}/S))$$

$$\quad \to \mathbb{H}^{-q-r}(\tilde{C}_h^*(X_{S'}/S')) = \text{HH}_{q+r}(X_{S'}/S').$$

Composing (2.19) and (2.21) yields (2.15). ■

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