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WEIGHTED COMPOSITION OPERATORS ON WEIGHTED LORENTZ SPACES

BY

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Abstract. The boundedness, compactness and closedness of the range of weighted composition operators acting on weighted Lorentz spaces $L(p,q,wd\mu)$ for $1 < p \leq \infty$, $1 \leq q \leq \infty$ are characterized.

1. Introduction. Let (X, Σ, μ) and (Y, Γ, υ) be two σ-finite measure space. A measurable transformation $T : Y \to X$ is said to be *nonsingular* if $\upsilon(T^{-1}(A)) = 0$, whenever $A \in \Sigma$ with $\mu(A) = 0$. In this case, we write $\upsilon \circ T^{-1} \ll \mu$. Let $u : Y \to \mathbb{C}$ be a measurable function. Then the linear transformation

$$W = W_{u,T} : L(X, \Sigma, \mu) \to L(Y, \Gamma, \upsilon)$$

is defined as

$$W(f)(x) = W_{u,T}(f)(x) = u(T(x)) \cdot f(T(x)) \tag{1.1}$$

for all $x \in Y$ and $f \in L(X, \Sigma, \mu)$ where $L(X, \Sigma, \mu)$ and $L(Y, \Gamma, \upsilon)$ are the linear spaces of all μ-measurable and υ-measurable functions on X and Y, respectively. Here, the nonsingularity of T guarantees that the operator W is well defined as a mapping of equivalence classes of functions. In the case when W maps $L(\mu)$ into $L(\upsilon)$, we call $W = W_{u,T}$ the *weighted composition operator* induced by the pair (u, T). If $u \equiv 1$, then $W \equiv C_T$ is called the *composition operator* induced by T. If T is the identity mapping, then $W \equiv M_u : f \mapsto u \cdot f$ is the multiplication operator induced by u.

These simple operators have a wide range of applications in ergodic theory, dynamical systems, etc. The study of (weighted) composition and multiplication operators has a long history. From books on functional analysis and papers related to these operators, one can learn many properties of these operators on various function spaces including Lebesgue and Lorentz spaces. For the study of these operators acting on Lebesgue and Lorentz spaces, see

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[C, HKK, JP, K1, K2, SM, T, TY] and [ADV1, ADV2, ADV3, KK1, KK2], respectively.

2. Preliminaries. Throughout the paper $X = (X, \Sigma, \mu)$ will stand for a σ-finite measure space, $L(\mu)$ will denote the linear space of all equivalence classes of Σ-measurable functions on X, and χ_A will be used for the characteristic function of a set A. For any two nonnegative expressions (i.e. functions or functionals), A and B, the symbol $A \prec B$ means that $A \le cB$ for some positive constant c independent of the variables in the expressions A and B. If $A \prec B$ and $B \prec A$, we write $A \approx B$ and say that A and B are *equivalent*.

Let w be a *weight function*, i.e. a measurable, complex-valued and locally bounded function on X, satisfying $w(x) \ge 1$ for all $x \in X$. Weighted Lorentz spaces (or Lorentz spaces over weighted measure spaces) $L(p, q, wd\mu)$ are studied and discussed in [DG, MNS]. The *distribution function* of a complex-valued measurable function f defined on the measure space $(X, wd\mu)$ is

$$\lambda_{f,w}(y) = w\{x \in X : |f(x)| > y\} = \int\limits_{\{x\in X:\,|f(x)|>y\}} w(x)\,d\mu(x), \qquad y \ge 0.$$

The *nonnegative rearrangement* of f is given by

$$f_w^*(t) = \inf\{y > 0 : \lambda_{f,w}(y) \le t\} = \sup\{y > 0 : \lambda_{f,w}(y) > t\}, \qquad t \ge 0,$$

where we assume that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. Also the *average function* of f on $(0, \infty)$ is given by

$$f_w^{**}(t) = \frac{1}{t}\int\limits_0^t f_w^*(s)\,ds.$$

Note that $\lambda_{f,w}(\cdot)$, $f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are nonincreasing and right continuous functions. The *weighted Lorentz space* $L(p, q, wd\mu)$ is the collection of all functions f such that $\|f\|_{p,q}^* < \infty$, where

$$\|f\|_{p,q,w}^* = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int\limits_0^\infty t^{q/p-1}[f_w^*(t)]^q\,dt\right)^{1/q}, & 0 < p, q < \infty, \\ \sup\limits_{t>0} t^{1/p} f_w^*(t), & 0 < p < q = \infty. \end{cases} \tag{2.1}$$

