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MAPS WITH DIMENSIONALLY RESTRICTED FIBERS

BY

VESKO VALOV (North Bay)

Abstract. We prove that if $f: X \to Y$ is a closed surjective map between metric spaces such that every fiber $f^{-1}(y)$ belongs to a class S of spaces, then there exists an F_σ-set $A \subset X$ such that $A \in \mathrm{S}$ and $\dim f^{-1}(y) \setminus A = 0$ for all $y \in Y$. Here, S can be one of the following classes: (i) $\{M : \text{e-dim}\, M \le K\}$ for some CW-complex K; (ii) C-spaces; (iii) weakly infinite-dimensional spaces. We also establish that if $\mathrm{S} = \{M : \dim M \le n\}$, then $\dim f \triangle g \le 0$ for almost all $g \in C(X, \mathbb{I}^{n+1})$.

1. Introduction. All spaces in the paper are assumed to be paracompact and all maps continuous. By $C(X, M)$ we denote all maps from X into M. Unless stated otherwise, all function spaces are endowed with the source limitation topology provided M is a metric space.

The paper is inspired by the results of Pasynkov [11], Toruńczyk [16], Sternfeld [15] and Levin [8]. Pasynkov announced in [11] and proved in [12] that if $f: X \to Y$ is a surjective map with $\dim f \le n$, where X and Y are finite-dimensional metric compacta, then $\dim f \triangle g \le 0$ for almost all maps $g \in C(X, \mathbb{I}^n)$ (see [10] for a non-compact version of this result). Toruńczyk [16] established (in a more general setting) that if f, X and Y are as in Pasynkov's theorem, then for each $0 \le k \le n-1$ there exists a σ-compact subset $A_k \subset X$ such that $\dim A_k \le k$ and $\dim f|(X \setminus A_k) \le n-k-1$.

Next results in this direction were established by Sternfeld and Levin. Sternfeld [15] proved that if in the above results Y is not necessarily finite-dimensional, then $\dim f \triangle g \le 1$ for almost all $g \in C(X, \mathbb{I}^n)$ and there exists a σ-compact subset $A \subset X$ such that $\dim A \le n-1$ and $\dim f|(X \setminus A) \le 1$. Levin [8] improved Sternfeld's results by showing that $\dim f \triangle g \le 0$ for almost all $g \in C(X, \mathbb{I}^{n+1})$, and showed that this is equivalent to the existence of an n-dimensional σ-compact subset $A \subset X$ with $\dim f|(X \setminus A) \le 0$.

The above results of Pasynkov and Toruńczyk were generalized in [18] to closed maps between metric spaces X and Y with Y being a C-space (recall that each finite-dimensional paracompact space is a C-space [6]).

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But the question whether the results of Pasynkov and Toruńczyk remain valid without the finite-dimensionality assumption on Y is still open.

In this paper we provide non-compact analogues of Levin's results for closed maps between metric spaces.

We say that a topological property of metrizable spaces is an S-*property* if the following conditions are satisfied:

(i) S is hereditary with respect to closed subsets;
(ii) if X is metrizable and $\{H_i\}_{i=1}^{\infty}$ is a sequence of closed S-subsets of X, then $\bigcup_{i=1}^{\infty} H_i \in \mathrm{S}$;
(iii) a metrizable space X belongs S provided there exists a closed surjective map $f\colon X \to Y$ such that Y is a 0-dimensional metrizable space and $f^{-1}(y) \in \mathrm{S}$ for all $y \in Y$;
(iv) any discrete union of S-spaces is an S-space.

Any map whose fibers have a given S-property is called an S-*map*.

Here are some examples of S-properties (we identify S with the class of spaces having the property S):

- $\mathrm{S} = \{X : \dim X \le n\}$ for some $n \ge 0$;
- $\mathrm{S} = \{X : \dim_G X \le n\}$, where G is an Abelian group and $\dim_G$ is the cohomological dimension;
- more generally, $\mathrm{S} = \{X : \text{e-dim}\, X \le K\}$, where K is a CW-complex and e-dim is the extension dimension (see [4], [5]);
- $\mathrm{S} = \{X : X \text{ is weakly infinite-dimensional}\}$;
- $\mathrm{S} = \{X : X \text{ is a } C\text{-space}\}$.

