COLLOQUIUM MATHEMATICUM

VOL. 123 2011 NO. 2

ON TWO TAME ALGEBRAS WITH SUPER-DECOMPOSABLE PURE-INJECTIVE MODULES

BY

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Abstract. Let k be a field of characteristic different from 2. We consider two important tame non-polynomial growth algebras: the incidence k-algebra of the garland $\mathcal{G}_3$ of length 3 and the incidence k-algebra of the enlargement of the Nazarova–Zavadskij poset $\mathcal{NZ}$ by a greatest element. We show that if Λ is one of these algebras, then there exists a special family of pointed Λ-modules, called an independent pair of dense chains of pointed modules. Hence, by a result of Ziegler, Λ admits a super-decomposable pure-injective module if k is a countable field.

1. Introduction. Let R be a ring with a unit. By a *module* we always mean a left unital module. An R-module is called *super-decomposable* if it has no indecomposable direct summand. For the concept of *pure-injectivity* we refer to [10] (see also [8] and [9, Chapter 7]).

The problem of the existence of super-decomposable pure-injective modules over finite-dimensional algebras is stated in [30, Chapter 3] and studied in particular in [18], [19]. In [17, Chapter 13] Prest considers the problem in connection with representation types and he proves that such modules exist for wild algebras [17, Theorem 13.7]. Since then it has turned out that there also exist tame algebras possessing such modules.

In [18] Puninski has proved that super-decomposable pure-injective modules exist over any non-polynomial growth string algebra over a countable field. This is obtained by applying a remarkable theorem of Ziegler which states that if the ring R is countable, then non-existence of width of the lattice of all pp-formulae over R is equivalent to R possessing a super-decomposable pure-injective module (see [30]).

There is a recent result of Harland [7] asserting that super-decomposable pure-injective modules exist over tubular algebras.

Our work extends the class of tame algebras that are known to possess super-decomposable pure-injective modules. Actually, we prove the existence of such modules for important examples of non-polynomial growth algebras

2010 *Mathematics Subject Classification*: Primary 16G20; Secondary 16G60, 03C60.

Key words and phrases: super-decomposable pure-injective module, non-polynomial growth algebras, Galois covering, Nazarova–Zavadskij poset.

DOI: 10.4064/cm123-2-9

by applying a criterion for non-existence of width of the lattice of all pp-formulae expressed in terms of some special family of pointed modules (see [20, Proposition 5.4]), called an *independent pair of dense chains of pointed modules* (Definition 3.2).

The algebras considered in this paper are, in a sense, minimal in certain classes of tame non-polynomial growth algebras (see [25]). Our main result, Theorem 1.1, is a basic step in proving the existence of super-decomposable pure-injective modules for strongly simply connected algebras of non-polynomial growth (see [27] for the definitions).

Our approach is purely module-theoretical. We do not explicitly use any model-theoretical concepts, like pp-formulae.

Let k be a field, $\mathcal{G} = \mathcal{G}_3$ be the quiver

$$\mathcal{G} = \begin{tikzcd} \bullet \arrow[d] \arrow[dr] & \bullet \arrow[d] \arrow[dl] \\ \bullet \arrow[d] \arrow[dr] & \bullet \arrow[d] \arrow[dl] \\ \bullet & \bullet \end{tikzcd}$$

and $\mathcal{I}$ be the ideal of the path algebra $k\mathcal{G}$ generated by all commutativity relations (see [25, Section 5]). Following Simson (see e.g. [24, Definition 15.29]) we call the poset represented by $\mathcal{G}$ a *garland* of length 3. The algebra $k\mathcal{G}/\mathcal{I}$ is isomorphic to the incidence k-algebra of this poset.

One of our main results is the following theorem proved in Section 6.

THEOREM 1.1. *Assume that k is a field of characteristic different from* 2. *Let Λ be the bound quiver algebra $k\mathcal{G}/\mathcal{I}$.*

(a) *There exists an independent pair of dense chains of pointed Λ-modules in the sense of Definition* 3.2.
(b) *If k is countable, then there exists a super-decomposable pure-injective Λ-module.*

We remark that we do not assume that the field k is algebraically closed. However, if it is, then Λ in Theorem 1.1 is of tame representation type but not of polynomial growth (see [24, Chapter 14] and [26, Chapter XIX] for the concept of representation type and growth of algebras).

The paper is organized as follows. In Section 2 we introduce basic concepts related to algebras and k-categories associated to bound quivers. Section 3 is devoted to recalling from [20] the concept of an independent pair of dense chains of pointed modules (Definition 3.2). In Section 4 we collect necessary facts about Galois coverings of k-categories in the restricted context we need. For more information on that subject the reader is referred to [2] and [5]. In Section 5 we construct some special independent pairs of dense chains of pointed modules over a certain string algebra and in the following section we prove that an independent pair of dense chains of pointed modules exists over $k\mathcal{G}/\mathcal{I}$. We finish the proof of Theorem 1.1 in Section 6. Finally, in Section 7, we collect some corollaries important for applications. To give a flavor of such application, we show in Corollary 7.2 that there exists a super-decomposable pure-injective module over the incidence algebra of the Nazarova–Zavadskij poset, another classical example of a non-polynomial growth algebra (see [12], [13], [24, Chapter 15]), provided k is a countable field of characteristic different from 2.

2. Algebras and k-categories associated to bound quivers. Let $Q = (Q_0, Q_1)$ be a finite quiver with the set Q_0 of vertices and the set Q_1 of arrows. Given an arrow $\alpha \in Q_1$ with the starting point $s(\alpha)$ and the terminal point $t(\alpha)$ we denote by α^{-1} or α^{-} its *formal inverse* and set $s(\alpha^{-1}) = t(\alpha)$, $t(\alpha^{-1}) = s(\alpha)$ and $(\alpha^{-1})^{-1} = \alpha$. The set of the formal inverses of all arrows in Q_1 is denoted by Q_1^{-1}; the elements of Q_1 are called *direct arrows* whereas those of Q_1^{-1} are *inverse arrows*.

By a *walk* from x to y of length $n \geq 1$ in Q we mean a sequence $c_1 \dots c_n$ in $Q_1 \cup Q_1^{-1}$ such that $s(c_n) = x \in Q_0$, $t(c_1) = y \in Q_0$, $s(c_i) = t(c_{i+1})$ and $c_i^{-1} \neq c_{i+1}$ for all $1 \leq i < n$; we agree that $(c_1 \dots c_n)^{-1} = c_n^{-1} \dots c_1^{-1}$. A walk $c_1 \dots c_n$ is called a *path* provided $c_i \in Q_1$ for $1 \leq i \leq n$. Furthermore, to each vertex $x \in Q_0$ we associate the *stationary path* e_x of length 0, with $s(e_x) = t(e_x) = x$.

Given a field k we associate to a quiver Q the *path k-category* $\underline{kQ}$, whose objects are the vertices of Q and the space of morphisms from x to y has a basis consisting of the paths from x to y. The composition is defined by concatenation of paths. If Q has no oriented cycles then $\underline{kQ}$ is a locally bounded k-category in the sense of [5, 1.1].

Let $\langle Q_1 \rangle$ be the two-sided ideal in $\underline{kQ}$ generated by the arrows of Q. A two-sided ideal I in $\underline{kQ}$ is *admissible* if

$$\langle Q_1 \rangle^n \subseteq I \subseteq \langle Q_1 \rangle^2$$

for some natural number n. Then the pair (Q, I) is called a *bound quiver* and the factor k-category $\underline{kQ}/I$ is the *bound quiver k-category* associated to (Q, I).

The *path algebra* over a field k associated to Q is the algebra kQ with the set of all paths in Q as a k-basis and with multiplication induced by concatenation of paths, i.e.

$$(\alpha_1 \dots \alpha_n) \cdot (\beta_1 \dots \beta_m) = \begin{cases} \alpha_1 \dots \alpha_n \beta_1 \dots \beta_m & \text{if } s(\alpha_n) = t(\beta_1), \\ 0 & \text{otherwise.} \end{cases}$$

Again, a two-sided ideal I of kQ is called *admissible* if $\langle Q_1 \rangle^n \subseteq I \subseteq \langle Q_1 \rangle^2$

for some natural number $n \geq 2$. The pair (Q, I) is also called a bound quiver and the associated *bound quiver algebra* is the quotient of kQ modulo I.

Given a k-algebra A, the categories of left A-modules and finite-dimensional left A-modules are denoted by A-Mod and A-mod, respectively.

By a (left) *module* over a k-category $\underline{A} = \underline{kQ}/I$ we mean a k-linear functor

$$M : \underline{A} \to k\text{-Mod},$$

and M is *finite-dimensional* if $M(x)$ is finite-dimensional for any $x \in Q_0$. Note that when Q is finite, finite-dimensional means the same as locally finite-dimensional [4]. The categories of left $\underline{A}$-modules and of finite-dimensional left $\underline{A}$-modules are denoted by $\underline{A}$-Mod and $\underline{A}$-mod, respectively. Clearly, modules in $\underline{A}$-Mod are "almost the same" as k-representations of the bound quiver (Q, I) (see [1, Chapter III]).

There is an equivalence of k-categories

$$\underline{A}\text{-Mod} \to A\text{-Mod},$$

where $\underline{A} = \underline{kQ}/I$, $A = kQ/I$ and the ideal in the k-category is denoted by the same letter I as the corresponding ideal in the k-algebra (formally, the latter is generated by the former). The above equivalence restricts to an equivalence of the categories of finite-dimensional modules.

From now on we identify the categories A-Mod and $\underline{A}$-Mod. We remark that the indecomposable projective A-module Ae_x associated to the vertex x is identified with the the functor $\underline{A}(-, x)$, and A, as an A-module, is identified with $\bigoplus_{x \in Q_0} \underline{A}(-, x)$.

Let $\underline{A} = \underline{kQ}/I$ and $\underline{B} = \underline{kQ'}/I'$ be bound quiver k-categories, where Q and Q' are finite quivers. Given a k-category homomorphism $\eta : \underline{A} \to \underline{B}$ we denote by

$$\eta_\bullet : \underline{B}\text{-mod} \to \underline{A}\text{-mod}$$

the induced functor $(-) \circ \eta$.

Following [28] we call a bound quiver (Q, I) and the corresponding bound quiver k-algebra *special biserial* if:

- any vertex of Q is the starting point of at most two arrows and the terminal point of at most two arrows,
- given an arrow β there is at most one arrow α with $s(\beta) = t(\alpha)$ and $\beta\alpha \notin I$ and at most one arrow γ with $s(\gamma) = t(\beta)$ and $\gamma\beta \notin I$.

A *string algebra* is a special biserial algebra kQ/I such that I is generated by paths. By a *string* in the string algebra kQ/I we mean a walk $c_1 \dots c_n$ in Q such that neither $c_i \dots c_{i+t}$ nor $c_{i+t}^{-1} \dots c_i^{-1}$ belongs to I for $1 \leq i < i+t \leq n$. Moreover, by a *band* in kQ/I we mean a string $S = c_1 \dots c_n$ such that:

- all powers of S are defined, i.e. $t(c_1) = s(c_n)$ and S^m is a string for all $m \in \mathbb{N}$,
- the string S is not a power of any string of a smaller length,
- c_1 is a direct arrow and c_n is an inverse arrow.

We can also speak about special biserial and string k-categories replacing kQ/I by $\underline{kQ}/I$.

Given a string $S = c_1 \dots c_n$ in the string algebra kQ/I the *string module* associated to S is by definition an $(n+1)$-dimensional kQ/I-module $M(S)$ with k-basis $\{z_1, \dots, z_{n+1}\}$ (called the *canonical* basis of $M(S)$) and with multiplication defined by the formulae:

$$a \cdot z_j = \begin{cases} z_{j-1} & \text{if } j \geq 2 \text{ and } a = c_{j-1}, \\ z_{j+1} & \text{if } j \leq n \text{ and } a^{-1} = c_j, \\ 0 & \text{otherwise} \end{cases}$$

for any direct arrow a and $j \in \{1, \dots, n+1\}$, and

$$e_x \cdot z_j = \begin{cases} z_j & \text{if } (j \geq 2 \text{ and } x = s(c_{j-1})) \text{ or } (j = 1 \text{ and } x = t(c_1)), \\ 0 & \text{otherwise} \end{cases}$$

for any vertex $x \in Q_0$ and $j \in \{1, \dots, n+1\}$.