In general, however, $\|\cdot\|_{p,q,w}^*$ is not a norm since the Minkowski inequality may fail. But by replacing f_w^* with f_w^{**} in (2.1), we find that $L(p, q, wd\mu)$ is a Banach space with the norm $\|\cdot\|_{p,q,w}$ defined by

$$\|f\|_{p,q,w} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int\limits_0^\infty t^{q/p-1}[f_w^{**}(t)]^q\,dt\right)^{1/q}, & 0 < p, q < \infty, \\ \sup\limits_{t>0} t^{1/p} f_w^{**}(t), & 0 < p < q = \infty. \end{cases}$$

If $1 < p \le \infty$ and $1 \le q \le \infty$ then

$$\|f\|^*_{p,q,w} \le \|f\|_{p,q,w} \le \frac{p}{p-1}\|f\|^*_{p,q,w},$$

where the first inequality is an immediate consequence of the fact that $f^*_w \le f^{**}_w$, and the second follows from the Hardy inequality. For more on weighted Lorentz spaces one can refer to [DG], [MNS] and references therein.

The main aim of this paper is to study the boundedness of weighted composition operators between two weighted Lorentz spaces. This paper is motivated by the interesting work of S. C. Arora, G. Datt, S. Verma and R. Kumar in [ADV2], [ADV3] and [K2]. We mostly and frequently benefit from the techniques used in these papers.

3. Boundedness

THEOREM 3.1. *Let $T : X \to X$ be a nonsingular measurable transformation and w_1, w_2 be two weight functions. Then T induces a (bounded) composition operator C_T from $L(p,q,w_1 d\mu)$ into $L(p,q,w_2 d\mu)$ for $1 < p \le \infty$ and $1 \le q \le \infty$ if and only if there exists some $M > 0$ such that*

$$(w_2 \circ T^{-1})(E) \le M w_1(E) \quad \text{for all } E \in \Sigma.$$

Proof. Assume that $1 < p \le \infty$, $1 \le q \le \infty$ and C_T is bounded as stated. Let $E \in \Sigma$ with $w_1(E) < \infty$. The nonincreasing rearrangement of χ_E equals

$$(\chi_E)^*_{w_1}(t) = \chi_{[0,w_1(E))}(t).$$

It follows that

$$(\chi_E)^{**}_{w_1}(t) = \frac{1}{t}\int_0^t (\chi_E)^*_{w_1}(s)\,ds = \begin{cases} 1 & \text{if } 0 \le t < w_1(E), \\ \frac{1}{t}w_1(E) & \text{if } t \ge w_1(E). \end{cases}$$

Therefore

$$\begin{aligned} \|\chi_E\|^q_{p,q,w_1} &= \frac{q}{p}\int_0^\infty t^{q/p-1}[(\chi_E)^{**}_{w_1}(t)]^q\,dt \\ &= \frac{q}{p}\int_0^{w_1(E)} t^{q/p-1}\,dt + \frac{q}{p}\int_{w_1(E)}^\infty t^{q/p-1}\left[\frac{1}{t}w_1(E)\right]^q dt \\ &= (w_1(E))^{q/p} + \frac{1}{p-1}(w_1(E))^{q/p} = p'(w_1(E))^{q/p}, \end{aligned} \tag{3.1}$$

where $1/p + 1/p' = 1$. Consequently, $\chi_E \in L(p,q,w_1 d\mu)$ and

$$\begin{aligned} (w_2 \circ T^{-1})(E) &= w_2(T^{-1}(E)) = (p')^{-p/q}\|\chi_{T^{-1}(E)}\|^p_{p,q,w_2} \\ &= (p')^{-p/q}\|\chi_E \circ T\|^p_{p,q,w_2} = (p')^{-p/q}\|C_T\chi_E\|^p_{p,q,w_2} \\ &\le (p')^{-p/q}\|C_T\|^p\|\chi_E\|^p_{p,q,w_1} = \|C_T\|^p w_1(E) \end{aligned}$$

by (3.1). Hence

$$(w_2 \circ T^{-1})(E) \le M w_1(E),$$

where $M = \|C_T\|^p$. If $w_1(E) = \infty$, then the inequality is trivial. For $q = \infty$, we write

$$\|\chi_E\|_{p,\infty,w_1} = \sup_{t>0} t^{1/p} (\chi_E)^{**}_{w_1}(t) = (w_1(E))^{1/p} \tag{3.2}$$

and by (3.2),

$$(w_2 \circ T^{-1})(E) = \|C_T \chi_E\|^p_{p,\infty,w_2} \le \|C_T\|^p w_1(E).$$

