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A NOTE ON NAKAI'S CONJECTURE FOR THE RING $K[X_1,\dots,X_n]/(a_1X_1^m+\dots+a_nX_n^m)$

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Abstract. Let k be a field of characteristic zero, $k[X_1,\dots,X_n]$ the polynomial ring, and B the ring $k[X_1,\dots,X_n]/(a_1X_1^m+\dots+a_mX_n^m)$, $0\neq a_i\in k$ for all i and $m,n\in\mathbb{N}$ with $n\geq 2$ and $m\geq 1$. Let $\mathrm{Der}_k^2(B)$ be the B-module of all second order k-derivations of B and $\mathrm{der}_k^2(B)=\mathrm{Der}_k^1(B)+\mathrm{Der}_k^1(B)\,\mathrm{Der}_k^1(B)$ where $\mathrm{Der}_k^1(B)$ is the B-module of k-derivations of B. If $m\geq 2$ we exhibit explicitly a second order derivation $D\in\mathrm{Der}_k^2(B)$ such that $D\notin\mathrm{der}_k^2(B)$ and thus we prove that Nakai's conjecture is true for the k-algebra B.

Introduction. Throughout this paper k denotes a field of characteristic zero.

Let $S=k[X_1,\dots,X_n]$ be the polynomial ring in n variables over a field k and let $A=S/J$ be an affine k-algebra. Let $\mathrm{Der}_k^n(A)$ be the A-module of k-derivations of order $\leq n$ where $1\leq n\in\mathbb{N}$. Let $\mathrm{Der}_k(A)$ be the k-algebra $\bigcup_n\mathrm{Der}_k^n(A)$ and $\mathrm{der}_k(A)$ the subalgebra generated by $\mathrm{Der}_k^1(A)$. The set $\mathrm{der}_k(A)\cap\mathrm{Der}_k^n(A)$ will be denoted by $\mathrm{der}_k^n(A)$. In [1], Grothendieck has shown that $\mathrm{Der}_k(A)=\mathrm{der}_k(A)$ if A is regular. The Nakai conjecture states the converse. In 1986 Singh [5] presented the following conjecture, which is stronger than Nakai's conjecture: If $A=S/(F)$ and $\mathrm{Der}_k^2(A)=\mathrm{der}_k^2(A)$ then A is regular.

Singh's conjecture for a generic affine k-algebra A is not valid. A counterexample can be found in [4].

In this work we prove that Singh's conjecture is true in the following cases:

(1) $B=S/(F)$, where $F=a_1X_1^m+\dots+a_mX_n^m$ with $0\neq a_i\in k$ for all i (Theorem 6).
(2) $C=S/(H)$ where $H\in S$ is homogeneous of degree ≤ 2 (Corollary 7).

1. A set of generators for $\mathrm{Der}_k^1(B)$. Let B be a ring $S/(F)$, where $F=a_1X_1^m+\dots+a_mX_n^m$ with $0\neq a_i\in k$. In this section we give a set of generators for the B-module $\mathrm{Der}_k^1(B)$.

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Let $A = S/I$ be a finitely generated k-algebra. Consider the S-submodule $\mathcal{D}_I = \{D \in \mathrm{Der}^1_k(S);\ D(I) \subseteq I\}$ of $\mathrm{Der}^1_k(S)$. It is well known that the S-homomorphism $\varphi : \mathcal{D}_I \to \mathrm{Der}^1_k(A)$ given by $\varphi(D)(g+I) = D(g)+I$ induces an A-isomorphism of $\mathcal{D}_I/I\,\mathrm{Der}^1_k(S)$ in $\mathrm{Der}^1_k(A)$. From this fact we deduce a version for n variables of a result in three variables, due to O. Zariski, presented by J. Lipman in 1965 in [2].

Before presenting this version let us establish some notation. Given $H \in S = k[X_1, \dots, X_n]$ and $1 \leq i \leq n$, the partial derivative $\partial H/\partial X_i$ is denoted by H_{X_i}. For all pairs $i, j \in \{1, \dots, n\}$ with $i \neq j$, we define the derivation

$$D^H_{ij} = H_{X_i}\frac{\partial}{\partial X_j} - H_{X_j}\frac{\partial}{\partial X_i}$$

on S. Note that $D^H_{ij}(H) = 0$.