To show that the property e-dim $\le K$ satisfies condition (iii), we apply [3, Corollary 2.5]. For the case of weakly infinite-dimensional spaces and C-spaces this follows from [7].

The question whether (strong) countable-dimensionality is an S-property was raised in the first version of this paper. The referee kindly informed us that, according to [14, Remark 2.2] (see also the remark after [6, Corollary 5.4.6], as well as [6, Problem 6.2.D(b)]), there exists a map with strongly countable-dimensional fibers from a metric compactum X onto the Cantor set such that X is not countable-dimensional. Hence, (strong) countable-dimensionality is not an S-property.

THEOREM 1.1. *Let $f\colon X \to Y$ be a closed surjective* S-*map with X and Y being metrizable spaces. Then there exists an F_σ-subset $A \subset X$ such that $A \in \mathrm{S}$ and $\dim f^{-1}(y) \setminus A = 0$ for all $y \in Y$. Moreover, if f is a perfect map, the conclusion remains true provided* S *is a property satisfying conditions* (i)–(iii).

Theorem 1.1 was established by Levin [9, Theorem 1.2] in the case when X and Y are metric compacta and S is the property e-dim $\le K$ for a given

CW-complex K. Levin's proof remains valid for any S-property, but it does not work for non-compact spaces.

We say that a map $f\colon X \to Y$ has a *countable functional weight* (notation $W(f) \leq \aleph_0$, see [10]) if there exists a map $g\colon X \to \mathbb{I}^{\aleph_0}$ such that $f \triangle g\colon X \to Y \times \mathbb{I}^{\aleph_0}$ is an embedding. For example [12, Proposition 9.1], $W(f) \leq \aleph_0$ for any closed map $f\colon X \to Y$ such that X is a metrizable space and every fiber $f^{-1}(y)$, $y \in Y$, is separable.

Theorem 1.2. *Let X and Y be paracompact spaces and $f\colon X \to Y$ a closed surjective map with* $\dim f \leq n$ *and* $W(f) \leq \aleph_0$. *Then* $C(X, \mathbb{I}^{n+1})$ *equipped with the uniform convergence topology contains a dense subset of maps g such that* $\dim f \triangle g \leq 0$.

This theorem was established by Levin [8, Theorem 1.6] for metric compacta X and Y, but Levin's arguments do not work for non-compact spaces. We use Pasynkov's technique from [10] to reduce the proof of Theorem 1.2 to the case of X and Y being metric compacta.

Our last results concern the function spaces $C(X, \mathbb{I}^n)$ and $C(X, \mathbb{I}^{\aleph_0})$ equipped with the source limitation topology. Recall that this topology on $C(X, M)$ with M being a metrizable space can be described as follows: the neighborhood base at a given map $h \in C(X, M)$ consists of the sets $B_\rho(h, \epsilon) = \{g \in C(X, M) : \rho(g, h) < \epsilon\}$, where ρ is a fixed compatible metric on M and $\epsilon : X \to (0, 1]$ runs over continuous positive functions on X. The symbol $\rho(h, g) < \epsilon$ means that $\rho(h(x), g(x)) < \epsilon(x)$ for all $x \in X$. It is well known that for paracompact spaces X this topology does not depend on the metric ρ and it has the Baire property provided M is completely metrizable.

Theorem 1.3. *Let $f\colon X \to Y$ be a perfect surjection between paracompact spaces and* $W(f) \leq \aleph_0$.

- (i) *The maps $g \in C(X, \mathbb{I}^{\aleph_0})$ such that $f \triangle g$ embeds X into $Y \times \mathbb{I}^{\aleph_0}$ form a dense G_δ-set in $C(X, \mathbb{I}^{\aleph_0})$ with respect to the source limitation topology.*
- (ii) *If there exists a map $g \in C(X, \mathbb{I}^n)$ with* $\dim f \triangle g \leq 0$, *then all maps having this property form a dense G_δ-set in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology.*

Corollary 1.4. *Let $f\colon X \to Y$ be a perfect surjection with* $\dim f \leq n$ *and* $W(f) \leq \aleph_0$, *where X and Y are paracompact spaces. Then all maps $g \in C(X, \mathbb{I}^{n+1})$ with* $\dim f \triangle g \leq 0$ *form a dense G_δ-set in $C(X, \mathbb{I}^{n+1})$ with respect to the source limitation topology.*

Corollary 1.4 follows directly from Theorem 1.2 and Theorem 1.3(ii). Corollary 1.5 below follows from Corollary 1.4 and [2, Corollary 1.1] (see Section 3).