Conversely, suppose there is a constant $M > 0$ such that $(w_2 \circ T^{-1})(E)$ $\le M w_1(E)$ for all $E \in \Sigma$. For any $f \in L(p, q, w_1 d\mu)$ the distribution of $C_T f = f \circ T$ is

$$\begin{aligned} \lambda_{f\circ T, w_2}(y) &= w_2\{x \in X : |f(T(x))| > y\} \\ &= (w_2 \circ T^{-1})\{x \in X : |f(x)| > y\} \\ &\le M w_1\{x \in X : |f(x)| > y\} = M \lambda_{f,w_1}(y). \end{aligned}$$

Therefore, we get

$$\{s > 0 : \lambda_{f,w_1}(s) \le t\} \subseteq \{s > 0 : \lambda_{f\circ T,w_2}(s) \le Mt\},$$

$f^*_{w_1}(t) \ge (f \circ T)^*_{w_2}(Mt)$ and so $f^{**}_{w_1}(t) \ge (f \circ T)^{**}_{w_2}(Mt)$ for all $t > 0$. From these inequalities, for $f \in L(p, q, w_1 d\mu)$, we deduce that

$$\begin{aligned} \|C_T f\|^q_{p,q,w_2} &= \frac{q}{p} \int_0^\infty t^{q/p-1} [(C_T f)^{**}_{w_2}(t)]^q \, dt = \frac{q}{p} \int_0^\infty t^{q/p-1} [(f \circ T)^{**}_{w_2}(t)]^q \, dt \\ &= M^{q/p} \frac{q}{p} \int_0^\infty t^{q/p-1} [(f \circ T)^{**}_{w_2}(Mt)]^q \, dt \\ &\le M^{q/p} \frac{q}{p} \int_0^\infty t^{q/p-1} [f^{**}_{w_1}(t)]^q \, dt = M^{q/p} \|f\|^q_{p,q,w_1}. \end{aligned}$$

This says that C_T is bounded for $q \ne \infty$. For $q = \infty$ and $1 < p \le \infty$, we have

$$\begin{aligned} \|C_T f\|_{p,\infty,w_2} &= \sup_{t>0} t^{1/p} (C_T f)^{**}_{w_2}(t) = \sup_{t>0} t^{1/p} (f \circ T)^{**}_{w_2}(t) \\ &= M^{1/p} \sup_{t>0} t^{1/p} (f \circ T)^{**}_{w_2}(Mt) \le M^{1/p} \sup_{t>0} t^{1/p} f^{**}_{w_1}(t) \\ &= M^{1/p} \|f\|_{p,\infty,w_1}. \end{aligned}$$

As a result, C_T is bounded from $L(p, q, w_1 d\mu)$ into $L(p, q, w_2 d\mu)$ for $1 < p$ $\le \infty$ and $1 \le q \le \infty$. Morever,

$$\|C_T\| = (\inf\{M > 0 : (w_2 \circ T^{-1})(E) \le M w_1(E)\})^{1/p}. \ \blacksquare$$

THEOREM 3.2. *Let $u : X \to \mathbb{C}$ be a measurable function and w_1, w_2 be two weight functions. Let $T : X \to X$ be a nonsingular measurable transformation such that the Radon–Nikodym derivative $f_T = d\mu(w_2(T^{-1}))/d\mu w_1$ is essentially bounded, i.e. $f_T \in L^\infty$. If $u \in L^\infty(\mu)$, then the pair (u,T) induces a bounded weighted composition operator $W = W_{u,T}$ from $L(p,q,w_1 d\mu)$ into $L(p,q,w_2 d\mu)$ for $1 < p \le \infty$ and $1 \le q \le \infty$.*

Proof. Let $M = \|f_T\|_\infty$. Then for any $f \in L(p,q,w_1 d\mu)$, the distribution function of $Wf = W_{u,T}(f)$, where $W_{u,T}(f) = u \circ T \cdot f \circ T$, satisfies

$$\begin{aligned}\lambda_{Wf,w_2}(y) &= w_2\{x \in X : |Wf(x)| > y\}\\ &= w_2\{x \in X : |u(T(x))f(T(x))| > y\}\\ &= w_2(T^{-1})\{x \in X : |u(x)f(x)| > y\}\\ &\le Mw_1\{x \in X : \|u\|_\infty |f(x)| > y\} = M\lambda_{\|u\|_\infty f,w_1}(y).\end{aligned}$$