Proposition 1. *Let k be a field and let $F \in S = k[X_1, \dots, X_n]$ $(n \geq 2)$ be such that $\{F_{X_1}, \dots, F_{X_n}\}$ is a regular sequence in S. If there is a derivation ∂ on S such that $\partial(F) = \alpha F$ for some α in k^*, then the S-module*

$$\mathcal{D}_F := \{D \in \mathrm{Der}^1_k(S);\ D(F) \in F \cdot S\}$$

is generated by the derivation ∂ and the derivations $D_{ij} = D^F_{ij}$ for $i < j$.

Proof. Let $D \in \mathrm{Der}^1_k(S)$ be such that $D(F) = HF$, with $H \in S$. Because $\partial(F) = \alpha F$, we have $(D-(H/\alpha)\partial)(F) = 0$. Thus, it is sufficient to show that the submodule $\mathcal{D}_0 := \{D \in \mathcal{D}_F;\ D(F) = 0\}$ is generated by $\{D_{ij};\ i < j\}$. Let $\tilde{\mathcal{D}}_0$ be the submodule of $\mathcal{D}_0$ generated by $\{D_{ij};\ i < j\}$. By induction on r, we first prove the following:

Claim. *If $D \in \mathcal{D}_0$ and $0 \leq r \leq n-2$ then there exists $D' \in \tilde{\mathcal{D}}_0$ such that*

$$D - D' = \sum_{j=1}^{n-(r+1)} H_j \frac{\partial}{\partial X_j} \qquad \textit{with } H_j \in S.$$

For $r = 0$, we know that $D = H'_n \frac{\partial}{\partial X_n} + \sum_{j=1}^{n-1} H'_j \frac{\partial}{\partial X_j}$ with $H'_j \in S$. Since $D(F) = 0$ we have $H'_n F_{X_n} \in (F_{X_1}, \dots, F_{X_{n-1}})$. Because $\{F_{X_1}, \dots, F_{X_n}\}$ is a regular sequence, we have $H'_n = \sum_{j=1}^{n-1} G_j F_{X_j}$ with $G_j \in S$. It then follows that

$$D = \Big(\sum_{j=1}^{n-1} G_j F_{X_j}\Big)\frac{\partial}{\partial X_n} + \sum_{j=1}^{n-1}(G_j F_{X_n} - G_j F_{X_n})\frac{\partial}{\partial X_j} + \sum_{j=1}^{n-1} H'_j \frac{\partial}{\partial X_j}$$

$$= \sum_{j=1}^{n-1} G_j\Big(F_{X_j}\frac{\partial}{\partial X_n} - F_{X_n}\frac{\partial}{\partial X_j}\Big) + \sum_{j=1}^{n-1}(H'_j + G_j F_{X_n})\frac{\partial}{\partial X_j} = D' + \sum_{j=1}^{n-1} H_j \frac{\partial}{\partial X_j}$$

with $D' \in \tilde{\mathcal{D}}_0$.

Now suppose that $0 \le r < n-2$ and $D - D' = \sum_{j=1}^{n-(r+1)} H'_j \frac{\partial}{\partial X_j}$ with $D' \in \tilde{\mathcal{D}}_0$. Since $D(F) = D'(F) = 0$, we have

$$H'_{n-(r+1)} F_{X_{n-(r+1)}} \in (F_{X_1}, \ldots, F_{X_{n-(r+2)}})$$

and so $H'_{n-(r+1)} = \sum_{j=1}^{n-(r+2)} G_j F_{X_j}$. By repeating the same process as used in the proof of $D = D' + \sum_{j=1}^{n-1} H_j \frac{\partial}{\partial X_j}$, we obtain

$$D - D' = D'' + \sum_{j=1}^{n-(r+2)} H_j \frac{\partial}{\partial X_j}$$

with $D'' \in \tilde{\mathcal{D}}_0$. This completes the proof of the Claim.