Corollary 1.5. *Let X, Y be paracompact spaces and $f\colon X \to Y$ a perfect surjection with $\dim f \le n$ and $W(f) \le \aleph_0$. Then for every metrizable ANR-space M the maps $g \in C(X, \mathbb{I}^{n+1} \times M)$ such that $\dim g(f^{-1}(y)) \le n+1$ for all $y \in Y$ form a dense G_δ-set E in $C(X, \mathbb{I}^{n+1} \times M)$ with respect to the source limitation topology.*

2. S-properties and maps into finite-dimensional cubes. This section contains the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We follow the proof of [19, Proposition 4.1]. Let us first reduce the proof to the case where f is a perfect map. Indeed, according to Vainstein's lemma, the boundary $\operatorname{Fr} f^{-1}(y)$ of every fiber $f^{-1}(y)$ is compact. Defining $F(y)$ to be $\operatorname{Fr} f^{-1}(y)$ if $\operatorname{Fr} f^{-1}(y) \ne \emptyset$, and an arbitrary point from $f^{-1}(y)$ otherwise, we obtain a set $X_0 = \bigcup\{F(y) : y \in Y\}$ such that $X_0 \subset X$ is closed and the restriction $f|X_0$ is a perfect map. Moreover, each $f^{-1}(y) \setminus X_0$ is open in X and has property S (as an F_σ-subset of the S-space $f^{-1}(y)$). Hence, $X \setminus X_0$, being the union of the discrete family $\{f^{-1}(y) \setminus X_0 : y \in Y\}$ of S-sets, is an S-set. At the same time $X \setminus X_0$ is open in X. Consequently, $X \setminus X_0$ is the union of countably many closed sets $X_i \subset X$, $i = 1, 2, \ldots$. Obviously, each X_i, $i \ge 1$, also has property S. Therefore, it suffices to prove Theorem 1.1 for the S-map $f|X_0\colon X_0 \to Y$.

So, we may suppose that f is perfect. According to [10], there exists a map $g\colon X \to \mathbb{I}^{\aleph_0}$ such that g embeds every fiber $f^{-1}(y)$, $y \in Y$. Let $g = \triangle_{i=1}^{\infty} g_i$ and $h_i = f \triangle g_i\colon X \to Y \times \mathbb{I}$, $i \ge 1$. Moreover, we choose countably many closed intervals $\mathbb{I}_j$ such that every open subset of $\mathbb{I}$ contains some $\mathbb{I}_j$. By [18, Lemma 4.1], for every j there exists a 0-dimensional F_σ-set $C_j \subset Y \times \mathbb{I}_j$ such that $C_j \cap (\{y\} \times \mathbb{I}_j) \ne \emptyset$ for every $y \in Y$. Now, consider the sets $A_{ij} = h_i^{-1}(C_j)$ for all $i, j \ge 1$ and let A be their union. Since f is an S-map, so is the map h_i for any i. Hence, A_{ij} has property S for all i, j. This implies that so does A.

It remains to show that $\dim f^{-1}(y) \setminus A \le 0$ for every $y \in Y$. Let $\dim f^{-1}(y_0) \setminus A > 0$ for some y_0. Since $g|f^{-1}(y_0)$ is an embedding, there exists an integer i such that $\dim g_i(f^{-1}(y_0) \setminus A) > 0$. Then $g_i(f^{-1}(y_0) \setminus A)$ has a non-empty interior in $\mathbb{I}$. So, $g_i(f^{-1}(y_0) \setminus A)$ contains some $\mathbb{I}_j$. Choose $t_0 \in \mathbb{I}_j$ with $c_0 = (y_0, t_0) \in C_j$. Then there exists $x_0 \in f^{-1}(y_0) \setminus A$ such that $g_i(x_0) = t_0$. On the other hand, $x_0 \in h_i^{-1}(c_0) \subset A_{ij} \subset A$, a contradiction. ■