We refer to [3], [28] for basic facts on string algebras and modules. We recall that, in particular, any string module is indecomposable and $M(S_1) \cong M(S_2)$ if and only if $S_1 = S_2$ or $S_1 = S_2^{-1}$.

3. A sufficient existence condition. We start with some basic facts on pointed modules.

Let R be a ring with a unit. By a *pointed R-module* we mean a pair (M, m), where M is a left R-module and $m \in M$. We say that a pointed module (M, m) is indecomposable (finitely generated, finitely presented etc.) if and only if so is the module M. Furthermore, by a *pointed R-homomorphism* from (M, m) to (N, n) we mean an R-homomorphism $f : M \to N$ such that $f(m) = n$; we write $f : (M, m) \to (N, n)$ in this case. A pointed homomorphism $f : (M, m) \to (N, n)$ is an isomorphism if and only if so is $f : M \to N$. Pointed modules (M, m) and (N, n) are isomorphic if and only if there exists a pointed isomorphism $f : (M, m) \to (N, n)$.

It is convenient to identify a pointed R-module (M, m) with the R-homomorphism $\chi_{(M,m)} : R \to M$ such that $\chi_{(M,m)}(1) = m$. Then an R-homomorphism $f : M \to N$ is a pointed R-homomorphism $f : (M, m) \to (N, n)$ if and only if

$$f\chi_{(M,m)} = \chi_{(N,n)}.$$

Given two pointed modules (M, m) and (N, n), the *pointed pushout* of (M, m) and (N, n) is by definition the pointed module (P, p), where P is the pushout of the homomorphisms $\chi_{(M,m)}$ and $\chi_{(N,n)}$ and p corresponds to m (and n), that is,

$$P = M \oplus N/\{(\chi_{(M,m)}(r), -\chi_{(N,n)}(r));\ r \in R\} = M \oplus N/\{(rm, -rn);\ r \in R\}$$

and $p = \overline{(m,0)} = \overline{(0,n)}$. We write $(M,m) * (N,n)$ for the pointed pushout of the pointed modules (M,m) and (N,n).

Let R be an arbitrary ring with a unit. The following concepts are introduced in [20].

DEFINITION 3.1. Assume that L is a countable dense chain without end points. A *dense chain of pointed R-modules* is a family $(M_q, m_q)_{q\in L}$ of pointed R-modules such that:

(a) the modules M_q are indecomposable and $m_q \neq 0$ for all $q \in L$,
(b) there exist pointed homomorphisms $\mu_{q,q'} : (M_q, m_q) \to (M_{q'}, m_{q'})$ for all $q, q' \in L$ such that $q < q'$,
(c) the pointed modules (M_q, m_q) and $(M_{q'}, m_{q'})$ are not isomorphic for any $q \neq q' \in L$.

DEFINITION 3.2. An *independent pair of dense chains of pointed R-modules* is a pair $((M_q, m_q)_{q\in L_1}, (N_t, n_t)_{t\in L_2})$ of dense chains of pointed R-modules such that:

(a) there is no pointed homomorphism from (M_q, m_q) to (N_t, n_t) and no pointed homomorphism from (N_t, n_t) to (M_q, m_q) for all $q \in L_1$, $t \in L_2$,
(b) the pointed pushout $(M_q, m_q) * (N_t, n_t)$ is an indecomposable R-module for all $q \in L_1$, $t \in L_2$,
(c) $(M_q, m_q) * (N_t, n_t)$ is not isomorphic to $(M_{q'}, m_{q'}) * (N_t, n_t)$ or to $(M_q, m_q) * (N_{t'}, n_{t'})$ for any $q \neq q' \in L_1$, $t \neq t' \in L_2$.

It is clear that the posets L, L_1, L_2 in the definitions above are isomorphic to the poset $\mathbb{Q}$ of rational numbers. The assumption that they are arbitrary countable dense chains without end points is only used for technical reasons in Section 5.

The following theorem due to Puninski, Puninskaya and Toffalori (see [20, Proposition 5.4]) shows the significance of independent pairs of dense chains of pointed R-modules.

THEOREM 3.3. *If there is an independent pair of dense chains of pointed R-modules in R*-mod *then the width* (see [17, pp. 205–206]) *of the lattice of all pp-formulae over R does not exist.* ∎

The following corollary is a consequence of the above theorem and Ziegler's construction [30, Lemma 7.8].

Corollary 3.4. *Assume that R is a countable ring with an identity. If there is an independent pair of dense chains of pointed R-modules in R-*mod *then there exists a super-decomposable pure-injective module over R.* ∎

Observe that the above criterion may be used in the case of bound quiver algebras over a countable field since $|kQ/I| \leq \max\{|k|, \aleph_0\}$ for any finite quiver Q and an admissible ideal I in kQ.

4. Three non-polynomial growth algebras. We recall that a finite-dimensional algebra Λ over an algebraically closed field k is *tame* if, for any natural number d, the set of all isomorphism classes of d-dimensional indecomposable Λ-modules can be parameterized, up to finitely many elements, by a finite number of regular 1-parameter families. Let $\mu_\Lambda(d)$ be the minimal number of 1-parameter families needed to parameterize the d-dimensional indecomposables in the above sense. A (tame) algebra Λ is of *polynomial growth* if there exists a natural number g such that $\mu_\Lambda(d) \leq d^g$ for all $d \geq 2$. We refer to [24, Chapter 14] and [26, Chapter XIX] for precise statements of the definitions.

We introduce three bound quiver algebras (k-categories) of non-polynomial growth, some associated functors and a certain Galois covering.

From now on we denote

$$Q := \begin{tikzcd} x_1 \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\ x_2 \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\gamma"] \\ x_3 \end{tikzcd} \qquad\qquad Q' := \begin{tikzcd} x_{11} \arrow[d, "\alpha_1"'] \arrow[dr, "\beta_1" near start] & x_{12} \arrow[d, "\alpha_2"] \arrow[dl, "\beta_2"' near start] \\ x_{21} \arrow[d, "\delta_1"'] \arrow[dr, "\gamma_1" near start] & x_{22} \arrow[d, "\delta_2"] \arrow[dl, "\gamma_2"' near start] \\ x_{31} & x_{32} \end{tikzcd}$$

and

$$\begin{aligned} \Lambda_1 &:= kQ/I_1, & I_1 &= \langle \delta\alpha, \gamma\beta \rangle, \\ \Lambda_2 &:= kQ/I_2, & I_2 &= \langle \gamma\alpha - \delta\beta, \delta\alpha - \gamma\beta \rangle, \\ \Lambda_3 &:= kQ'/I_3, & I_3 &= \langle \gamma_1\alpha_1 - \delta_2\beta_1, \delta_1\alpha_1 - \gamma_2\beta_1, \gamma_2\alpha_2 - \delta_1\beta_2, \delta_2\alpha_2 - \gamma_1\beta_2 \rangle. \end{aligned}$$

Note that Q' and I_3 are denoted by $\mathcal{G}$ and $\mathcal{I}$, respectively, in Section 1.

It is well known that if $\operatorname{char}(k) \neq 2$ then the k-categories Λ_1 and Λ_2 are isomorphic. Namely, a k-category isomorphism $\iota : \Lambda_2 \to \Lambda_1$ is given by

$$\iota(\alpha) = \alpha + \beta, \quad \iota(\beta) = -\alpha + \beta, \quad \iota(\gamma) = -\gamma + \delta, \quad \iota(\delta) = \gamma + \delta.$$

Furthermore, $\iota : \Lambda_2 \to \Lambda_1$ induces an isomorphism

$$\iota_\bullet : \Lambda_1\text{-mod} \to \Lambda_2\text{-mod}$$

of the associated module categories.

Observe that Λ_1 is a string algebra with two different bands starting with the same direct arrow and ending with the same inverse arrow (for example: $\gamma\delta^-$ and $\gamma\alpha\beta^-\delta^-$), and, by [4], [29], Λ_1 is of tame representation type and, by [21–23], of non-polynomial growth when k is algebraically closed. Consequently, so is Λ_2 if $\mathrm{char}(k) \neq 2$. For the proof that Λ_3 is also tame of non-polynomial growth we refer to [6] or [14].

Consider the k-category automorphisms $\rho : \Lambda_1 \to \Lambda_1$, $\sigma : \Lambda_2 \to \Lambda_2$ and $g : \Lambda_3 \to \Lambda_3$ defined as follows:

$$\begin{array}{llll} \rho(\alpha) = \beta, & \rho(\beta) = \alpha, & \rho(\gamma) = \delta, & \rho(\delta) = \gamma, \\ \sigma(\alpha) = \alpha, & \sigma(\beta) = -\beta, & \sigma(\gamma) = -\gamma, & \sigma(\delta) = \delta, \\ g(\alpha_i) = \alpha_j, & g(\beta_i) = \beta_j, & g(\gamma_i) = \gamma_j, & g(\delta_i) = \delta_j \end{array}$$

for $(i,j) \in \{(1,2),(2,1)\}$.

The automorphism g induces an action on Λ_3-modules. Given a Λ_3-module M we denote by ${}^gM$ the module $(g^{-1})_\bullet(M)$ (see [5, 3.2], [4]).

It is clear that Λ_2 is isomorphic to the orbit category of Λ_3 modulo the action of g ([5, 3.1]). Therefore there is a Galois covering functor

$$F : \Lambda_3 \to \Lambda_2$$

with covering group $\mathbb{Z}_2$ such that $F(s_i) = s$ for $s \in \{\alpha,\beta,\gamma,\delta\}$ and $i \in \{1,2\}$ (see [6]).

The functor $F_\bullet$ is called the *pull-up* functor associated to F. It has a left adjoint F_λ and a right adjoint F_ρ (called the *push-down* functors). Since the covering F has finite fibres, the two functors coincide and they are defined on the objects of Λ_3-Mod as follows:

$$\begin{tikzcd} V_{11} \arrow[d, "v_{\alpha_1}"'] \arrow[dr, "v_{\beta_1}" near start] & V_{12} \arrow[d, "v_{\alpha_2}"] \arrow[dl, "v_{\beta_2}"' near start] & & V_{11}\oplus V_{12} \arrow[d, shift right, "w_\alpha"'] \arrow[d, shift left, "w_\beta"] \\ V_{21} \arrow[d, "v_{\delta_1}"'] \arrow[dr, "v_{\gamma_1}" near start] & V_{22} \arrow[d, "v_{\delta_2}"] \arrow[dl, "v_{\gamma_2}"' near start] & \longmapsto & V_{21}\oplus V_{22} \arrow[d, shift right, "w_\delta"'] \arrow[d, shift left, "w_\gamma"] \\ V_{31} & V_{32} & & V_{31}\oplus V_{32} \end{tikzcd}$$

where

$$w_\alpha = \begin{bmatrix} v_{\alpha_1} & 0 \\ 0 & v_{\alpha_2} \end{bmatrix}, \quad w_\beta = \begin{bmatrix} 0 & v_{\beta_2} \\ v_{\beta_1} & 0 \end{bmatrix},$$
$$w_\gamma = \begin{bmatrix} 0 & v_{\gamma_2} \\ v_{\gamma_1} & 0 \end{bmatrix}, \quad w_\delta = \begin{bmatrix} v_{\delta_1} & 0 \\ 0 & v_{\delta_2} \end{bmatrix}.$$

PROPOSITION 4.1. *The following two diagrams are commutative:*

$$\begin{tikzcd} \Lambda_1 \arrow[d, "\rho"] & \Lambda_2 \arrow[l, "\iota"'] \arrow[d, "\sigma"] \\ \Lambda_1 & \Lambda_2 \arrow[l, "\iota"'] \end{tikzcd} \qquad \begin{tikzcd} \Lambda_1\text{-mod} \arrow[r, "\iota_\bullet"] & \Lambda_2\text{-mod} \\ \Lambda_1\text{-mod} \arrow[u, "\rho_\bullet"] \arrow[r, "\iota_\bullet"] & \Lambda_2\text{-mod} \arrow[u, "\sigma_\bullet"] \end{tikzcd}$$

Proof. The commutativity of the left diagram is an easy consequence of the definitions, and it implies the commutativity of the right one since $(fg)_\bullet = g_\bullet f_\bullet$ for all composable k-category homomorphisms f and g. ∎

Now we collect basic properties of the Galois covering $F : \Lambda_3 \to \Lambda_2$ and its pull-up and push-down functors. The following lemma is essentially contained in [5, 3.4].

