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# DENSITY OF SOME SEQUENCES MODULO 1

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**Abstract.** Recently, Cilleruelo, Kumchev, Luca, Rué and Shparlinski proved that for each integer $a \geq 2$ the sequence of fractional parts $\{a^n/n\}_{n=1}^{\infty}$ is everywhere dense in the interval $[0,1]$. We prove a similar result for all Pisot numbers and Salem numbers $\alpha$ and show that for each $c>0$ and each sufficiently large $N$, every subinterval of $[0,1]$ of length $cN^{-0.475}$ contains at least one fractional part $\{Q(\alpha^n)/n\}$, where $Q$ is a nonconstant polynomial in $\mathbb{Z}[z]$ and $n$ is an integer satisfying $1 \leq n \leq N$.

**1. Introduction.** Throughout, let $\{x\}$ be the fractional part of $x \in \mathbb{R}$. In a recent paper [3] Cilleruelo, Kumchev, Luca, Rué and Shparlinski showed that for each integer $a \geq 2$,

$$\text{the sequence } \{a^n/n\}_{n=1}^{\infty} \text{ is everywhere dense in } [0,1] \tag{1.1}$$

and, furthermore, for any $c>0$ and any sufficiently large integer $N$ every interval $J \subseteq [0,1]$ of length $|J| \geq cN^{-0.475}$ contains an element of this sequence with the index $n$ satisfying $1 \leq n \leq N$. In the proof of (1.1) they considered a subsequence $A$ of the sequence $\{a^n/n\}_{n=1}^{\infty}$ with indices $n=pq$, where both $p$ and $q$ are primes satisfying $q \leq \log p/\log a$. Using exponential sums and other tools from analytic number theory they first proved an upper bound for the discrepancy of the sequence $A$ which implies (1.1) (see Theorem 1 in [3]) and then gave an alternative (much shorter) argument which implies (1.1) as well (see Theorem 2 in [3]). The main result of this note (see Theorem 1.2 below) generalizes Theorem 2 of [3].

A reader familiar with the literature in analytic number theory may guess, from the constant 0.475 and the fact that prime numbers are involved in $A_1$, that the authors of [3] used some results on gaps between consecutive primes. A well-known result of Baker, Harman and Pintz [1] asserts there is a constant $\theta < 0.525$ such that for each sufficiently large $x$ the interval $(x-x^{\theta}, x)$ contains a prime number. (Note that $0.475 = 1-0.525$.) We shall use a version of this result which follows from a more general Lemma 2 of [3] (which itself is extracted from Theorem 10.8 in [7]):

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LEMMA 1.1. *If $C$ is a positive constant and $h$ is a positive integer satisfying $h \le (\log x)^C$ then for each sufficiently large $x$ the interval $(x - x^\theta, x)$, where $\theta < 0.525$ is some constant, contains a prime number which is equal to $1$ modulo $h$.*

Before stating our result we recall that an algebraic integer $\alpha > 1$ is a *Pisot number* (resp. a *Salem number*) if all of its conjugates over $\mathbb{Q}$ (if any) lie strictly inside the unit circle $|z| = 1$ (resp. in the disc $|z| \le 1$ with at least one conjugate lying on the circle $|z| = 1$). See [2] for some basic properties of Pisot and Salem numbers. For example, all rational integers greater than or equal to 2, the golden section $(1 + \sqrt{5})/2 = 1.61803\ldots$ and the number $1.32471\ldots$ which is a root of the polynomial $z^3 - z - 1$ are Pisot numbers. (Siegel [9] proved that the latter is the smallest Pisot number.) The smallest known Salem number $1.17628\ldots$ is a root of the Lehmer polynomial $z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.

We can now state the main result of this paper.