Proof of Theorem 1.2. We first prove the next proposition, which is a small modification of [10, Theorem 8.1]. For any map $f\colon X \to Y$ we consider the set $C(X, Y \times \mathbb{I}^{n+1}, f)$ consisting of all maps $g\colon X \to Y \times \mathbb{I}^{n+1}$ such that $f = \pi_n \circ g$, where $\pi_n\colon Y \times \mathbb{I}^{n+1} \to Y$ is the projection onto Y. We also consider the other projection $\varpi_n\colon Y \times \mathbb{I}^{n+1} \to \mathbb{I}^{n+1}$. It is easily seen that the formula $g \mapsto \varpi_n \circ g$ provides one-to-one correspondence between

$C(X, Y\times\mathbb{I}^{n+1}, f)$ and $C(X,\mathbb{I}^{n+1})$. So, we may assume that $C(X, Y\times\mathbb{I}^{n+1}, f)$ is a metric space isometric with $C(X,\mathbb{I}^{n+1})$, where $C(X,\mathbb{I}^{n+1})$ is equipped with the supremum metric.

Proposition 2.1. *Let $f\colon X \to Y$ be an n-dimensional surjective map between compact spaces with $n > 0$ and $\lambda\colon X \to Z$ a map into a metric compactum Z. Then the maps $g \in C(X, Y\times\mathbb{I}^{n+1}, f)$ satisfying the condition below form a dense subset of $C(X, Y\times\mathbb{I}^{n+1}, f)$: there exists a compact space H and maps $\varphi\colon X \to H$, $h\colon H \to Y\times\mathbb{I}^{n+1}$ and $\mu\colon H \to Z$ such that $\lambda = \mu\circ\varphi$, $g = h\circ\varphi$, $W(h) \le \aleph_0$ and* $\dim h = 0$.

Proof. We fix a map $g_0 \in C(X, Y\times\mathbb{I}^{n+1}, f)$ and $\epsilon > 0$. Let $g_1 = \varpi_n\circ g_0$. Then $\lambda \triangle g_1 \in C(X, Z\times\mathbb{I}^{n+1})$. Consider also the constant maps $f'\colon Z\times\mathbb{I}^{n+1} \to \mathrm{Pt}$ and $\eta\colon Y \to \mathrm{Pt}$, where Pt is the one-point space. So, we have $\eta\circ f = f'\circ(\lambda\triangle g_1)$. According to Pasynkov's factorization theorem [13, Theorem 13], there exist metrizable compacta K, T and maps $f^*\colon K \to T$, $\xi_1\colon X \to K$, $\xi_2\colon K \to Z\times\mathbb{I}^{n+1}$ and $\eta^*\colon Y \to T$ such that:

- $\eta^*\circ f = f^*\circ\xi_1$;
- $\xi_2\circ\xi_1 = \lambda\triangle g_1$;
- $\dim f^* \le \dim f \le n$.

If $p\colon Z\times\mathbb{I}^{n+1} \to Z$ and $q\colon Z\times\mathbb{I}^{n+1} \to \mathbb{I}^{n+1}$ denote the corresponding projections, we have

$$p\circ\xi_2\circ\xi_1 = \lambda \quad\text{and}\quad q\circ\xi_2\circ\xi_1 = g_1.$$

Since $\dim f^* \le n$, by Levin's result [8, Theorem 1.6], there exists a map $\phi\colon K \to \mathbb{I}^{n+1}$ such that ϕ is ϵ-close to $q\circ\xi_2$ and $\dim f^*\triangle\phi \le 0$. Then the map $\phi\circ\xi_1$ is ϵ-close to g_1, so $g = f\triangle(\phi\circ\xi_1)$ is ϵ-close to g_0. Denote $\varphi = f\triangle\xi_1$, $H = \varphi(X)$ and $h = (\mathrm{id}_Y\times\phi)|H$. If $\varpi_H\colon H \to K$ is the restriction of the projection $Y\times K \to K$ on H, we have

$$\lambda = p\circ\xi_2\circ\xi_1 = p\circ\xi_2\circ\varpi_H\circ\varphi,\ \text{so}\ \lambda = \mu\circ\varphi,\ \text{where}\ \mu = p\circ\xi_2\circ\varpi_H.$$

Moreover, $g = f\triangle(\phi\circ\xi_1) = (\mathrm{id}_Y\times\phi)\circ(f\triangle\xi_1) = h\circ\varphi$. Since K is a metrizable compactum, $W(\phi) \le \aleph_0$. Hence, $W(h) \le \aleph_0$.