Therefore, for each $t \ge 0$, we have

$$\{s > 0 : \lambda_{\|u\|_\infty f,w_1}(s) \le t/M\} \subseteq \{s > 0 : \lambda_{Wf,w_2}(s) \le t\}$$

and this gives

$$\begin{aligned}(Wf)^*_{w_2}(t) &= \inf\{s > 0 : \lambda_{Wf,w_2}(s) \le t\}\\ &\le \inf\{s > 0 : \lambda_{\|u\|_\infty f,w_1}(s) \le t/M\}\\ &= \inf\{s > 0 : w_1\{x \in X : \|u\|_\infty |f(x)| > s\} \le t/M\}\\ &= \|u\|_\infty f^*_{w_1}(t/M).\end{aligned}$$

Also, we have

$$(Wf)^{**}_{w_2}(t) \le \|u\|_\infty f^{**}_{w_1}(t/M).$$

For $q \ne \infty$, we get

$$\begin{aligned}\|Wf\|^q_{p,q,w_2} = \|W_{u,T}f\|^q_{p,q,w_2} &= \frac{q}{p}\int_0^\infty t^{q/p-1}[(Wf)^{**}_{w_2}(t)]^q\,dt\\ &\le \|u\|^q_\infty \frac{q}{p}\int_0^\infty t^{q/p-1}[f^{**}_{w_1}(t/M)]^q\,dt\\ &= \|u\|^q_\infty M^{q/p}\|f\|^q_{p,q,w_1},\end{aligned}$$

and for $q = \infty$,

$$\begin{aligned}\|Wf\|_{p,\infty,w_2} &= \sup_{t>0} t^{1/p}(Wf)^{**}_{w_2}(t) \le \sup_{t>0} t^{1/p}\|u\|_\infty f^{**}_{w_1}(t/M)\\ &= \|u\|_\infty \sup_{t>0} t^{1/p} f^{**}_{w_1}(t/M) = M^{1/p}\|u\|_\infty \|f\|_{p,\infty,w_1}.\end{aligned}$$

Therefore, $\|W\| \le M^{1/p}\|u\|_\infty$ and so $W = W_{u,T}$ is a bounded weighted composition operator from $L(p,q,w_1 d\mu)$ into $L(p,q,w_2 d\mu)$ for $1 < p \le \infty$ and $1 \le q \le \infty$. ∎

THEOREM 3.3. *Let $u : X \to \mathbb{C}$ be a measurable function and w_1, w_2 be two weight functions. Let $T : X \to X$ be a nonsingular measurable transformation such that $T(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon > 0$ where $E_\varepsilon = \{x \in X : |u(x)| > \varepsilon\}$. If $w_1 \approx w_2$ and the weighted composition operator $W = W_{u,T}$ from $L(p,q,w_1 d\mu)$ into $L(p,q,w_2 d\mu)$ is bounded for some $1 < p \le \infty$ and $1 \le q \le \infty$, then $u \in L^\infty(\mu)$.*

Proof. Assume the contrary, i.e. $u \notin L^\infty(\mu)$. Then, for each $n \in \mathbb{N}$, the set $E_n = \{x \in X : |u(x)| > n\}$ has a positive measure. Since $T(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon > 0$, we have $T(E_n) \subseteq E_n$ and equivalently $\chi_{E_n} \le \chi_{T^{-1}(E_n)}$. From this, we get

$$\begin{aligned}(3.3)\qquad \{x \in X : |\chi_{E_n}(x)| > s\} &\subseteq \{x \in X : |\chi_{T^{-1}(E_n)}(x)| > s\}\\ &\subseteq \{x \in X : |u(T(x))\chi_{T^{-1}(E_n)}(x)| > ns\}.\end{aligned}$$