LEMMA 4.2. *If* $\operatorname{char}(k) \neq 2$, *then there is an isomorphism of functors*

$$F_\lambda F_\bullet \cong \mathrm{id}_{\Lambda_2\text{-Mod}} \oplus \sigma_\bullet.$$

Proof. We prove that $F_\lambda(F_\bullet(M)) \cong M \oplus \sigma_\bullet(M)$ for any Λ_2-module M. Let M be a Λ_2-module identified with the representation

$$\begin{tikzcd} V_1 \arrow[d, shift right, "v_\alpha"'] \arrow[d, shift left, "v_\beta"] \\ V_2 \arrow[d, shift right, "v_\delta"'] \arrow[d, shift left, "v_\gamma"] \\ V_3 \end{tikzcd}$$

Then it is easy to see that

$$F_\bullet(M) = \begin{tikzcd} V_1 \arrow[d, "v_\alpha"'] \arrow[dr, "v_\beta"] & V_1 \arrow[d, "v_\alpha"] \arrow[dl, "v_\beta"'] \\ V_2 \arrow[d, "v_\delta"'] \arrow[dr, "v_\gamma"] & V_2 \arrow[d, "v_\delta"] \arrow[dl, "v_\gamma"'] \\ V_3 & V_3 \end{tikzcd} \quad \text{and} \quad F_\lambda(F_\bullet(M)) = \begin{tikzcd} V_1 \oplus V_1 \arrow[d, shift right, "w_\alpha"'] \arrow[d, shift left, "w_\beta"] \\ V_2 \oplus V_2 \arrow[d, shift right, "w_\delta"'] \arrow[d, shift left, "w_\gamma"] \\ V_3 \oplus V_3 \end{tikzcd}$$

where

$$w_s = \begin{bmatrix} v_s & 0 \\ 0 & v_s \end{bmatrix} \quad \text{for } s \in \{\alpha, \delta\}, \qquad w_t = \begin{bmatrix} 0 & v_t \\ v_t & 0 \end{bmatrix} \quad \text{for } t \in \{\beta, \gamma\}.$$

Observe that $\sigma_\bullet(M)$ corresponds to the representation

$$\begin{tikzcd} V_1 \arrow[d, shift right, "v_\alpha"'] \arrow[d, shift left, "-v_\beta"] \\ V_2 \arrow[d, shift right, "v_\delta"'] \arrow[d, shift left, "-v_\gamma"] \\ V_3 \end{tikzcd}$$

We show that the map $h : F_\lambda(F_\bullet(M)) \to M \oplus \sigma_\bullet(M)$ given by

$$F_\lambda(F_\bullet(M)) = \begin{tikzcd} V_1 \oplus V_1 \arrow[r, "f_1"] \arrow[d, shift right, "w_\alpha"'] \arrow[d, shift left, "w_\beta"] & V_1 \oplus V_1 \arrow[d, shift right, "w'_\alpha"'] \arrow[d, shift left, "w'_\beta"] \\ V_2 \oplus V_2 \arrow[r, "f_2"] \arrow[d, shift right, "w_\delta"'] \arrow[d, shift left, "w_\gamma"] & V_2 \oplus V_2 \arrow[d, shift right, "w'_\delta"'] \arrow[d, shift left, "w'_\gamma"] \\ V_3 \oplus V_3 \arrow[r, "f_3"] & V_3 \oplus V_3 \end{tikzcd} = M \oplus \sigma_\bullet(M)$$

with $w'_\alpha = w_\alpha$, $w'_\delta = w_\delta$,

$$w'_t = \begin{bmatrix} v_t & 0 \\ 0 & -v_t \end{bmatrix} \quad \text{for } t \in \{\beta, \gamma\}, \qquad f_i = \begin{bmatrix} \mathrm{id}_{V_i} & \mathrm{id}_{V_i} \\ -\mathrm{id}_{V_i} & \mathrm{id}_{V_i} \end{bmatrix} \quad \text{for } i = 1, 2, 3,$$

is an isomorphism.

Since $\mathrm{char}(k) \neq 2$, we have $\det h_i = 2^{\dim V_i} \neq 0$, for $i = 1, 2, 3$, and therefore $h : F_\lambda(F_\bullet(M)) \to M \oplus \sigma_\bullet(M)$ is an isomorphism. Obviously, it is functorial, that is, induces an isomorphism of functors $F_\lambda F_\bullet \cong \mathrm{id}_{\Lambda_2\text{-Mod}} \oplus \sigma_\bullet$. ■

As a consequence of Lemma 4.2 we get the following corollary.

Corollary 4.3.

(a) *If M is an indecomposable Λ_2-module, $M \ncong \sigma_\bullet(M)$ and* $\mathrm{char}(k) \neq 2$, *then $F_\bullet(M)$ is an indecomposable Λ_3-module.*
(b) *If M and N are non-isomorphic and indecomposable Λ_2-modules, $M \ncong \sigma_\bullet(N)$ and* $\mathrm{char}(k) \neq 2$, *then $F_\bullet(M) \ncong F_\bullet(N)$.*

Proof. (a) Let M be an indecomposable Λ_2-module and assume that $F_\bullet(M) = N_1 \oplus \cdots \oplus N_s$ for some $s \geq 2$ and indecomposable Λ_3-modules N_i. According to Lemma 4.2 we get the following isomorphisms:

$$(*) \qquad M \oplus \sigma_\bullet(M) \cong F_\lambda(F_\bullet(M)) \cong F_\lambda(N_1 \oplus \cdots \oplus N_s) \cong F_\lambda(N_1) \oplus \cdots \oplus F_\lambda(N_s).$$

Observe that the Krull–Remak–Schmidt Theorem yields $s = 2$ and the Λ_2-modules $F_\lambda(N_1)$ and $F_\lambda(N_2)$ are indecomposable. It follows that $M \cong$

$F_\lambda(N)$, where $N = N_1$ or $N = N_2$, and obviously

$$N_1 \oplus N_2 = F_\bullet(M) \cong F_\bullet(F_\lambda(N)) \cong N \oplus {}^g N$$

by the well-known property of Galois coverings (see [5, 3.2]). Hence $F_\lambda(N_1) \cong F_\lambda(N_2)$, and finally $M \cong \sigma_\bullet(M)$ by $(*)$, a contradiction.

(b) Assume, to the contrary, that $F_\bullet(M) \cong F_\bullet(N)$. According to Lemma 4.2 we get the isomorphisms

$$M \oplus \sigma_\bullet(M) = F_\lambda(F_\bullet(M)) \cong F_\lambda(F_\bullet(N)) \cong N \oplus \sigma_\bullet(N).$$

Since M and N are not isomorphic and indecomposable, there is an isomorphisms $M \cong \sigma_\bullet(N)$ and we get a contradiction. ∎

Now we prove that the pull-up functor $F_\bullet : \Lambda_2\text{-mod} \to \Lambda_3\text{-mod}$ of the covering functor $F : \Lambda_3 \to \Lambda_2$ preserves pointed pushouts (in the sense of Lemma 4.4). This is a consequence of the fact that any functor having a right adjoint preserves the colimits which exist in its domain (see [11, Chapter V, Theorem 5.1]).

First observe that the pull-up functor $F_\bullet : \Lambda_2\text{-mod} \to \Lambda_3\text{-mod}$ induces a functor between categories of pointed modules. Indeed, let $(M, m), (N, n)$ be pointed Λ_2-modules and $f : (M, m) \to (N, n)$ a pointed Λ_2-homomorphism. If $\chi_{(M,m)} : \Lambda_2 \to M$ and $\chi_{(N,n)} : \Lambda_2 \to N$ are the corresponding homomorphisms (see Section 3), then $f\chi_{(M,m)} = \chi_{(N,n)}$ and thus $F_\bullet(f)F_\bullet(\chi_{(M,m)}) = F_\bullet(\chi_{(N,n)})$.

There is an isomorphism

$$\omega : \Lambda_3 \to F_\bullet(\Lambda_2)$$

of left Λ_3-modules. We fix ω for the remainder of the paper. The homomorphism $F_\bullet(f) : F_\bullet(M) \to F_\bullet(N)$ is a pointed Λ_3-homomorphism from $F_\bullet(\chi_{(M,m)})\omega$ to $F_\bullet(\chi_{(N,n)})\omega$, since obviously

$$F_\bullet(f)(F_\bullet(\chi_{(M,m)})\omega) = F_\bullet(\chi_{(N,n)})\omega.$$

The pointed Λ_3-module corresponding to $F_\bullet(\chi_{(M,m)})\omega$ will be denoted by $(F_\bullet(M), \widetilde{m})$, that is,

$$\widetilde{m} := (F_\bullet(\chi_{(M,m)})\omega)(1_{\Lambda_3}) \in F_\bullet(M).$$

Moreover, we set

$$F_\bullet(M, m) := (F_\bullet(M), \widetilde{m})$$

for any pointed Λ_2-module (M, m). We point out that the construction depends on ω.

Lemma 4.4. *If (M, m) and (N, n) are pointed Λ_2-modules, then*

$$F_\bullet((M, m) * (N, n)) \cong F_\bullet(M, m) * F_\bullet(N, n).$$

Proof. Let (M,m) and (N,n) be pointed Λ_2-modules. Assume that the diagram

$$\begin{tikzcd} \Lambda_2 \arrow[r, "\chi_{(M,m)}"] \arrow[d, "\chi_{(N,n)}"'] & M \arrow[d, "f"] \\ N \arrow[r, "h"] & P \end{tikzcd}$$

is the pushout of $\chi_{(M,m)}$ and $\chi_{(N,n)}$ in Λ_2-mod. Moreover, assume that $f(m) = p = h(n)$, that is, $(M,m) * (N,n) = (P,p)$ and $f\chi_{(M,m)} = \chi_{(P,p)} = h\chi_{(N,n)}$.

Since $F_\bullet : \Lambda_2\text{-mod} \to \Lambda_3\text{-mod}$ preserves pushouts (as a left adjoint, see [11, Chapter V, Theorem 5.1]) we conclude that the diagram

$$\begin{tikzcd} F_\bullet(\Lambda_2) \arrow[r, "F_\bullet(\chi_{(M,m)})"] \arrow[d, "F_\bullet(\chi_{(N,n)})"'] & F_\bullet(M) \arrow[d, "F_\bullet(f)"] \\ F_\bullet(N) \arrow[r, "F_\bullet(h)"] & F_\bullet(P) \end{tikzcd}$$

is the pushout of $F_\bullet(\chi_{(M,m)})$ and $F_\bullet(\chi_{(N,n)})$. Then it is easy to see that $F_\bullet(P)$ is the pushout of $F_\bullet(\chi_{(M,m)})\omega$ and $F_\bullet(\chi_{(N,n)})\omega$, so

$$F_\bullet(M,m) * F_\bullet(N,n) \cong (F_\bullet(P), x),$$

where $x = (F_\bullet(f)F_\bullet(\chi_{(M,m)})\omega)(1_{\Lambda_3})$. On the other hand,

$$\widetilde{p} = (F_\bullet(\chi_{(P,p)})\omega)(1_{\Lambda_3}) = (F_\bullet(f\chi_{(M,m)})\omega)(1_{\Lambda_3}) = (F_\bullet(f)F_\bullet(\chi_{(M,m)})\omega)(1_{\Lambda_3}).$$

This yields

$$F_\bullet((M,m) * (N,n)) = F_\bullet(P,p) = (F_\bullet(P), \widetilde{p}) \cong F_\bullet(M,m) * F_\bullet(N,n). \qquad \blacksquare$$

We finish this section with another important property of the pull-up functor $F_\bullet : \Lambda_2\text{-mod} \to \Lambda_3\text{-mod}$.