To show that $\dim h \le 0$, it suffices to prove that $\dim h \le \dim f^*\triangle\phi$. To this end, we show that any fiber $h^{-1}((y,v))$, where $(y,v) \in Y\times\mathbb{I}^{n+1}$, is homeomorphic to a subset of the fiber $(f^*\triangle\phi)^{-1}((\eta^*(y),v))$. Indeed, let π_Y be the restriction of the projection $Y\times K \to Y$ on the set H. Since $\eta^*\circ f = f^*\circ\xi_1$, H is a subset of the pullback of Y and K with respect to the maps η^* and f^*. Therefore, ϖ_H embeds every fiber $\pi_Y^{-1}(y)$ into $(f^*)^{-1}(y)$, $y \in Y$. Let $a_i = (y_i, k_i) \in H \subset Y\times K$, $i = 1,2$, be such that $h(a_1) = h(a_2)$. Then $(y_1, \phi(k_1)) = (y_2, \phi(k_2))$, so $y_1 = y_2 = y$ and $\phi(k_1) = \phi(k_2) = v$. This implies $\varpi_H(a_i) = k_i \in (f^*)^{-1}(\eta^*(\pi_Y(a_i))) = (f^*)^{-1}(\eta^*(y))$, $i = 1,2$.

Hence, ϖ_H embeds the fiber $h^{-1}((y,v))$ into the fiber $(f^*\triangle\phi)^{-1}((\eta^*(y),v))$. Consequently, $\dim h \le \dim f^*\triangle\phi = 0$. ■

We can now finish the proof of Theorem 1.2. It suffices to show that every map from $C(X, Y\times\mathbb{I}^{n+1}, f)$ can be approximated by maps $g \in C(X, Y\times\mathbb{I}^{n+1}, f)$ with $\dim g \le 0$. We fix $g_0 \in C(X, Y\times\mathbb{I}^{n+1}, f)$ and $\epsilon > 0$. Since $W(f) \le \aleph_0$, there exists a map $\lambda\colon X \to \mathbb{I}^{\aleph_0}$ such that $f\triangle\lambda$ is an embedding. Let $\beta f\colon \beta X \to \beta Y$ be the Čech–Stone extension of the map f. Then $\dim \beta f \le n$ (see [13, Theorem 15]). Consider also the maps $\beta\lambda\colon \beta X \to \mathbb{I}^{\aleph_0}$ and $\bar g_0 = \beta f\triangle\beta g_1$, where $g_1 = \varpi_n\circ g_0$. According to Proposition 2.1, there exists a map $\bar g \in C(\beta X, \beta Y\times\mathbb{I}^{n+1}, \beta f)$ which is ϵ-close to $\bar g_0$ and satisfies the following condition: there exists a compact space H and maps $\varphi\colon \beta X\to H$, $h\colon H\to \beta Y\times\mathbb{I}^{n+1}$ and $\mu\colon H\to\mathbb{I}^{\aleph_0}$ such that $\beta\lambda = \mu\circ\varphi$, $\bar g = h\circ\varphi$, $W(h)\le\aleph_0$ and $\dim h = 0$. We have the equalities

$$\begin{aligned}\beta f\triangle\beta\lambda &= (\pi_n\circ\bar g)\triangle(\mu\circ\varphi) = (\pi_n\circ h\circ\varphi)\triangle(\mu\circ\varphi)\\ &= ((\pi_n\circ h)\triangle\mu)\circ\varphi,\end{aligned}$$

where π_n denotes the projection $\beta Y\times\mathbb{I}^{n+1}\to\beta Y$. This implies that φ embeds X into H because $f\triangle\lambda$ embeds X into $Y\times\mathbb{I}^{\aleph_0}$. Let g be the restriction of $\bar g$ over X. Identifying X with $\varphi(X)$, we find that h is an extension of g. Hence, $\dim g\le\dim h = 0$. Observe also that g is ϵ-close to g_0, which completes the proof. ■