Now take $D \in \mathcal{D}_0$ and $r = n-2$. By the Claim there exists $D' \in \tilde{\mathcal{D}}_0$ and $H_1 \in K^{[n]}$ such that $D - D' = H_1 \frac{\partial}{\partial X_1}$. Since $(D - D')(F) = 0$ we have $H_1 = 0$ and so $D = D' \in \tilde{\mathcal{D}}_0$. ∎

Let us return to the case of the ring B. From now on the derivations D^F_{ij} where $F = a_1 X_1^m + \cdots + a_n X_n^m$ will be denoted by D_{ij}. Note that

$$D_{ij}(X_k) = \begin{cases} -m a_j X_j^{m-1} & \text{if } k = i, \\ m a_i X_i^{m-1} & \text{if } k = j, \\ 0 & \text{if } k \notin \{i, j\}. \end{cases}$$

Observe that $D_{ij}/m \in \mathrm{Der}^1_k(S)$ induces $d_{ij} = a_j x_j^{m-1} \frac{\partial}{\partial x_i} - a_i x_i^{m-1} \frac{\partial}{\partial x_j} \in \mathrm{Der}^1_k(B)$. The derivation

$$E = X_1 \frac{\partial}{\partial X_1} + \cdots + X_n \frac{\partial}{\partial X_n}$$

is called the *Euler derivation* of S. As F is homogeneous of degree m we have $E(F) = mF$. Thus $E \in \mathrm{Der}^1_k(S)$ induces $\varepsilon = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \in \mathrm{Der}^1_k(B)$.

As a consequence of Proposition 1 we obtain the following result:

PROPOSITION 2. *If $F = a_1 X_1^m + \cdots + a_n X_n^m$ then*

$$\mathcal{D}_F := \{D \in \mathrm{Der}^1_k(S);\ D(F) \in F \cdot S\}$$

is generated by the Euler derivation E and the derivations D_{ij}, $i < j$. In particular the B-module $\mathrm{Der}^1_k(B)$ is generated by the derivation ε and by the derivations d_{ij} for $i < j$.

Proof. Given Proposition 1 it is sufficient to observe that $\{X_1^{m-1}, \ldots, X_n^{m-1}\}$ is a regular sequence and $E(F) = mF$. ∎

2. Nakai's conjecture for the ring B. In this section, if $m \ge 2$, we exhibit explicitly a derivation $D \in \mathrm{Der}^2_k(B)$ such that $D \notin \mathrm{der}^2_k(B)$. From this, we will be able to verify that Singh's conjecture is true for the cases (1) and (2) as mentioned in the introduction.

The main result of this section will be a consequence of several auxiliary results.

LEMMA 3. *Let k be a field, $S = k[X_1, \dots, X_n]$, and $F = a_1X_1^m + a_2X_2^m + \dots + a_nX_n^m \in S$ with $m \geq 2$ and $a_i \in k \setminus \{0\}$. Let $d, d' \in \mathrm{Der}_k^1(S)$ and $D \in \mathrm{Der}_k^2(S)$.*

(a) *If $d(F) \in (F)$ then, for every i,*

$$d(X_i) \in J_i = (X_1^{m-1}, \dots, X_{i-1}^{m-1}, X_i, X_{i+1}^{m-1}, \dots, X_n^{m-1}) \text{ and } d(J_i) \subseteq J_i.$$

(b) *If $d(F), d'(F) \in (F)$ then $(d' \circ d)(X_1) \in J_1 = (X_1, X_2^{m-1}, \dots, X_n^{m-1})$.*

(c) *If $d(F), d'(F) \in (F)$ then $d(X_1F) = HF$ and $(d' \circ d)(X_1F) = GF$, with $H, G \in J_1$.*

(d) *If $2 \leq l \in \mathbb{N}$ then, for every $j \in \{1, \dots, n\}, D(X_1X_j^l) \in (X_1, X_j^{l-1})$ and $D(X_1F) \in J_1$.*

Proof. (a) By Proposition 2 we have $d = \sum_{i<j} H_{ij}D_{ij} + GE$, where $H_{ij} \in S$,

$$D_{ij} = a_iX_i^{m-1}\frac{\partial}{\partial X_j} - a_jX_j^{m-1}\frac{\partial}{\partial X_i} \quad \text{and} \quad E = X_1\frac{\partial}{\partial X_1} + \dots + X_n\frac{\partial}{\partial X_n}.$$

Observe that given i we have $E(X_i) = X_i$, $D_{lj}(X_i) = 0$ if $i \notin \{l, j\}$, $D_{ij}(X_i) = -a_jX_j^{m-1}$ if $i < j$, and $D_{ji}(X_i) = a_jX_j^{m-1}$ if $j < i$. Thus $d(X_i) \in J_i$. Now because for every $j \neq i$ we have $d(X_j^{m-1}) = (m-1)X_j^{m-2}d(X_j)$ and $d(X_j) \in J_j$, we conclude that $d(J_i) \subseteq J_i$.