THEOREM 1.2. *If $\alpha$ is a Pisot number or a Salem number and $Q(z)$ is a nonconstant polynomial with integer coefficients then the sequence $\{Q(\alpha^n)/n\}_{n=1}^\infty$ is everywhere dense in $[0, 1]$. Furthermore, for any $c > 0$ and any sufficiently large integer $N$ every interval $J \subseteq [0, 1]$ of length $|J| \ge cN^{-0.475}$ contains at least one element of this sequence with the index $n$ in the range $1 \le n \le N$.*

By the same method Theorem 1.2 can be proved for nonconstant polynomials $Q$ with rational coefficients. It would be of interest to extend this result to sequences of the form $\{Q(\alpha^n)/P(n)\}_{n=1}^\infty$, where $P \in \mathbb{Q}[z]$ is a polynomial of degree at least 2, e.g., to the sequence $\{2^n/(n^3 + 1)\}_{n=1}^\infty$.

**2. Preparation for the proof of Theorem 1.2.** We begin with a short proof of (1.1) (following [3], i.e. taking $n = pg$, although without assuming that $g$ is a prime) and then continue the proof of Theorem 1.2 along the same lines with a more subtle choice of $g$ (see (2.3) and (3.1)) and $p$.

To prove (1.1) it suffices to show that the sequence $\{a^n/n\}_{n=1}^\infty$, where $a \ge 2$ is an integer, is everywhere dense in the open interval $(0, 1)$. Fix any $\lambda$ in the interval $(0, 1)$. We will show that for each $\varepsilon$ satisfying $0 < \varepsilon < \lambda$ there is $n \in \mathbb{N}$ of the form $n = pg$, where $g$ is a large integer and $p$ is a prime number, such that $\lambda - \varepsilon < \{a^n/n\} < \lambda$. Indeed, for each sufficiently large integer $g > g_0(a, \lambda, \varepsilon)$ (which is assumed to be relatively prime to $a$) there is a prime number $p > g$ which satisfies

$$\frac{a^g}{g\lambda} < p < \frac{a^g}{g(\lambda - \varepsilon)} \tag{2.1}$$

and $\varphi(g) \mid (p-1)$, where $\varphi(g)$ is Euler's function. With this choice of $p$ and $g$, by Euler's theorem, we see that the difference $a^{(p-1)g} - 1$ is divisible by $p$ and by $g$. Hence their product $pg$ divides $a^{pg} - a^g$. Using (2.1) we find that for $n = pg$,

$$\{a^n/n\} = \{a^{pg}/pg\} = \{a^g/pg\} = a^g/pg \in (\lambda - \varepsilon, \lambda),$$

as claimed.

Let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ be the full set of conjugates of $\alpha$ over $\mathbb{Q}$ with minimal polynomial

$$F(z) = (z - \alpha_1) \cdots (z - \alpha_d) = z^d + b_{d-1} z^{d-1} + \cdots + b_0 \in \mathbb{Z}[z].$$

Put

$$S_n := \alpha_1^n + \cdots + \alpha_d^n \quad \text{and} \quad R_n := S_n - \alpha_1^n = S_n - \alpha^n. \tag{2.2}$$

Note that, by the Newton formula,

$$S_n + b_{d-1} S_{n-1} + \cdots + b_0 S_{n-d} = 0$$

for each integer $n \geq d + 1$.

Suppose that $g$ is a positive integer satisfying

$$\gcd(b_0, g) = 1. \tag{2.3}$$

Then $(S_n)_{n=1}^{\infty}$ is a sequence of integers which is purely periodic modulo $g$ with period $h$ in the range $1 \leq h \leq g^d$. (This result is known and can be easily proved in few lines; see, for instance, Lemma 2 in [5].) In particular, this implies that

$$g \mid (S_l - S_k) \quad \text{if } h \mid (l - k). \tag{2.4}$$

Another useful result concerning $S_n$ is that

$$p \mid (S_{pk} - S_k) \tag{2.5}$$

for every $k \in \mathbb{N}$ and every prime number $p$. This is an old 1839 result of Schönemann [8], several times rediscovered by different authors. See, e.g., [4] and also [6], [10] for some generalizations; e.g., the latter paper contains the proof of $n \mid \sum_{t|n} \mu(n/t) S_{tk}$ for each $n \in \mathbb{N}$, where $\mu$ is the Möbius function, which gives (2.5) when $n$ is a prime number. We remark that the properties (2.4) and (2.5) hold for all algebraic integers $\alpha$ (and not just for Pisot and Salem numbers).