LEMMA 4.5. *Assume that (M,m) and (N,n) are pointed Λ_2-modules. If there exists a pointed Λ_3-homomorphism from $(F_\bullet(M), \widetilde{m})$ to $(F_\bullet(N), \widetilde{n})$, then there exists a pointed Λ_2-homomorphism from (M,m) to (N,n).*

Proof. Assume that $f : (F_\bullet(M), \widetilde{m}) \to (F_\bullet(N), \widetilde{n})$ is a pointed homomorphism in Λ_3-mod. Then

$$fF_\bullet(\chi_{(M,m)}) = F_\bullet(\chi_{(N,n)})$$

and therefore

$$F_\lambda(f)F_\lambda(F_\bullet(\chi_{(M,m)})) = F_\lambda(fF_\bullet(\chi_{(M,m)})) = F_\lambda(F_\bullet(\chi_{(N,n)})).$$

Since $F_\lambda F_\bullet \cong \mathrm{id}_{\Lambda_2\text{-Mod}} \oplus \sigma_\bullet$, by Lemma 4.2, we finally get the equality

$$F_\lambda(f)(\chi_{(M,m)} \oplus \sigma_\bullet(\chi_{(M,m)})) = \chi_{(N,n)} \oplus \sigma_\bullet(\chi_{(N,n)}).$$

Assume that

$$F_\lambda(f) = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} : M \oplus \sigma_\bullet(M) \to N \oplus \sigma_\bullet(N).$$

It follows that

$$\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \cdot \begin{bmatrix} \chi_{(M,m)} & 0 \\ 0 & \sigma_\bullet(\chi_{(M,m)}) \end{bmatrix} = \begin{bmatrix} \chi_{(N,n)} & 0 \\ 0 & \sigma_\bullet(\chi_{(N,n)}) \end{bmatrix}$$

and thus $f_1\chi_{(M,m)} = \chi_{(N,n)}$, so $f_1 : M \to N$ is a pointed Λ_2-homomorphism from (M,m) to (N,n). ■

5. Independent pairs of dense chains in Λ_1-mod. In this section we present a construction of independent pairs of dense chains of pointed modules in Λ_1-mod, and examine their properties that are fundamental for constructing an independent pair of dense chains of pointed modules in Λ_3-mod. The contents of this section refine some results of [18] and [23].

Throughout the section, $A = k\Delta/I$ is an arbitrary string algebra. Given an arrow $a \in \Delta_1$, we define $\mathcal{S}(a)$ to be the set of strings over A that start with a. We recall from [3] that there is a linear ordering $<$ on $\mathcal{S}(a)$ such that $S < T$ if and only if one of the following conditions is satisfied:

- $S\varphi^{-1}U = T$ for an arrow φ and a string U,
- $S = T\psi V$ for an arrow ψ and a string V,
- $S = S'\psi W$ and $T = S'\varphi^{-1}X$ for some arrows φ, ψ and strings S', W, X.

Assume that a string algebra A possesses two different bands U and V starting with the same direct arrow and ending with the same inverse arrow such that U is not a prolongation of V and V is not a prolongation of U, i.e. $U \neq VX$ and $V \neq UY$ for any strings X and Y. Moreover, assume that $U < V$ and let $\Sigma(U,V)$ be the set of all finite words over the alphabet $\{U,V\}$, including the empty word $\emptyset$.

The following technical lemmata describe the properties of elements of $\Sigma(U,V)$ that we use in the proof of Theorem 5.3.

LEMMA 5.1. *Assume that U and V are two different bands starting with the same direct arrow and ending with the same inverse arrow such that U is not a prolongation of V and V is not a prolongation of U. Moreover, assume that $U < V$. Then the following conditions are satisfied:*

(a) $U^{p_1}V^{q_1}\ldots U^{p_n}V^{q_n} = U^{p'_1}V^{q'_1}\ldots U^{p'_m}V^{q'_m}$ *if and only if $n = m$ and $p_i = p'_i$, $q_i = q'_i$ for any $1 \le i \le n = m$.*

(b) *Assume that $W_1, W_2 \in \Sigma(U,V)$ and $W_1 = XY_1$, $W_2 = XY_2$ for some $X, Y_1, Y_2 \in \Sigma(U,V)$ such that Y_1 and Y_2 do not start with the same letter U or V. Then $W_1 < W_2$ if and only if either Y_1 starts with U and Y_2 starts with V, or $Y_1 \neq \emptyset$ and $Y_2 = \emptyset$.*

Proof. (a) It is enough to show that there is no word $W \in \Sigma(U,V)$ such that $US = W = VS'$ for some strings S, S'. But this is obvious since the assumption $US = VS'$ implies that U is a prolongation of V or vice versa, a contradiction.

(b) Assume that $W_1, W_2 \in \Sigma(U,V)$, $W_1 < W_2$ and there are $X, Y_1, Y_2 \in \Sigma(U,V)$ such that $W_1 = XY_1, W_2 = XY_2$ and Y_1, Y_2 do not start with the same letter.

Obviously $Y_1 \neq \emptyset$, since $Y_1 = \emptyset$ implies $W_1 \geq W_2$, because either $Y_2 = \emptyset$ or Y_2 starts with a direct arrow (as an element of $\Sigma(U,V)$). Therefore either ($Y_1 \neq \emptyset$ and $Y_2 = \emptyset$) or ($Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$).

Assume that $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$, and Y_1 starts with V, while Y_2 starts with U. Since U is not a prolongation of V or vice versa, we may assume that $U = St_1 \ldots t_n$ and $V = Sr_1 \ldots r_m$ for some string S of non-zero length and arrows (direct or inverse) t_i, r_j such that $t_1 \neq r_1$. Now observe that $U < V$ implies that t_1 is a direct arrow and r_1 is an inverse arrow. Thus $W_1 > W_2$, because W_1 starts with the string XSr_1 and W_2 starts with XSt_1, a contradiction. This proves that Y_1 starts with U and Y_2 starts with V.

Finally, either $Y_1 \neq \emptyset$ and $Y_2 = \emptyset$, or Y_1 starts with U and Y_2 starts with V. The converse implication follows easily by the definition of $<$. ∎

If $W \in \Sigma(U,V)$, we denote by $I(W) \in \{U,V\}$ the initial letter of W. If W is an empty word, we set $I(W) = \emptyset$.

Observe that these definitions are correct, because Lemma 5.1(a) shows that any element $W = U^{p_1}V^{q_1} \ldots U^{p_n}V^{q_n}$ of $\Sigma(U,V)$ determines the sequence $p_1, q_1, \ldots, p_n, q_n$ uniquely. Then we denote $l(W) = \sum_{i=1}^{n}(p_i + q_i)$.

In Lemma 5.2 and Theorem 5.3 below we assume that U and V are two different bands such that $U < V$ and U is not a prolongation of V or vice versa. Thus the premises of Lemma 5.1 hold true.

LEMMA 5.2. *Assume that $W_1, W_2 \in \Sigma(U,V)$ and $W_1 < W_2$. If there exists a word W_3 such that $W_1 \geq W_2W_3$, then $W_1 = W_2Y$ for some non-empty word $Y \in \Sigma(U,V)$.*

Proof. We may assume that $W_1 = XY_1$ and $W_2 = XY_2$ for some $X, Y_1, Y_2 \in \Sigma(U,V)$ such that $I(Y_1) \neq I(Y_2)$. Obviously $Y_1 \neq \emptyset$ or $Y_2 \neq \emptyset$ since $W_1 \neq W_2$. Moreover, $Y_1 = \emptyset$ or $Y_2 = \emptyset$. Indeed, otherwise $I(Y_1) = U$ and $I(Y_2) = V$ by Lemma 5.1(b). But then $W_1 < W_2W_3$ for any word $W_3 \in \Sigma(U,V)$, a contradiction. Therefore

$$(Y_1 \neq \emptyset \text{ or } Y_2 \neq \emptyset) \text{ and } (Y_1 = \emptyset \text{ or } Y_2 = \emptyset).$$

Assume that $Y_1 = \emptyset$. This yields $Y_2 \neq \emptyset$ and thus $I(Y_2) \in \{U,V\}$, so $W_2 = XY_2 < X = W_1$, a contradiction.

Hence $Y_1 \neq \emptyset$ and $Y_2 = \emptyset$, which finishes the proof of the lemma (just put $Y = Y_1$). ∎

We now present a more general version of a result of Schröer (see [23, Proposition 6.2]).

THEOREM 5.3. *Assume that U and V are two different bands such that $U < V$ and U is not a prolongation of V or vice versa. Let $S, T \in \Sigma(U, V)$. Then the set*

$$\mathcal{L}_S^T(U, V) := \{SXTU;\ X \in \Sigma(U, V)\}$$

is a dense chain without end points.

Proof. First observe that $\mathcal{L}_S^T(U, V)$ has no end points. Indeed, Lemma 5.1(b) yields

$$SXTUTU < SXTU < SXV^{l(T)+1}TU,$$

because $U < V$, and thus $\mathcal{L}_S^T(U, V)$ has neither the smallest nor the greatest element.

To prove that $\mathcal{L}_S^T(U, V)$ is dense, assume that $SX_1TU < SX_2TU$ for some words X_1, X_2. We consider two cases:

(1) If $SX_1TU < SX_2TUTU$, then obviously

$$SX_1TU < SX_2TUTU < SX_2TU.$$

(2) Assume that $SX_1TU \geq SX_2TUTU$. Since $SX_1TU < SX_2TU$, Lemma 5.2 yields $SX_1TU = SX_2TUY$ for some $Y \neq \emptyset$. Since the string Y ends with U, it may be written in the form $Y = V^nUZ$ with $l(Z) \geq 0$ and $n \geq 0$. Observe that

$$SX_2TUY = SX_2TUV^nUZ < SX_2TUV^{n+1}TU,$$

since $U < V$. Moreover,

$$SX_2TUV^{n+1}TU < SX_2TU$$

and finally

$$\begin{aligned} SX_1TU = SX_2TUY = SX_2TUV^nUZ &< SX_2TUV^{n+1}TU \\ &< SX_2TU. \end{aligned}$$

Therefore, in both cases there exists a string $X \in \mathcal{L}_S^T(U, V)$ such that $SX_1TU < X < SX_2TU$. This proves the density of $\mathcal{L}_S^T(U, V)$. ∎

Note that if we set $S = T = \emptyset$ in Theorem 5.3, then the set

$$\mathcal{L}(U, V) := \mathcal{L}_\emptyset^\emptyset(U, V) = \{XU;\ X \in \Sigma(U, V)\}$$

is a dense chain without end points. This is exactly what Schröer shows in the proof of Proposition 6.2 in [23].

We denote by $\mathbb{Q}$ the poset of rational numbers.

Definition 5.4. A pair (U, V) of two different bands over the string algebra A starting with the same direct arrow and ending with the same inverse arrow is $\mathbb{Q}$-*generating* provided $U < V$ and U is not a prolongation of V or vice versa.

Thus, if (U, V) is a $\mathbb{Q}$-generating pair of bands over the string algebra A, then Theorem 5.3 implies that the set $\mathcal{L}_S^T(U, V)$ is isomorphic to the poset $\mathbb{Q}$ of rational numbers for any strings $S, T \in \Sigma(U, V)$.