(b) and (c) are direct consequences of (a).

(d) Induction on $l \geq 2$. By Nakai's definition of higher order derivations [3], we have, for $l = 2$,

$$D(X_1X_j^2) = X_1D(X_j^2) + 2X_jD(X_1) - 2X_1X_jD(X_j) - X_j^2D(X_1) \in (X_1, X_j),$$

and for $l \geq 3$,

$$\begin{aligned} D(X_1X_j^l) &= D(X_1X_jX_j^{l-1}) \\ &= X_1D(X_j^l) + X_jD(X_1X_j^{l-1}) + X_j^{l-1}D(X_1X_j) \\ &\quad - X_1X_jD(X_j^{l-1}) - X_1X_j^{l-1}D(X_j) - X_j^lD(X_1). \end{aligned}$$

Since $D(X_1X_j^{l-1}) \in (X_1, X_j^{l-2})$, we have $D(X_1X_j^l) \in (X_1, X_j^{l-1})$. Therefore for $l = m \geq 2$ we obtain $D(X_1F) \in J_1$. ■

LEMMA 4. *In the notation of Lemma 3, let*

$$D = d + \sum_{i=1}^{s} d_i' \circ d_i + FD',$$

where $\{d, d_i', d_i;\ 1 \leq i \leq s\} \subset \mathrm{Der}_k^1(S)$ and $D' \in \mathrm{Der}_k^2(S)$. If $d(F) \in (F)$ and

$\{d_i(F), d_i'(F)\} \subset (F)$ *for every* i *then*

$$D(X_1F) = HF \quad \text{with } H \in J_1 = (X_1, X_2^{m-1}, \dots, X_n^{m-1}).$$

Proof. By Lemma 3(c), we have

$$d(X_1F) + \sum_{i=1}^{s}(d_i' \circ d_i)(X_1F) = H_1F \quad \text{with } H_1 \in J_1,$$

By Lemma 3(d), $D'(X_1F) \in J_1$. Thus

$$D(X_1F) = d(X_1F) + \sum_{i=1}^{s} d_i' \circ d_i(X_1F) + FD'(X_1F) = HF,$$

where $H = H_1 + D'(X_1F) \in J_1$. ∎

Since $B = S/(F) = S/(G)$ where $S = k[X_1, \dots, X_n]$, $F = a_1X_1^m + a_2X_2^m + \dots + a_nX_n^m$ and $G = X_1^m + (a_2/a_1)X_2^m + \dots + (a_n/a_1)X_n^m$, henceforth we will assume that

$$F = X_1^m + a_2X_2^m + \dots + a_nX_n^m \quad \text{with } a_j \neq 0.$$

LEMMA 5. *Assume that* $m \geq 2$ *and let* D *be the second order* k*-derivation of* S *given by*

$$D = -(m-1)(n-2)G\frac{\partial}{\partial X_1} - X_1G\frac{\partial^2}{\partial X_1^2} - 2G\sum_{j=2}^{n} X_j \frac{\partial^2}{\partial X_1 \partial X_j} + X_1^{m-1}\sum_{j=2}^{n} \frac{G}{a_jX_j^{m-2}}\frac{\partial^2}{\partial X_j^2}$$

where

$$G = \prod_{j\geq 2} \frac{1}{m(m-1)}\frac{\partial^2 F}{\partial X_j^2} = a_2 \cdots a_n X_2^{m-2} \cdots X_n^{m-2}.$$

Then

$$D(F) = 0, \quad D(X_1F) = -(2m + (m+1)(n-1))GF, \quad D(X_jF) = 0,\ j \geq 2.$$