3. Proof of Theorem 1.3 and Corollary 1.5

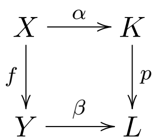
Proof of Theorem 1.3(ii). Since $W(f)\le\aleph_0$, there is a map $\lambda\colon X\to\mathbb{I}^{\aleph_0}$ such that $f\triangle\lambda$ embeds X into $Y\times\mathbb{I}^{\aleph_0}$. Choose a sequence $\{\gamma_k\}_{k\ge1}$ of open covers of $\mathbb{I}^{\aleph_0}$ with $\mathrm{mesh}(\gamma_k)\le 1/k$, and let $\omega_k = \lambda^{-1}(\gamma_k)$ for all k. We denote by $C_{(\omega_k,0)}(X,\mathbb{I}^n,f)$ the set of all maps $g\in C(X,\mathbb{I}^n)$ with the following property: every $z\in(f\triangle g)(X)$ has a neighborhood V_z in $Y\times\mathbb{I}^n$ such that $(f\triangle g)^{-1}(V_z)$ can be represented as the union of a disjoint open (in X) family refining the cover ω_k. According to [18, Lemma 2.5], each of the sets $C_{(\omega_k,0)}(X,\mathbb{I}^n,f)$, $k\ge1$, is open in $C(X,\mathbb{I}^n)$ with respect to the source limitation topology. It follows from the definition of the covers ω_k that $\bigcap_{k\ge1} C_{(\omega_k,0)}(X,\mathbb{I}^n,f)$ consists of maps g with $\dim f\triangle g\le 0$. Since $C(X,\mathbb{I}^n)$ with the source limitation topology has the Baire property, it remains to show that any $C_{(\omega_k,0)}(X,\mathbb{I}^n,f)$ is dense in $C(X,\mathbb{I}^n)$.

To this end, we need the following result established in our forthcoming book [1] with T. Banakh: Suppose $h_0\colon Z\to E$ is a map from a Tikhonov space Z into an ANR-space E and $O(h_0)$ is a neighborhood of h_0 in $C(Z,E)$ equipped with the source limitation topology. Then there exists an open cover γ of Z such that for any γ-map $h_1\colon Z\to P$ into a paracompact space P (i.e., $h_0^{-1}(\omega)$ refines γ for some open cover ω of P) there exists a map

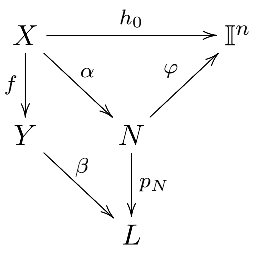
$h_2 \colon G \to E$ with $h_2 \circ h_1 \in O(h_0)$, where G is an open neighborhood of the closure of $h(Z)$ in P.

We apply the above result for a fixed cover ω_m, a map $g_0 \in C(X, \mathbb{I}^n)$ and a neighborhood $B_\rho(g_0, \epsilon)$ of g_0 in $C(X, \mathbb{I}^n)$, where $\epsilon \colon X \to (0,1]$ is a continuous function and ρ is the Euclidean metric on $\mathbb{I}^n$. More precisely, we are going to find $h \in C_{(\omega_m,0)}(X, \mathbb{I}^n, f)$ such that $\rho(g_0(x), h(x)) < \epsilon(x)$ for all $x \in X$. According to the result formulated above, there exists an open cover $\mathcal{U}$ of X satisfying the following condition: if $\alpha \colon X \to K$ is a $\mathcal{U}$-map into a paracompact space K, then there exists a map $q \colon G \to \mathbb{I}^n$, where G is an open neighborhood of $\overline{\alpha(X)}$ in K, such that g_0 and $q \circ \alpha$ are $\epsilon/2$-close with respect to the metric ρ. Let $\mathcal{U}_1$ be an open cover of X refining both $\mathcal{U}$ and ω_m such that $\inf\{\epsilon(x) : x \in U\} > 0$ for all $U \in \mathcal{U}_1$.

Since $\dim f \triangle g \leq 0$ for some $g \in C(X, \mathbb{I}^n)$, according to [1, Theorem 6] there exists an open cover $\mathcal{V}$ of Y such that for any $\mathcal{V}$-map $\beta \colon Y \to L$ into a simplicial complex L we can find a $\mathcal{U}_1$-map $\alpha \colon X \to K$ into a simplicial complex K and a perfect PL-map $p \colon K \to L$ with $\beta \circ f = p \circ \alpha$ and $\dim p \leq n$. We can assume that $\mathcal{V}$ is locally finite. Take L to be the nerve of the cover $\mathcal{V}$ and $\beta \colon Y \to L$ the corresponding natural map. Then there exist a simplicial complex K and maps p and α satisfying the above conditions. Hence, the following diagram is commutative:



Since K is paracompact, the choice of the cover $\mathcal{U}$ guarantees the existence of a map $\varphi \colon G \to \mathbb{I}^n$, where $G \subset K$ is an open neighborhood of $\overline{\alpha(X)}$, such that g_0 and $h_0 = \varphi \circ \alpha$ are $\epsilon/2$-close with respect to ρ. Replacing the triangulation of K by a suitable subdivision, we may additionally assume that no simplex of K meets both $\overline{\alpha(X)}$ and $K \setminus G$. So, the union N of all simplexes $\sigma \in K$ with $\sigma \cap \overline{\alpha(X)} \neq \emptyset$ is a subcomplex of K and $N \subset G$. Moreover, since N is closed in K, $p_N = p|N \colon N \to L$ is a perfect map. Therefore, we have the following commutative diagram:



Since α is a $\mathcal{U}_1$-map and $\inf\{\epsilon(x) : x \in U\} > 0$ for all $U \in \mathcal{U}_1$, we can construct a continuous function $\epsilon_1 : N \to (0,1]$ and an open cover γ of N

such that $\epsilon_1 \circ \alpha \le \epsilon$ and $\alpha^{-1}(\gamma)$ refines $\mathcal{U}_1$. Since $\dim p_N \le \dim p \le n$ and L, being a simplicial complex, is a C-space, we can apply [18, Theorem 2.2] to find a map $\varphi_1 \in C_{(\gamma,0)}(N, \mathbb{I}^n, p_N)$ which is $\epsilon_1/2$-close to φ. Let $h = \varphi_1 \circ \alpha$. Then h and h_0 are $\epsilon/2$-close because $\epsilon_1 \circ \alpha \le \epsilon$. On the other hand, h_0 is $\epsilon/2$-close to g_0. Hence, g_0 and h are ϵ-close.

It remains to show that $h \in C_{(\omega_m,0)}(X, \mathbb{I}^n, f)$. To this end, fix a point $z = (f(x), h(x)) \in (f \triangle h)(X) \subset Y \times \mathbb{I}^n$ and let $y = f(x)$. Then $w = (p_N \triangle \varphi_1)(\alpha(x)) = (\beta(y), h(x))$. Since $\varphi_1 \in C_{(\gamma,0)}(N, \mathbb{I}^n, p_N)$, there exists a neighborhood V_w of w in $L \times \mathbb{I}^n$ such that $W = (p_N \triangle \varphi_1)^{-1}(V_w)$ is the union of a disjoint open family in N refining γ. We can assume that $V_w = V_{\beta(y)} \times V_{h(x)}$, where $V_{\beta(y)}$ and $V_{h(x)}$ are neighborhoods of $\beta(y)$ and $h(x)$ in Y and $\mathbb{I}^n$, respectively. Consequently, $(f \triangle h)^{-1}(\Gamma) = \alpha^{-1}(W)$, where $\Gamma = \beta^{-1}(V_{\beta(y)}) \times V_{h(x)}$. Finally, observe that $\alpha^{-1}(W)$ is the disjoint union of an open (in X) family refining ω_m. Therefore, $h \in C_{(\omega_m,0)}(X, \mathbb{I}^n, f)$. ∎

Proof of Theorem 1.3(i). Let λ and ω_k be as in the proof of Theorem 1.3(ii). Denote by $C_{\omega_k}(X, \mathbb{I}^{\aleph_0}, f)$ the set of all $g \in C(X, \mathbb{I}^{\aleph_0})$ such that $f \triangle g$ is an ω_k-map. It can be shown that every $C_{\omega_k}(X, \mathbb{I}^{\aleph_0}, f)$ is open in $C(X, \mathbb{I}^{\aleph_0})$ with the source limitation topology (see [17, Proposition 3.1]). Moreover, $\bigcap_{k\ge 1} C_{\omega_k}(X, \mathbb{I}^{\aleph_0}, f)$ consists of maps g with $f \triangle g$ embedding X into $Y \times \mathbb{I}^{\aleph_0}$. So, we need to show that each $C_{\omega_k}(X, \mathbb{I}^{\aleph_0}, f)$ is dense in $C(X, \mathbb{I}^{\aleph_0})$ equipped with the source limitation topology.