Our next aim is to prove that given two $\mathbb{Q}$-generating pairs (U, V) and (U^{-1}, V^{-1}) and $S, T \in \Sigma(U, V)$, $S', T' \in \Sigma(U^{-1}, V^{-1})$ we can produce an independent pair of dense chains of pointed modules.

Assume that $S = s_1 \dots s_n$ is a string over the string algebra $A = k\Delta/I$, $M(S)$ is the associated string module and $z_1^S \in M(S)$ is the first element of the canonical k-basis of $M(S)$. We call the pointed A-module $(M(S), z_1^S)$ the *canonical pointed string module* associated with S.

We remark that, although $M(S) \cong M(S^{-1})$, usually the pointed modules $(M(S), z_1^S)$ and $(M(S^{-1}), z_1^{S^{-1}})$ are not isomorphic.

Lemma 5.5 ([19, 3.1]). *Assume that $a \in \Delta_1$, $S, T \in \mathcal{S}(a)$ and $S < T$. Then there is a pointed A-homomorphism $f_{(T,S)} : (M(T), z_1^T) \to (M(S), z_1^S)$ of the canonical pointed string modules $(M(T), z_1^T)$ and $(M(S), z_1^S)$.* ∎

Assume that T, S are strings over A such that $T = t_1 \dots t_k$, $S = s_1 \dots s_m$ and TS is also a string. We denote by $z^{(T,S)}$ the element z_{k+1}^{TS} of the canonical basis $(z_1^{TS}, \dots, z_{k+1}^{TS}, \dots, z_{k+m+1}^{TS})$ of $M(TS)$.

Lemma 5.6 ([19, 3.2]). *Assume that T, S are strings over A such that $T^{-1}S$ is also a string. Then the pointed module $(M(T^{-1}S), z^{(T^{-1},S)})$ is the pointed pushout of the pointed modules $(M(S), z_1^S)$ and $(M(T), z_1^T)$.* ∎

We refer to [23, Section 2] for a combinatorial description of pointed homomorphisms between string modules.

Theorem 5.7. *Assume that (U, V) and (U^{-1}, V^{-1}) are $\mathbb{Q}$-generating over the string algebra A, and let $S, T \in \Sigma(U, V)$, $S', T' \in \Sigma(U^{-1}, V^{-1})$. Then*

$$((M(X), z_1^X)_{X \in \mathcal{L}_S^T(U,V)}, (M(Y), z_1^Y)_{Y \in \mathcal{L}_{S'}^{T'}(U^{-1},V^{-1})})$$

is an independent pair of dense chains of pointed modules in A-mod.

Proof. We prove that $(M(X), z_1^X)_{X \in \mathcal{L}_S^T(U,V)}$, $(M(Y), z_1^Y)_{Y \in \mathcal{L}_{S'}^{T'}(U^{-1},V^{-1})}$ are dense chains of pointed A-modules.

By Theorem 5.3, the sets $\mathcal{L}_S^T(U, V)$, $\mathcal{L}_{S'}^{T'}(U^{-1}, V^{-1})$ are dense chains without end points.

The modules $M(X)$ and $M(Y)$ are indecomposable for any $X \in \mathcal{L}_S^T(U, V)$, $Y \in \mathcal{L}_{S'}^{T'}(U^{-1}, V^{-1})$, since they are string modules over A (see [3]).

The existence of pointed homomorphisms in $(M(X), z_1^X)_{X\in\mathcal{L}_S^T(U,V)}$ and $(M(Y), z_1^Y)_{Y\in\mathcal{L}_{S'}^{T'}(U^{-1},V^{-1})}$ follows from Lemma 5.5.

The pointed modules $(M(X_1), z^{X_1})$ and $(M(X_2), z^{X_2})$ are not isomorphic for any $X_1, X_2 \in \mathcal{L}_S^T(U,V)$ such that $X_1 \neq X_2$. Indeed, $X_1 \neq X_2^{-1}$ since X_2^{-1} starts with a different direct arrow than X_1; $X_1 \neq X_2$ by assumption. It follows that $M(X_1)$ and $M(X_2)$ are not isomorphic (see [3]), and thus $(M(X_1), z_1^{X_1})$ and $(M(X_2), z_1^{X_2})$ are not either.

Similar arguments show that $(M(Y_1), z_1^{Y_1})$ and $(M(Y_2), z_1^{Y_2})$ are not isomorphic for any $Y_1, Y_2 \in \Sigma(U^{-1}, V^{-1})$ such that $Y_1 \neq Y_2$.

Consequently, $(M(X), z_1^X)_{X\in\mathcal{L}_S^T(U,V)}$ and $(M(Y), z_1^Y)_{Y\in\mathcal{L}_{S'}^{T'}(U^{-1},V^{-1})}$ are dense chains of pointed modules in A-mod. Now we prove that these chains are independent.

First we show that there is no pointed homomorphism from $(M(X), z_1^X)$ to $(M(Y), z_1^Y)$ for any $X \in \mathcal{L}_S^T(U,V)$, $Y \in \mathcal{L}_{S'}^{T'}(U^{-1},V^{-1})$. Indeed, assume, to the contrary, that there is a pointed A-homomorphism $f : (M(X), z_1^X) \to (M(Y), z_1^Y)$. Observe that X and Y start with different direct arrows a and b. This yields

$$af(z_2^X) = f(az_2^X) = f(z_1^X) = z_1^Y = bz_2^Y.$$

Assume that

$$f(z_2^X) = \lambda_1 z_1^Y + \lambda_2 z_2^Y + \cdots + \lambda_{n+1} z_{n+1}^Y,$$

where $\lambda_i \in k$ and $z_1^Y, \ldots, z_{n+1}^Y$ is the canonical basis of $M(Y)$. Thus

$$z_1^Y = af(z_2^X) = \lambda_1 az_1^Y + \lambda_2 az_2^Y + \cdots + \lambda_{n+1} az_{n+1}^Y.$$

Observe that a is a direct arrow and Y starts with $b \neq a$, hence $az_1^Y = 0$, $az_2^Y = 0$. Moreover, $az_i^Y \in \{0, z_{i-1}^Y\}$ for $i \geq 3$. It follows that z_1^Y is a linear combination of $z_2^Y, \ldots, z_{n+1}^Y$, a contradiction with the fact that $z_1^Y, \ldots, z_{n+1}^Y$ are linearly independent. Hence there is no pointed homomorphism from $(M(X), z_1^X)$ to $(M(Y), z_1^Y)$.

Similar arguments show that there is no pointed homomorphism from $(M(Y), z_1^Y)$ to $(M(X), z_1^X)$.

The pointed pushout of $(M(X), z_1^X)$ and $(M(Y), z_1^Y)$ is indecomposable since it is isomorphic to $(M(X^{-1}Y), z^{(X^{-1},Y)})$ by Lemma 5.6.

The pointed pushouts $(M(X_1^{-1}Y), z^{(X_1^{-1},Y)})$ and $(M(X_2^{-1}Y), z^{(X_2^{-1},Y)})$ are not isomorphic for any $X_1, X_2 \in \mathcal{L}_S^T(U,V)$ such that $X_1 \neq X_2$. Indeed, $X_1^{-1}Y \neq X_2^{-1}Y$ since $X_1 \neq X_2$; $X_1^{-1}Y \neq (X_2^{-1}Y)^{-1} = Y^{-1}X_2$, because X_1^{-1} starts with a different direct arrow than Y^{-1}.

Similar arguments show that the pointed pushouts $(M(X^{-1}Y_1), z^{(X^{-1},Y_1)})$ and $(M(X^{-1}Y_2), z^{(X^{-1},Y_2)})$ are not isomorphic for any $Y_1, Y_2 \in \mathcal{L}_S^T(U^{-1}, V^{-1})$ such that $Y_1 \neq Y_2$.

Consequently, $(M(X), z_1^X)_{X\in\mathcal{L}_S^T(U,V)}$ and $(M(Y), z_1^Y)_{Y\in\mathcal{L}_{S'}^{T'}(U^{-1},V^{-1})}$ are independent dense chains of pointed A-modules, and the proof is complete. ■

Recall from Section 4 that we have defined the automorphism $\rho : \Lambda_1 \to \Lambda_1$ of the k-category Λ_1. Now we interpret ρ in terms of the set $\mathcal{S}(\Lambda_1)$ of strings over Λ_1.

Let $\overline{\rho} : \mathcal{S}(\Lambda_1) \to \mathcal{S}(\Lambda_1)$ be defined by $\overline{\rho}(a) = \rho(a)$ and $\overline{\rho}(a^{-1}) = \rho(a)^{-1}$ for any direct arrow $a \in Q_1$. It is easy to see that the sequence $\overline{\rho}(a_1)\ldots\overline{\rho}(a_n)$ is a well-defined string over Λ_1 provided $a_1\ldots a_n$ is a string, and we set $\overline{\rho}(a_1\ldots a_n) = \overline{\rho}(a_1)\ldots\overline{\rho}(a_n)$ for any $a_1\ldots a_n \in \mathcal{S}(\Lambda_1)$.

Observe that $\overline{\rho}\circ\overline{\rho} = 1_{\mathcal{S}(\Lambda_1)}$, $\overline{\rho}_{|Q_1} = \rho_{|Q_1}$ and $s(\overline{\rho}(a)) = s(a), t(\overline{\rho}(a)) = t(a)$ for any direct or inverse arrow a.

Proposition 5.8. *If $S \in \mathcal{S}(\Lambda_1)$, then $\rho_\bullet(M(S)) \cong M(\overline{\rho}(S))$.*

Proof. Assume that $S = s_1\ldots s_n$ and let $\{z_1,\ldots,z_{n+1}\}$ be the canonical k-basis of the string module $M(S)$, and $\{\overline{z_1},\ldots,\overline{z_{n+1}}\}$ the canonical k-basis of $M(\overline{\rho}(S))$.

We have the following formulae:

$$(M(\overline{\rho}(S))(a))(\overline{z_j}) = \begin{cases} \overline{z_{j-1}} & \text{if } j\geq 2 \text{ and } a = \overline{\rho}(s_{j-1}),\\ \overline{z_{j+1}} & \text{if } j\leq n \text{ and } a^{-1} = \overline{\rho}(s_j),\\ 0 & \text{otherwise}\end{cases}$$

and

$$(\rho_\bullet M(S)(a))(z_j) = (M(S)(\rho(a)))(z_j) = \begin{cases} z_{j-1} & \text{if } j\geq 2 \text{ and } \rho(a) = s_{j-1},\\ z_{j+1} & \text{if } j\leq n \text{ and } \rho(a)^{-1} = s_j,\\ 0 & \text{otherwise,}\end{cases}$$

for any direct arrow a and $j \in \{1,\ldots,n+1\}$.

Moreover, it is easy to see that

$$(M(\overline{\rho}(S))(e_x))(\overline{z_j}) = \overline{z_j} \quad \text{if and only if} \quad (M(S)(\rho(e_x)))(z_j) = z_j$$

and

$$(M(\overline{\rho}(S))(e_x))(\overline{z_j}) = 0 \quad \text{if and only if} \quad (M(S)(\rho(e_x)))(z_j) = 0$$

for any vertex $x \in Q_0$ and $j \in \{1,\ldots,n+1\}$.

This shows that $z_i \mapsto \overline{z_i}$ induces an isomorphism $\rho_\bullet(M(S)) \cong M(\overline{\rho}(S))$. ■

As a corollary we get the following combinatorial lemma.

Lemma 5.9. *Assume that (U,V) is a $\mathbb{Q}$-generating pair of bands over Λ_1 such that the pair (U^{-1},V^{-1}) is also $\mathbb{Q}$-generating. Moreover, assume that $\overline{\rho}(U) = U^{-1}$ and $\overline{\rho}(V) = V^{-1}$. Then:*

(a) $\rho_\bullet(M(S_1)) \ncong M(S_2)$ *for any* $S_1, S_2 \in \mathcal{L}_V^V(U,V)$.