Since $w_1 \approx w_2$, this yields

$$\begin{aligned}(3.4)\qquad (W\chi_{E_n})^*_{w_2}(t) &= \inf\{s > 0 : \lambda_{W\chi_{E_n},w_2}(s) \le t\}\\ &= \inf\{s > 0 : w_2\{x \in X : |u(T(x)) \cdot \chi_{E_n}(T(x))| > s\} \le t\}\\ &= n\inf\{s > 0 : w_2\{x \in X : |u(T(x)) \cdot \chi_{T^{-1}(E_n)}(x)| > ns\} \le t\}\\ &\ge n\inf\{s > 0 : cw_1\{x \in X : |u(T(x)) \cdot \chi_{T^{-1}(E_n)}(x)| > ns\} \le t\}\\ &\ge nc\inf\{s > 0 : w_1\{x \in X : |\chi_{T^{-1}(E_n)}(x)| > s\} \le t\}\\ &\ge nc\inf\{s > 0 : w_1\{x \in X : |\chi_{E_n}(x)| > s\} \le t\} = nc(\chi_{E_n})^*_{w_1}(t).\end{aligned}$$

Consequently,

$$(W\chi_{E_n})^{**}_{w_2}(t) \ge nc(\chi_{E_n})^{**}_{w_1}(t)$$

and

$$\|W\chi_{E_n}\|_{p,q,w_2} \ge nc\|\chi_{E_n}\|_{p,q,w_1},$$

which contradicts our assumption. ■

4. Compactness and closed range. In this section, we discuss the compactness and the closed range of the weighted composition operator $W = W_{u,T} : f \mapsto u \circ T \cdot f \circ T$ from $L(p,q,w_1 d\mu)$ into $L(p,q,w_2 d\mu)$. We will use two conditions:

(i) $w_1 \prec w_2$ if $p = 1$, $0 < q \le 1$ or $1 \le q \le p < \infty$;
(ii) $w_1 \approx w_2$ if $1 < p \le \infty$, $1 \le q \le \infty$.

Suppose $w_1 \prec w_2$, i.e. there is a constant c such that $w_1(x) \le cw_2(x)$ for all $x \in X$. By using [DG, Proposition 2.7], we know that $L(p,q,w_2 d\mu) \subseteq L(p,q,w_1 d\mu)$ whenever $p = 1$, $0 < q \le 1$ or $1 \le q \le p < \infty$, with $\|f\|_{p,q,w_1} \prec \|f\|_{p,q,w_2}$ for all $f \in L(p,q,w_2 d\mu)$.

Assume that (i) or (ii) is satisfied. Let $T : X \to X$ be a nonsingular measurable transformation with the Radon–Nikodym derivative $f_T = d\mu(w_2(T^{-1}))/d\mu w_1$. If $f_T \in L^\infty(\mu)$ with $\|f_T\|_\infty = k$, then we get

$$\begin{aligned}(Wf)^*_{w_2}(kt) &= \inf\{s>0:\lambda_{Wf,w_2}(s)\le kt\}\\ &= \inf\{s>0: w_2\{x\in X:|u(T(x))f(T(x))|>s\}\le kt\}\\ &= \inf\{s>0:(w_2\circ T^{-1})\{x\in X:|(u\cdot f)(x)|>s\}\le kt\}\\ &\le \inf\{s>0:w_1\{x\in X:|(u\cdot f)(x)|>s\}\le t\} = (M_uf)^*_{w_1}(t),\end{aligned}$$

and similarly $(Wf)^{**}_{w_2}(kt)\le (M_uf)^{**}_{w_1}(t)$ for all $f\in L(p,q,w_1d\mu)$ and $t>0$. Therefore, we obtain

$$\begin{aligned}\|Wf\|^q_{p,q,w_2} &= \frac{q}{p}\int_0^\infty t^{q/p-1}[(Wf)^{**}_{w_2}(t)]^q\,dt\\ &= \frac{q}{p}\int_0^\infty k^{q/p-1}t^{q/p-1}[(Wf)^{**}_{w_2}(kt)]^q k\,dt\\ &\le k^{q/p}\,\frac{q}{p}\int_0^\infty t^{q/p-1}[(M_uf)^{**}_{w_1}(t)]^q\,dt = k^{q/p}\|M_uf\|^q_{p,q,w_1}\\ &\prec k^{q/p}\|M_uf\|^q_{p,q,w_2}\end{aligned} \tag{4.1}$$

under condition (i) or (ii). If f_T is bounded away from zero on $S=\{x: u(x)\neq 0\}$, i.e. $f_T>\delta$ almost everywhere on S for some $\delta>0$, then we can write