Proof. Since $\frac{\partial^2(F)}{\partial X_1 \partial X_j} = 0$ and $\frac{\partial^2(F)}{\partial X_j^2} = m(m-1)a_jX_j^{m-2}$ for $j \geq 2$, we have

$$D(F) = -m(m-1)(n-2)GX_1^{m-1} - m(m-1)GX_1^{m-1} + X_1^{m-1}\sum_{j=2}^{n}\frac{G}{a_jX_j^{m-2}}\frac{\partial^2 F}{\partial X_j^2}.$$

Thus

$$D(F) = -m(m-1)(n-1)GX_1^{m-1} + X_1^{m-1}\sum_{j=2}^{n} m(m-1)G = 0.$$

Now we calculate

$$\begin{aligned} D(X_1F) &= D(X_1^{m+1}) + D\Big(X_1\Big(\sum_{j=2}^{n} a_jX_j^m\Big)\Big) \\ &= -(m-1)(n-2)G((m+1)X_1^m - m(m+1)X_1^mG + \sum_{j=2}^{n} a_jX_j^m \\ &\quad - 2mG\sum_{j=2}^{n} a_jX_j^m + X_1^m\sum_{j=2}^{n} m(m-1)G \\ &= -(2m+(m+1)(n-2))GX_1^m \\ &\quad - (2m+(m+1)(n-2))G\Big(\sum_{j=2}^{n} a_jX_j^m\Big), \end{aligned}$$

thus $D(X_1F) = -(2m+(m+1)(n-2))GF$.

Analogously one can prove that $D(X_jF) = 0$ for $j \geq 2$. ∎

The main result of this paper is

THEOREM 6. *Let $B = S/(F)$ where $S = k[X_1,\dots,X_n]$, $n \geq 2$ and $F = X_1^m + a_2X_2^m + \cdots + a_nX_n^m$ with $0 \neq a_j \in k$ and $m \geq 1$. Then*

$$\mathrm{Der}_k^2(B) = \mathrm{der}_k^2(B) \quad \textit{if and only if} \quad m = 1.$$

Proof. The sufficiency is a direct consequence of the fact that for $m = 1$, the ring $B = k[X_2,\dots,X_n]$ is a polynomial ring over k.

For the proof of the necessity suppose $m \geq 2$ and take the derivation $D \in \mathrm{Der}_k^2(S)$ defined in Lemma 5. Thus by Lemma 5, we have $D(F) = 0$ and $[D, X_i](F) \in (F)$ for every i. Then by [5, Prop. 2.10] we get $D((F)) \subseteq (F)$. Therefore D induces a derivation $\overline{D} \in \mathrm{Der}_k^2(B)$ defined by

$$\overline{D}(G + (F)) = D(G) + (F).$$

We claim that $\overline{D} \notin \mathrm{der}_k^2(B)$. Suppose that $\overline{D} \in \mathrm{der}_k^2(B)$. Then

$$D = d + \sum_{i=1}^{s} d_i' \circ d_i + FD'$$

with $D' \in \mathrm{Der}_k^2(S)$ and $\{d, d_i', d_i;\ 1 \leq i \leq s\} \subset \mathrm{Der}_k^1(S)$ with $d(F) \in (F)$ and $\{d_i(F), d_i'(F)\} \subset (F)$ for every i. Thus by Lemma 4,

$$D(X_1F) = HF \quad \text{with } H \in J_1 = (X_1, X_2^{m-1},\dots,X_n^{m-1}).$$

But, by Lemma 5, $D(X_1F) = -(2m+(m+1)(n-2))GF$, where

$$G = a_2\cdots a_nX_2^{m-2}\cdots X_n^{m-2} \notin J_1.$$

This contradiction proves our claim, and this proves the theorem. ∎

Corollary 7. *If* $H \in S = k[X_1, \ldots, X_n]$ *is a homogeneous polynomial of degree* m *with* $1 \leq m \leq 2$ *and* $C = S/(H)$ *then*

$$\mathrm{Der}_k^2(C) = \mathrm{der}_k^2(C) \quad \textit{if and only if} \quad m = 1.$$

Proof. This is a consequence of Theorem 6 and of the fact that if $m = 2$ then there exists a linear change of variables $X_i = \sum_j b_{ij} Y_j$, $b_{ij} \in k$, $1 \leq i \leq n$, such that $H = Y_1^2 + a_2 Y_2^2 + \cdots + a_s Y_s^2$, $s \leq n$. ∎

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