(b) $\rho_\bullet(M(T_1)) \ncong M(T_2)$ *for any* $T_1, T_2 \in \mathcal{L}_{V^{-1}}^{U^{-1}}(U^{-1},V^{-1})$.

(c) $\rho_\bullet(M(T_1^{-1}S_1)) \not\cong M(T_2^{-1}S_2)$ *for any* $S_1, S_2 \in \mathcal{L}_V^V(U,V)$ *and* $T_1, T_2 \in \mathcal{L}_{V^{-1}}^{U^{-1}}(U^{-1}, V^{-1})$.

Proof. (a) Assume that $S_1 = VX_1VU$ and $S_2 = VX_2VU$ for some $X_1, X_2 \in \Sigma(U,V)$. Thanks to Proposition 5.8 it is enough to show that $S_2 \neq \overline{\rho}(S_1)$ and $S_2 \neq \overline{\rho}(S_1)^{-1}$.

Obviously

$$\overline{\rho}(S_1) = \overline{\rho}(VX_1VU) = \overline{\rho}(V)\overline{\rho}(X_1)\overline{\rho}(VU) = V^{-1}\overline{\rho}(X_1)V^{-1}U^{-1}$$

and

$$\overline{\rho}(S_1)^{-1} = (V^{-1}\overline{\rho}(X_1)V^{-1}U^{-1})^{-1} = UV\overline{\rho}(X_1)^{-1}V.$$

Therefore $S_2 \neq \overline{\rho}(S_1)$ since V, as a band, starts with a different arrow than V^{-1}. Moreover, $S_2 \neq \overline{\rho}(S_1)^{-1}$ by Lemma 5.1(a), since $S_2, \overline{\rho}(S_1)^{-1} \in \Sigma(U,V)$ and S_2 starts with V, while $\overline{\rho}(S_1)^{-1}$ starts with U.

(b) Assume that $T_1 = V^{-1}Y_1U^{-2}$ and $T_2 = V^{-1}Y_2U^{-2}$ for some $Y_1, Y_2 \in \Sigma(U^{-1}, V^{-1})$. It is enough to show that $T_2 \neq \overline{\rho}(T_1)$ and $T_2 \neq \overline{\rho}(T_1)^{-1}$.

First, $\overline{\rho} \circ \overline{\rho} = 1_{\mathcal{S}(\Lambda_1)}$ yields $\overline{\rho}(W^{-1}) = W$ for any $W \in \{U, V\}$. Therefore

$$\overline{\rho}(T_1) = \overline{\rho}(V^{-1}Y_1U^{-2}) = \overline{\rho}(V^{-1})\overline{\rho}(Y_1)\overline{\rho}(U^{-2}) = V\overline{\rho}(Y_1)U^2$$

and

$$\overline{\rho}(T_1)^{-1} = (V\overline{\rho}(Y_1)U^2)^{-1} = U^{-2}\overline{\rho}(Y_1)^{-1}V^{-1}.$$

Thus the assertion follows from the fact that V starts with a different arrow than V^{-1}, and from Lemma 5.1(a) (since $T_2, \overline{\rho}(T_1)^{-1} \in \Sigma(U^{-1}, V^{-1})$).

(c) Assume that $S_1 = VX_1VU, S_2 = VX_2VU$ for some $X_1, X_2 \in \Sigma(U,V)$ and $T_1 = V^{-1}Y_1U^{-2}, T_2 = V^{-1}Y_2U^{-2}$ for some $Y_1, Y_2 \in \Sigma(U^{-1}, V^{-1})$. We have to show that $T_2^{-1}S_2 \neq \overline{\rho}(T_1^{-1}S_1)$ and $T_2^{-1}S_2 \neq \overline{\rho}(T_1^{-1}S_1)^{-1}$.

Obviously

$$T_2^{-1}S_2 = (V^{-1}Y_2U^{-2})^{-1}VX_2VU = U^2Y_2^{-1}V^2X_2VU,$$
$$\overline{\rho}(T_1^{-1}S_1) = \overline{\rho}(U^2Y_1^{-1}V^2X_1VU) = U^{-2}\overline{\rho}(Y_1^{-1})V^{-2}\overline{\rho}(X_1)V^{-1}U^{-1},$$
$$(\overline{\rho}(T_1^{-1}S_1))^{-1} = UV\overline{\rho}(X_1)^{-1}V^2\overline{\rho}(Y_1^{-1})^{-1}U^2,$$

so the assertion follows from the fact that U starts with a different arrow than U^{-1}, and from Lemma 5.1(a) (since $T_2^{-1}S_2, \overline{\rho}(T_1^{-1}S_1)^{-1} \in \Sigma(U,V)$). ■

Now we state the main theorem of the section.

THEOREM 5.10. *Assume that*

$$(U,V) = (\gamma\alpha\beta^{-1}\delta^{-1}, \gamma\delta^{-1}) \quad \textit{or} \quad (U,V) = (\beta\alpha^{-1}\beta\alpha^{-1}, \beta\alpha^{-1}\gamma^{-1}\delta\beta\alpha^{-1}).$$

Then (U,V) *and* (U^{-1}, V^{-1}) *are* $\mathbb{Q}$*-generating pairs of bands over* Λ_1. *Moreover, the pair*

$$((M(X), z_1^X)_{X \in \mathcal{L}_V^V(U,V)}, (M(Y), z_1^Y)_{Y \in \mathcal{L}_{V^{-1}}^{U^{-1}}(U^{-1}, V^{-1})})$$

is an independent pair of dense chains of pointed modules in Λ_1-mod *such that:*

(a) $\rho_\bullet(M(S_1)) \not\cong M(S_2)$ *for any* $S_1, S_2 \in \mathcal{L}_V^V(U,V)$.
(b) $\rho_\bullet(M(T_1)) \not\cong M(T_2)$ *for any* $T_1, T_2 \in \mathcal{L}_{V^{-1}}^{U^{-1}}(U^{-1},V^{-1})$.
(c) $\rho_\bullet(M(T_1^{-1}S_1)) \not\cong M(T_2^{-1}S_2)$ *for any* $S_1, S_2 \in \mathcal{L}_V^V(U,V)$ *and* $T_1, T_2 \in \mathcal{L}_{V^{-1}}^{U^{-1}}(U^{-1},V^{-1})$.

Proof. We check directly that (U,V) and (U^{-1},V^{-1}) are $\mathbb{Q}$-generating pairs of bands over Λ_1. It is easy to show by direct calculations that $\overline{\rho}(U) = U^{-1}$ and $\overline{\rho}(V) = V^{-1}$. Therefore the assertion is an immediate consequence of Theorem 5.7 and Lemma 5.9. ∎

We remark that there are many other examples of $\mathbb{Q}$-generating pairs of bands (U,V) over Λ_1 such that the pair (U^{-1},V^{-1}) is also $\mathbb{Q}$-generating.

6. Independent pairs of dense chains in Λ_3-mod. The aim of this section is to prove the existence of an independent pair of dense chains of pointed modules in Λ_3-mod. This will be a consequence of the main results of Sections 4 and 5.

Definition 6.1. An independent pair $((M_q,m_q)_{q\in L_1},(N_t,n_t)_{t\in L_2})$ of dense chains of pointed modules in Λ_2-mod is *non-symmetric* provided the following conditions are satisfied:

(a) $\sigma_\bullet(M_{q_1}) \not\cong M_{q_2}$ for any $q_1, q_2 \in L_1$,
(b) $\sigma_\bullet(N_{t_1}) \not\cong N_{t_2}$ for any $t_1, t_2 \in L_2$,
(c) $\sigma_\bullet(P_{q_1t_1}) \not\cong P_{q_2t_2}$ for any $q_1, q_2 \in L_1$, $t_1, t_2 \in L_2$, where we set
$$(P_{qt},p_{qt}) := (M_q,m_q) * (N_t,n_t)$$
for any $q \in L_1$, $t \in L_2$.

Lemma 6.2. *If* $\operatorname{char}(k) \neq 2$, *then there is a non-symmetric independent pair of dense chains of pointed modules in* Λ_2-mod.

Proof. Recall from Proposition 4.1 that the following diagram of k-category isomorphisms is commutative:

$$\begin{tikzcd} \Lambda_1\text{-mod} \arrow[r, "\iota_\bullet"] & \Lambda_2\text{-mod} \\ \Lambda_1\text{-mod} \arrow[r, "\iota_\bullet"] \arrow[u, "\sigma_\bullet"] & \Lambda_2\text{-mod} \arrow[u, "\sigma_\bullet"] \end{tikzcd}$$

It is clear that $\iota_\bullet(M) = M$ as k-vector spaces for any Λ_1-module M, and thus $(\iota_\bullet(M),m)$ is a well-defined pointed module in Λ_2-mod for any pointed module (M,m) in Λ_1-mod. Moreover, $\iota_\bullet(f) = f$ for any Λ_1-homomorphism f, and thus $\iota_\bullet(f) : (\iota_\bullet(M),m) \to (\iota_\bullet(N),n)$ is a well-defined pointed Λ_2-homomorphism for any pointed Λ_1-homomorphism $f : (M,m) \to (N,n)$.

Assume that (U, V) is one of the $\mathbb{Q}$-generating pairs of bands from Theorem 5.10, that is,

$$(U,V) = (\gamma\alpha\beta^{-1}\delta^{-1}, \gamma\delta^{-1}) \quad \text{or} \quad (U,V) = (\beta\alpha^{-1}\beta\alpha^{-1}, \beta\alpha^{-1}\gamma^{-1}\delta\beta\alpha^{-1}).$$

We set $L_1 := \mathcal{L}_V^V(U,V)$, $L_2 := \mathcal{L}_{V^{-1}}^{U^{-1}}(U^{-1}, V^{-1})$ and

$$(M_q, m_q) := (\iota_\bullet(M(q)), z_1^q) \quad \text{and} \quad (N_t, n_t) := (\iota_\bullet(M(t)), z_1^t),$$

for any $q \in L_1$, $t \in L_2$. Then $((M_q, m_q)_{q\in L_1}, (N_t, n_t)_{t\in L_2})$ is an independent pair of dense chains of pointed Λ_2-modules by Theorems 5.3 and 5.7. This pair is non-symmetric by Theorem 5.10 and Lemma 5.6. ∎

Now we prove the existence of an independent pair of dense chains of pointed modules in Λ_3-mod.

THEOREM 6.3. *Assume that* $\mathrm{char}(k) \neq 2$ *and* $((M_q, m_q)_{q\in L_1}, (N_t, n_t)_{t\in L_2})$ *is a non-symmetric pair of dense chains of pointed modules in* Λ_2-mod. *Then*

$$((F_\bullet(M_q), \widetilde{m_q})_{q\in L_1}, (F_\bullet(N_t), \widetilde{n_t})_{t\in L_2})$$

is an independent pair of dense chains of pointed modules in Λ_3-mod.

Proof. First we prove that $(F_\bullet(M_q), \widetilde{m_q})_{q\in L_1}$ and $(F_\bullet(N_t), \widetilde{n_t})_{t\in L_2}$ are dense chains of pointed modules in Λ_3-mod.

The modules $F_\bullet(M_q)$ and $F_\bullet(N_t)$ are indecomposable for any $q \in L_1$, $t \in L_2$ by Corollary 4.3(a), since M_q and N_t are indecomposable and $M_q \ncong \sigma_\bullet(M_q)$, $N_t \ncong \sigma_\bullet(N_t)$ for any $q \in L_1$, $t \in L_2$.