$$w_2(T^{-1}(E)) = \int_E f_Tw_1\,d\mu \ge \delta w_1(E)$$

for all $E\in\Sigma$ with $E\subseteq S$. Therefore, we get

$$\|Wf\|^q_{p,q,w_2}\succ \delta^{q/p}\|M_uf\|^q_{p,q,w_2} \tag{4.2}$$

by simple computations. Hence, for each $f\in L(p,q,w_1d\mu)$, under condition (i) or (ii), we have

$$\|Wf\|_{p,q,w_2}\approx \|M_uf\|_{p,q,w_2} \tag{4.3}$$

by (4.1) and (4.2) whenever $f_T\in L^\infty(\mu)$ and f_T is bounded away from zero. By [ADV1, Theorem 3.1], [HKK, Theorem 2.4] and (4.3), we obtain the following theorem.

THEOREM 4.1. *Let $T: X\to X$ be a nonsingular measurable transformation with f_T in $L^\infty(\mu)$ and bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded from $L(p,q,w_1d\mu)$ into $L(p,q,w_2d\mu)$ under condition* (i) *or* (ii)*. Then the following are equivalent:*

(a) *$W_{u,T}$ is compact,*
(b) *M_u is compact,*
(c) *$L(p,q,w_1d\mu)(A)$ is finite-dimensional for each $\varepsilon>0$, where*

$$L(p,q,w_1d\mu)(A)=\{f\chi_A: f\in L(p,q,w_1d\mu)\},\qquad A=\{x\in X:|u(x)|\ge\varepsilon\}.$$

THEOREM 4.2. *Let $T : X \to X$ be a nonsingular measurable transformation with f_T in $L^\infty(\mu)$ and bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded from $L(p, q, w_1 d\mu)$ into $L(p, q, w_2 d\mu)$ under condition* (i) *or* (ii). *Then $W_{u,T}$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on the support of u.*

Proof. Suppose that $W = W_{u,T}$ has closed range. Then there exists an $\varepsilon > 0$ such that $\|Wf\|_{p,q,w_2} \geq \varepsilon \|f\|_{p,q,w_1}$ for all $f \in L(p, q, w_1 d\mu)(S)$ where S is the support of u and $L(p, q, w_1 d\mu)(S) = \{f\chi_S : f \in L(p, q, w_1 d\mu)\}$.

Now, choose $\delta > 0$ such that $k^{1/p}\delta < \varepsilon$ where $k = \|f_T\|_\infty$. Assume that the set $E = \{x \in X : |u(x)| < \delta\}$ has positive measure, i.e. $0 < w_1(E) < \infty$. Then $\chi_E \in L(p, q, w_1 d\mu)(S)$ and

$$\begin{aligned}\|W\chi_E\|_{p,q,w_2} &\leq k^{1/p}\|u \cdot \chi_E\|_{p,q,w_2} \leq k^{1/p}\delta\|\chi_E\|_{p,q,w_1} \\ &< \varepsilon\|\chi_E\|_{p,q,w_1}\end{aligned}$$

by (4.1). This contradiction says that $|u(x)| \geq \delta$ a.e. on the support of u.

Conversely, assume that there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on S. By assumption, $f_T > m$ for some $m > 0$. By using this fact and (4.2), we get

$$\|Wf\|_{p,q,w_2} \geq m^{1/p}\|u \cdot f\|_{p,q,w_1} \geq m^{1/p}\delta\|f\|_{p,q,w_1}$$

for all $f \in L(p, q, w_1 d\mu)(S)$. Therefore W has closed range equal to $\ker(W) = L(p, q, w_1 d\mu)(X \setminus S)$. ∎

COROLLARY 4.3. *If $T^{-1}(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon > 0$ and $W_{u,T}$ has closed range, then $|u(x)| \geq \delta$ a.e. on S, the support of u, for some $\delta > 0$.*

Using the equivalence (4.3) and [ADV1, Theorem 4.1], we can state the following theorem:

THEOREM 4.4. *Let $T : X \to X$ be a nonsingular measurable transformation with f_T in $L^\infty(\mu)$ and bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded from $L(p, q, w_1 d\mu)$ into $L(p, q, w_2 d\mu)$ under condition* (i) *or* (ii). *Then the following are equivalent:*

(a) *$W_{u,T}$ has closed range,*
(b) *M_u has closed range,*
(c) *$|u(x)| \geq \delta$ a.e. for some $\delta > 0$ on S, the support of u.*

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