To prove this, we follow the notation and arguments from the proof of Theorem 1.3(ii) (that $C_{(\omega_k,0)}(X, \mathbb{I}^n, f)$ are dense in $C(X, \mathbb{I}^n)$) by considering $\mathbb{I}^{\aleph_0}$ instead of $\mathbb{I}^n$. We fix a cover ω_m, a map $g_0 \in C(X, \mathbb{I}^{\aleph_0})$ and a function $\epsilon \in C(X, (0,1])$. Since $W(f) \le \aleph_0$, we can apply Theorem 6 from [1] to find an open cover $\mathcal{V}$ of Y such that for any $\mathcal{V}$-map $\beta\colon Y \to L$ into a simplicial complex L there exists a $\mathcal{U}_1$-map $\alpha\colon X \to K$ into a simplicial complex K and a perfect PL-map $p\colon K \to L$ with $\beta \circ f = p \circ \alpha$. Proceeding as before, we find a map $h = \varphi_1 \circ \alpha$ which is ϵ-close to g_0, where $\varphi_1 \in C_\gamma(N, \mathbb{I}^{\aleph_0}, p_N)$. It is easily seen that $\varphi_1 \in C_\gamma(N, \mathbb{I}^{\aleph_0}, p_N)$ implies $h \in C_{\omega_m}(X, \mathbb{I}^{\aleph_0}, f)$. So, $C_{\omega_m}(X, \mathbb{I}^{\aleph_0}, f)$ is dense in $C(X, \mathbb{I}^{\aleph_0})$. ∎

Proof of Corollary 1.5. It follows from [2, Proposition 2.1] that the set E is G_δ in $C(X, \mathbb{I}^{n+1} \times M)$. So, we need to show it is dense in $C(X, \mathbb{I}^{n+1} \times M)$. To this end, we fix $g^0 = (g_1^0, g_2^0) \in C(X, \mathbb{I}^{n+1} \times M)$ with $g_1^0 \in C(X, \mathbb{I}^{n+1})$ and $g_2^0 \in C(X, M)$. Since, by Corollary 1.4, the set

$$G_1 = \{g_1 \in C(X, \mathbb{I}^{n+1}) : \dim f \triangle g_1 \le 0\}$$

is dense in $C(X, \mathbb{I}^{n+1})$, we may approximate g_1^0 by an $h_1 \in G_1$. Then, by [2, Corollary 1.1], the maps $g_2 \in C(X, M)$ with $\dim g_2((f \triangle h_1)^{-1}(z)) = 0$ for all $z \in Y \times \mathbb{I}^{n+1}$ form a dense subset G_2 of $C(X, M)$. So, we can approximate g_2^0 by a map $h_2 \in G_2$. Let us show that $h = (h_1, h_2) \in C(X, \mathbb{I}^{n+1}) \times M$

belongs to E. We define the map $\pi_h\colon (f \triangle h)(X) \to (f \triangle h_1)(X)$ by setting $\pi_h(f(x), h_1(x), h_2(x)) = (f(x), h_1(x))$, $x \in X$. Because f is perfect, so is π_h. Moreover,

$$(\pi_h)^{-1}(f(x), h_1(x)) = h_2(f^{-1}(f(x)) \cap h_1^{-1}(h_1(x))), \qquad x \in X.$$

So, every fiber of π_h is 0-dimensional. We also observe that $\pi_h(h(f^{-1}(y))) = (f \triangle h_1)(f^{-1}(y))$ and the restriction $\pi_h|h(f^{-1}(y))$ is a perfect surjection between the compact spaces $h(f^{-1}(y))$ and $(f \triangle h_1)(f^{-1}(y))$ for any $y \in Y$. Since $(f \triangle h_1)(f^{-1}(y)) \subset \{y\} \times \mathbb{I}^{n+1}$, we have $\dim(f \triangle h_1)(f^{-1}(y)) \leq n+1$, $y \in Y$. Consequently, applying Hurewicz's dimension-lowering theorem [6] for the map $\pi_h|h(f^{-1}(y))$, we have $\dim h(f^{-1}(y)) \leq n+1$. Therefore, $h \in E$, which completes the proof. ∎

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Vesko Valov
Department of Computer Science and Mathematics
Nipissing University
100 College Drive
P.O. Box 5002
North Bay, ON, P1B 8L7, Canada
E-mail: veskov@nipissingu.ca

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