Assume that $\mu_{q,q'} : (M_q, m_q) \to (M_{q'}, m_{q'})$ and $\nu_{t,t'} : (N_t, n_t) \to (N_{t'}, n_{t'})$ are pointed homomorphisms for some $q < q' \in L_1$ and $t < t' \in L_2$. Then there exist pointed homomorphisms

$$F_\bullet(\mu_{q,q'}) : (F_\bullet(M_q), \widetilde{m_q}) \to (F_\bullet(M_{q'}), \widetilde{m_{q'}})$$

and

$$F_\bullet(\nu_{t,t'}) : (F_\bullet(N_t), \widetilde{n_t}) \to (F_\bullet(N_{t'}), \widetilde{n_{t'}}).$$

The pointed modules $(F_\bullet(M_q), \widetilde{m_q})$ and $(F_\bullet(M_{q'}), \widetilde{m_{q'}})$ are not isomorphic by Corollary 4.3(b) for any $q \neq q' \in L_1$. Indeed, $F_\bullet(M_q)$ and $F_\bullet(M_{q'})$ are not isomorphic since M_q, $M_{q'}$ are indecomposable and $M_q \ncong \sigma_\bullet(M_{q'})$; thus $(F_\bullet(M_q), \widetilde{m_q})$ and $(F_\bullet(M_{q'}), \widetilde{m_{q'}})$ are not isomorphic.

Similarly, the pointed modules $(F_\bullet(N_t), \widetilde{n_t})$ and $(F_\bullet(N_{t'}), \widetilde{n_{t'}})$ are not isomorphic for any $t \neq t' \in L_2$.

Consequently, $(F_\bullet(M_q), \widetilde{m_q})_{q\in L_1}$ and $(F_\bullet(N_t), \widetilde{n_t})_{t\in L_2}$ are dense chains of pointed modules in Λ_3-mod. Now we prove that these chains are independent.

There is no pointed homomorphism from $(F_\bullet(M_q), \widetilde{m_q})$ to $(F_\bullet(N_t), \widetilde{n_t})$ and no pointed homomorphism from $(F_\bullet(N_t), \widetilde{n_t})$ to $(F_\bullet(M_q), \widetilde{m_q})$ for any $q \in L_1$, $t \in L_2$ by Lemma 4.5, since there is no pointed homomorphism from (M_q, m_q) to (N_t, n_t) and none from (N_t, n_t) to (M_q, m_q).

The pointed pushout $(F_\bullet(M_q),\widetilde{m_q}) * (F_\bullet(N_t),\widetilde{n_t})$ is indecomposable for any $q \in L_1$, $t \in L_2$ by Lemma 4.4 and Corollary 4.3(a). Indeed, assume that $(M_q, m_q) * (N_t, n_t) = (P_{qt}, p_{qt})$. Then $(F_\bullet(M_q),\widetilde{m_q}) * (F_\bullet(N_t),\widetilde{n_t}) = (F_\bullet(P_{qt}),\widetilde{p_{qt}})$ by Lemma 4.4 and $F_\bullet(P_{qt})$ is indecomposable by Corollary 4.3(a), since P_{qt} is indecomposable and $P_{qt} \ncong \sigma_\bullet(P_{qt})$.

The pointed pushouts $(F_\bullet(M_q),\widetilde{m_q}) * (F_\bullet(N_t),\widetilde{n_t})$ and $(F_\bullet(M_{q'}),\widetilde{m_{q'}}) * (F_\bullet(N_t),\widetilde{n_t})$ are not isomorphic for any $q \neq q' \in L_1$ and $t \in L_2$ by Lemma 4.4 and Corollary 4.3(b). Indeed, $F_\bullet(P_{qt}) \ncong F_\bullet(P_{q't})$ since $P_{qt}, P_{q't}$ are indecomposable and $P_{qt} \ncong \sigma_\bullet(P_{q't})$.

Similar arguments show that the pointed pushouts $(F_\bullet(M_q),\widetilde{m_q}) * (F_\bullet(N_t),\widetilde{n_t})$ and $(F_\bullet(M_q),\widetilde{m_q}) * (F_\bullet(N_{t'}),\widetilde{n_{t'}})$ are not isomorphic for any $q \in L_1$ and $t \neq t' \in L_2$.

Thus, $(F_\bullet(M_q),\widetilde{m_q})_{q\in L_1}$ and $(F_\bullet(N_t),\widetilde{n_t})_{t\in L_2}$ are independent dense chains of pointed modules in Λ_3-mod, and the proof is complete. ■

Proof of Theorem 1.1. The assertion (a) is an immediate consequence of Theorem 6.3, whereas (b) follows by applying (a) and Corollary 3.4. ■

7. Final remarks. In this section we collect some consequences of our results. In particular, we present a technical refinement of Theorem 6.3 (Corollary 7.1), which is important for future applications. Moreover, we show that there exists an independent pair of dense chains of pointed modules over the incidence algebra of the Nazarova–Zavadskij poset $\mathcal{NZ}$ enlarged by a unique maximal element.

We recall from [15] the concept of a prinjective module.

Given two k-algebras A, B and a B-A-bimodule $\mathcal{M}$, we consider the algebra

$$R = \begin{bmatrix} A & 0 \\ \mathcal{M} & B \end{bmatrix}$$

of matrices $\left[\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right]$, $a \in A$, $b \in B$, $m \in \mathcal{M}$, where multiplication is given by

$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \cdot \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ ma' + bm' & bb' \end{bmatrix}.$$

Let

$$e_A = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_B = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}.$$

Observe that if R is the bound quiver algebra kQ/I, then there are convex subquivers Q_A and Q_B of Q such that

$$A \cong kQ_A/(I \cap kQ_A) \quad \text{and} \quad B \cong kQ_B/(I \cap kQ_B)$$

and there are no oriented paths from Q_B to Q_A.

Moreover, every vertex of Q belongs to exactly one of the subquivers Q_A, Q_B, and $\mathcal{M}$ can be identified with the linear subspace of kQ/I generated by the cosets of the paths starting from Q_A and terminating in Q_B, equipped with the natural bimodule structure.

A left finitely generated R-module X is called $\mathcal{M}$-*prinjective* (see [15]) provided $e_A X$ is a projective A-module and $e_B X$ is an injective B-module. We remark that this is a left-module version of the concept of a right prinjective module introduced in [15] and [24, 17.4].

The algebra Λ_3 can be realized in this manner as follows (we are consistent with the notation of Section 4):

$$\begin{bmatrix} A_1 & 0 \\ \mathcal{M}_1 & B_1 \end{bmatrix} \cong \Lambda_3 \cong \begin{bmatrix} A_2 & 0 \\ \mathcal{M}_2 & B_2 \end{bmatrix},$$

where

$$\begin{aligned} A_1 &= (e_{x_{11}} + e_{x_{12}} + e_{x_{21}} + e_{x_{22}})\Lambda_3(e_{x_{11}} + e_{x_{12}} + e_{x_{21}} + e_{x_{22}}), \\ B_1 &= (e_{x_{31}} + e_{x_{32}})\Lambda_3(e_{x_{31}} + e_{x_{32}}), \\ \mathcal{M}_1 &= (e_{x_{31}} + e_{x_{32}})\Lambda_3(e_{x_{11}} + e_{x_{12}} + e_{x_{21}} + e_{x_{22}}), \end{aligned}$$

and

$$\begin{aligned} A_2 &= (e_{x_{11}} + e_{x_{12}})\Lambda_3(e_{x_{11}} + e_{x_{12}}), \\ B_2 &= (e_{x_{21}} + e_{x_{22}} + e_{x_{31}} + e_{x_{32}})\Lambda_3(e_{x_{21}} + e_{x_{22}} + e_{x_{31}} + e_{x_{32}}), \\ \mathcal{M}_2 &= (e_{x_{21}} + e_{x_{22}} + e_{x_{31}} + e_{x_{32}})\Lambda_3(e_{x_{11}} + e_{x_{12}}). \end{aligned}$$

The following corollary is a consequence of Lemma 6.2 and Theorem 6.3.

Corollary 7.1.

(a) *If* $\operatorname{char}(k) \neq 2$, *then there exists an independent pair*

$$((\widetilde{M}_q, \widetilde{m}_q)_{q\in L_1}, (\widetilde{N}_t, \widetilde{n}_t)_{t\in L_2})$$

of dense chains of pointed modules in Λ_3-mod *such that the modules* $\widetilde{M}_q, \widetilde{N}_t$ *are* $\mathcal{M}_1$-*prinjective and* $\widetilde{m}_q$ *(respectively,* $\widetilde{n}_t$*) belongs to* $e_{B_1}\widetilde{M}_q$ *(respectively,* $e_{B_1}\widetilde{N}_t$*) for any* $q \in L_1$, $t \in L_2$.

(b) *If* $\operatorname{char}(k) \neq 2$, *then there exists an independent pair*

$$((\widetilde{M}'_q, \widetilde{m}'_q)_{q\in L'_1}, (\widetilde{N}'_t, \widetilde{n}'_t)_{t\in L'_2})$$

of dense chains of pointed modules in Λ_3-mod *such that the modules* $\widetilde{M}'_q, \widetilde{N}'_t$ *are* $\mathcal{M}_2$-*prinjective and* $\widetilde{m}'_q$ *(respectively,* $\widetilde{n}'_t$*) belongs to* $(e_{x_{21}} + e_{x_{22}})\operatorname{soc}_{B_2}(e_{B_2}\widetilde{M}'_q)$ *(respectively,* $(e_{x_{21}} + e_{x_{22}})\operatorname{soc}_{B_2}(e_{B_2}\widetilde{N}'_t)$*) for any* $q \in L'_1$, $t \in L'_2$.

Proof. (a) We conclude from the proof of Lemma 6.2 that there is a nonsymmetric pair $((M_q, m_q)_{q\in L_1}, (N_t, n_t)_{t\in L_2})$ of dense chains of pointed modules in Λ_2-mod associated to the $\mathbb{Q}$-generating pair of bands $(\gamma\alpha\beta^{-1}\delta^{-1}, \gamma\delta^{-1})$.

We set

$$((\widetilde{M}_q, \widetilde{m}_q)_{q\in L_1}, (\widetilde{N}_t, \widetilde{n}_t)_{t\in L_2}) := ((F_\bullet(M_q), \widetilde{m_q})_{q\in L_1}, (F_\bullet(N_t), \widetilde{n_t})_{t\in L_2}).$$

This is an independent pair of dense chains of pointed modules in Λ_3-mod by Theorem 6.3. One checks directly that, for any string $S \in \Sigma(U, V)$, where $U = \gamma\alpha\beta^{-1}\delta^{-1}$, $V = \gamma\delta^{-1}$, the module $(e_{x_1} + e_{x_2})M(S)$ is a projective $(e_{x_1} + e_{x_2})\Lambda_1(e_{x_1} + e_{x_2})$-module. It follows that the $(e_{x_1} + e_{x_2})\Lambda_2(e_{x_1} + e_{x_2})$-modules $(e_{x_1} + e_{x_2})M_q$, $(e_{x_1} + e_{x_2})N_t$ are projective. Since the algebra B_1 is semisimple, it follows that the modules $\widetilde{M}_q, \widetilde{N}_t$ are $\mathcal{M}_1$-prinjective for any $q \in L_1$, $t \in L_2$.

The conditions $\widetilde{m}_q \in e_{B_1}\widetilde{M}_q$ and $\widetilde{n}_t \in e_{B_1}\widetilde{N}_t$ for any $q \in L_1$, $t \in L_2$, follow easily from the facts that $m_q \in e_{x_3}M_q$ and $n_t \in e_{x_3}N_t$.

(b) From the proof of Lemma 6.2, there is a non-symmetric pair $((M'_q, m'_q)_{q\in L'_1}, (N'_t, n'_t)_{t\in L'_2})$ of dense chains of pointed modules in Λ_2-mod associated to the $\mathbb{Q}$-generating pair of bands $(\beta\alpha^{-1}\beta\alpha^{-1}, \beta\alpha^{-1}\gamma^{-1}\delta\beta\alpha^{-1})$.

We set

$$((\widetilde{M}'_q, \widetilde{m}'_q)_{q\in L'_1}, (\widetilde{N}'_t, \widetilde{n}'_t)_{t\in L'_2}) := ((F_\bullet(M'_q), \widetilde{m'_q})_{q\in L'_1}, (F_\bullet(N'_t), \widetilde{n'_t})_{t\in L'_2}).$$

This is an independent pair of dense chains of pointed modules in Λ_3-mod by Theorem 6.3. As in the previous case, the modules $\widetilde{M}'_q, \widetilde{N}'_t$ are $\mathcal{M}_2$-prinjective for any $q \in L'_1$, $t \in L'_2$, since the modules $(e_{x_2} + e_{x_3})M'_q$, $(e_{x_2} + e_{x_3})N'_t$ are injective over the algebra $(e_{x_2} + e_{x_3})\Lambda_2(e_{x_2} + e_{x_3})$ and A_2 is semisimple.

The conditions

$$\widetilde{m}'_q \in (e_{x_{21}} + e_{x_{22}})\operatorname{soc}_{B_2}(e_{B_2}\widetilde{M}'_q) \quad \text{and} \quad \widetilde{n}'_t \in (e_{x_{21}} + e_{x_{22}})\operatorname{soc}_{B_2}(e_{B_2}\widetilde{N}'_t),$$

for any $q \in L'_1$, $t \in L'_2$ follow easily from the facts that $m'_q \in e_{x_2}M'_q$, $n'_t \in e_{x_2}N'_t$ and $\gamma m'_q = \delta m'_q = 0$, $\gamma n'_t = \delta n'_t = 0$. ∎

The specific independent pairs of dense chains of pointed modules from Corollary 7.1 play a role in applications of our results to pg-critical algebras (see [14]), which will be presented in a subsequent paper.

Now we apply Corollary 7.1(a) to show that there exists an independent pair of dense chains of pointed modules over the incidence algebra of the Nazarova–Zavadskij poset, denoted by $\mathcal{NZ}$ in [24, 15.31], enlarged by a unique maximal element.

More precisely, let $\Gamma := k(\mathcal{NZ})^*$ be the bound quiver algebra kQ/I, where

$$Q := \begin{tikzcd} 1 \arrow[d, "\eta_1"'] \arrow[dr, "\eta_2"] & 2 \arrow[dl, "\eta_3"'] \arrow[d, "\eta_4"] & & \\ 3 \arrow[drr, "\eta_5"'] & 4 \arrow[dr, "\eta_6"] & 5 \arrow[d, "\eta_7"] & 6 \arrow[dl, "\eta_8"] \\ & & * & \end{tikzcd}$$

and $I = \langle \eta_5\eta_1 - \eta_6\eta_2, \eta_5\eta_3 - \eta_6\eta_4 \rangle$. Actually, Γ is just the algebra opposite to the usual incidence algebra of the poset $(\mathcal{NZ})^*$ considered in the context of right modules. But our approach coincides with the traditional (right-module) one on the level of quiver representations.

Assume that $G : \Lambda_3\text{-mod} \to \Gamma\text{-mod}$ is the functor defined on objects of Λ_3-mod as follows:

$$
\begin{tikzcd}
V_{11} \arrow[d, "v_{\alpha_1}"'] \arrow[dr, "v_{\beta_1}" near start] & V_{12} \arrow[d, "v_{\alpha_2}"] \arrow[dl, "v_{\beta_2}"' near start] \\
V_{21} \arrow[d, "v_{\delta_1}"'] \arrow[dr, "v_{\gamma_1}" near start] & V_{22} \arrow[d, "v_{\delta_2}"] \arrow[dl, "v_{\gamma_2}"' near start] \\
V_{31} & V_{32}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
V_{11} \arrow[d, "v_{\alpha_1}"'] \arrow[dr, "v_{\beta_1}" near start] & V_{12} \arrow[d, "v_{\alpha_2}"] \arrow[dl, "v_{\beta_2}"' near start] & & \\
V_{21} \arrow[drr, "w_1"'] & V_{22} \arrow[dr, "w_2"] & V_{31} \arrow[d, "w_3"] & V_{32} \arrow[dl, "w_4"] \\
& & V_{31} \oplus V_{32} &
\end{tikzcd}
$$

where

$$
w_1 = \begin{bmatrix} v_{\delta_1} \\ v_{\gamma_1} \end{bmatrix}, \quad w_2 = \begin{bmatrix} v_{\gamma_2} \\ v_{\delta_2} \end{bmatrix}, \quad w_3 = \begin{bmatrix} \mathrm{id}_{V_{31}} \\ 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 0 \\ \mathrm{id}_{V_{32}} \end{bmatrix}.
$$

The functor is defined on homomorphisms in Λ_3-mod in a natural way.

It is easy to see that $G : \Lambda_3\text{-mod} \to \Gamma\text{-mod}$ is full, faithful and exact.

Assume that (M, m) is a pointed Λ_3-module such that $m \in e_{B_1}M = e_{x_{31}}M \oplus e_{x_{32}}M$. Then there is a Λ_3-homomorphism

$$
\widetilde{\chi}_{(M,m)} : P(x_{31}) \oplus P(x_{32}) \to M
$$

such that $\widetilde{\chi}_{(M,m)}(e_{x_{31}} + e_{x_{32}}) = m$, where $P(x_{31}) = \Lambda_3 e_{x_{31}}$ and $P(x_{32}) = \Lambda_3 e_{x_{32}}$ are indecomposable projectives associated to x_{31} and x_{32}, respectively.

Observe that

$$
G(P(x_{31})) \cong \Gamma e_5, \quad G(P(x_{32})) \cong \Gamma e_6,
$$

hence there is an epimorphism of Γ-modules

$$
\upsilon : \Gamma \to G(P(x_{31})) \oplus G(P(x_{32})).
$$

We set

$$
G(M, m) := (G(M), \overline{m}), \quad \text{where} \quad \overline{m} := (G(\widetilde{\chi}_{(M,m)})\upsilon)(1_\Gamma)
$$

for any pointed Λ_3-module (M, m) such that $m \in e_{B_1}M = e_{x_{31}}M \oplus e_{x_{32}}M$.

Assume that (M, m), (N, n) are pointed Λ_3-modules such that $m \in e_{B_1}M = e_{x_{31}}M \oplus e_{x_{32}}M$, $n \in e_{B_1}N = e_{x_{31}}N \oplus e_{x_{32}}N$ and $f : (M, m) \to (N, n)$ is a pointed Λ_3-homomorphism. Then

$$
f\widetilde{\chi}_{(M,m)} = \widetilde{\chi}_{(N,n)} \quad \text{and thus} \quad G(f)(G(\widetilde{\chi}_{(M,m)})\upsilon) = G(\widetilde{\chi}_{(N,n)})\upsilon.
$$

This implies that $G(f)$ is a pointed Γ-homomorphism $G(M, m) \to G(N, n)$. Observe that since G is full and faithful, any pointed homomorphism from

$G(M,n)$ to $G(N,n)$ is of the form $G(f)$ for some pointed homomorphism $f:(M,m)\to(N,n)$.

In what follows, we write $\overline{m_q},\overline{n_t}$ instead of $\overline{\widetilde{m}_q},\overline{\widetilde{n}_t}$.

Corollary 7.2. *Assume* $\mathrm{char}(k)\neq 2$ *and* $((\widetilde{M}_q,\widetilde{m}_q)_{q\in L_1},(\widetilde{N}_t,\widetilde{n}_t)_{t\in L_2})$ *is an independent pair of dense chains of pointed modules in* Λ_3-mod *satisfying the conditions of Corollary* 7.1(a). *Then*

$$((G(\widetilde{M}_q),\overline{m_q})_{q\in L_1},(G(\widetilde{N}_t),\overline{n_t})_{t\in L_2})$$

is an independent pair of dense chains of pointed modules in Γ-mod. *If, in addition, k is countable, then there exists a super-decomposable pure-injective* Γ*-module.*

Proof. First observe that the pointed modules $(G(\widetilde{M}_q),\overline{m_q})$ and $(G(\widetilde{N}_t),\overline{n_t})$ are well-defined for any $q\in L_1$, $t\in L_2$, since $\widetilde{m}_q\in e_{B_1}\widetilde{M}_q$ and $\widetilde{n}_t\in e_{B_1}\widetilde{N}_t$ by Corollary 7.1(a).

We prove that $(G(\widetilde{M}_q),\overline{m_q})_{q\in L_1}$ and $(G(\widetilde{N}_t),\overline{n_t})_{t\in L_2}$ are dense chains of pointed Γ-modules.

The modules $G(\widetilde{M}_q)$, $G(\widetilde{N}_t)$ are indecomposable for any $q\in L_1$, $t\in L_2$, since $G:\Lambda_3\text{-mod}\to\Gamma\text{-mod}$ preserves indecomposability.

The pointed modules $(G(\widetilde{M}_{q_1}),\overline{m_{q_1}})$ and $(G(\widetilde{N}_{t_2}),\overline{n_{t_2}})$ are not isomorphic for any $q_1,q_2\in L_1$ such that $q_1\neq q_2$, since $(\widetilde{M}_{q_1}\widetilde{m}_{q_1})$ and $(\widetilde{M}_{q_2},\widetilde{m}_{q_2})$ are not isomorphic and G is full and faithful.

Similarly, the pointed modules $(G(\widetilde{N}_{t_1}),\overline{n_{t_1}})$ and $(G(\widetilde{N}_{t_2}),\overline{n_{t_2}})$ are not isomorphic for any $t_1,t_2\in L_2$ such that $t_1\neq t_2$.

Consequently, $(G(\widetilde{M}_q),\overline{m_q})_{q\in L_1}$ and $(G(\widetilde{N}_t),\overline{n_t})_{t\in L_2}$ are dense chains of pointed modules in Γ-mod. Now we prove that these chains are independent.

First, there is no pointed homomorphism from $(G(\widetilde{M}_q),\overline{m_q})$ to $(G(\widetilde{N}_t),\overline{n_t})$ since there is none from $(\widetilde{M}_q,\widetilde{m}_q)$ to $(\widetilde{N}_t,\widetilde{n}_t)$ for any $q\in L_1$, $t\in L_2$. Similarly, there is no pointed homomorphism $(G(\widetilde{N}_t),\overline{n_t})\to(G(\widetilde{M}_q),\overline{m_q})$ for any $q\in L_1$, $t\in L_2$.

We observe that the pointed pushout

$$(P'_{qt},p'_{qt}):=(G(\widetilde{M}_q),\overline{m_q})*(G(\widetilde{N}_t),\overline{n_t})$$

of $(G(\widetilde{M}_q),\overline{m_q})$ and $(G(\widetilde{N}_t),\overline{n_t})$ is isomorphic to $G(P_{qt},p_{qt})$, where (P_{qt},p_{qt}) is the pointed pushout of $(\widetilde{M}_q,\widetilde{m}_q)$ and $(\widetilde{N}_t,\widetilde{n}_t)$. Indeed, since G is exact it preserves pushouts by [16, Proposition 2.1].

It follows that the pair of dense chains $(G(\widetilde{M}_q),\overline{m_q})_{q\in L_1}$, $(G(\widetilde{N}_t),\overline{n_t})_{t\in L_2}$ satisfies the condition (c) of Definition 3.2, since the pair $(\widetilde{M}_q,\widetilde{m}_q)_{q\in L_1}$, $(\widetilde{N}_t,\widetilde{n}_t)_{t\in L_2}$ does.

The remaining assertion follows now by applying Corollary 3.4. ∎

REMARK 7.3. (a) It is easy to see that the poset $\mathcal{NZ}$ in Corollary 7.2 can be replaced by any finite poset I containing $\mathcal{NZ}$ as a full subposet. Thus, in view of [24, Theorem 15.89] and [17, Theorem 13.7], super-decomposable pure-injective modules do exist over the incidence algebra of I^* for any poset I which is wild or of non-polynomial growth in the sense of [24, 15.10], where I^* denotes the enlargement of I by a unique maximal element, provided k is a countable field of characteristic different than 2.

(b) The assumption that k is countable in Theorem 1.1(b) seems to be essential, at least for a proof based on Ziegler's result. The assumption on the characteristic can perhaps be omitted.

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Received 8 March 2011;
revised 21 April 2011 (5476)