Let

$$Q(z) = a_t z^t + \cdots + a_0 \in \mathbb{Z}[z],$$

where $t \in \mathbb{N}$ and $a_t \neq 0$. Without restriction of generality we may assume that $a_t > 0$, since otherwise one can consider the polynomial $-Q$ instead of $Q$. Put

$$D_n := Q(\alpha^n) - \sum_{j=1}^{t} a_j S_{jn}. \tag{2.6}$$

From (2.2) and (2.6) it follows that $D_n = a_0 - \sum_{j=1}^{t} a_j R_{jn}$. Since $\alpha$ is a Pisot or a Salem number, all its conjugates lie in $|z| \le 1$, so $|R_{jn}| \le d-1$. Hence

$$|D_n| \le K := |a_0| + (d-1)\sum_{j=1}^{t} |a_j|. \tag{2.7}$$

As we already observed above, for any positive integer $g$ as in (2.3), the sequence $(S_n)_{n=1}^{\infty}$ is purely periodic modulo $g$ with period $h \le g^d$. Assume that $p > g$ is a prime which is equal to 1 modulo $h$. Take $n = pg$. Then $p \,|\, (S_{jpg} - S_{jg})$, by (2.5). Also, $g \,|\, (S_{jpg} - S_{jg})$, by (2.4), because $jpg - jg = jg(p-1)$ is divisible by the period $h$. Hence $pg \,|\, (S_{jpg} - S_{jg})$, because $\gcd(p, g) = 1$. It follows that $pg$ divides the difference between $\sum_{j=1}^{t} a_j S_{jpg}$ and $\sum_{j=1}^{t} a_j S_{jg}$. Thus, if $g < p$ is a positive integer satisfying (2.3) then in view of (2.6) we obtain

$$\{Q(\alpha^{pg})/pg\} = \Big\{(pg)^{-1} D_{pg} + (pg)^{-1} \sum_{j=1}^{t} a_j S_{jpg}\Big\} = \{y(p)\}, \tag{2.8}$$

where

$$y(p) := (pg)^{-1}\Big(D_{pg} + \sum_{j=1}^{t} a_j S_{jg}\Big). \tag{2.9}$$

In the next section we will select appropriate prime numbers $p$ and using (2.8) complete the proof of Theorem 1.2.

**3. Proof of Theorem 1.2.** Fix a large positive integer $N$ and take the largest $g \in \mathbb{N}$ satisfying (2.3) for which

$$\sum_{j=1}^{t} a_j S_{jg} - K \le N \tag{3.1}$$

with $K$ given in (2.7). Observe that the main term of the expression on the left hand side of (3.1) is $a_t \alpha^{tg}$ and at least one of $|b_0|$ consecutive integers $g$ satisfies the condition (2.3). Hence there are two positive constants $c_1 \le 1$ and $c_2$ (depending on $t, a_t, \alpha, b_0$ only and not on $N$) such that

$$c_1 N \le \sum_{j=1}^{t} a_j S_{jg} - K, \tag{3.2}$$

$$g \le c_2 \log N \tag{3.3}$$

for $N$ large enough. In particular, in view of $h \le g^d$ the inequality (3.3) implies that

$$h \le (\log N)^{d+1} \tag{3.4}$$

for each sufficiently large $N$.

For $g$ chosen as in (3.1) we set

$$L_1 := (g^{-1} \sum_{j=1}^{t} a_j S_{jg} + g^{-1}K)/2, \qquad L_2 := g^{-1} \sum_{j=1}^{t} a_j S_{jg} - g^{-1}K. \tag{3.5}$$

Clearly, by (3.1), (3.2) and (3.5),

$$c_1 N/g \le L_2 \le N/g \tag{3.6}$$

and, since $2L_1 = L_2 + 2K/g$,

$$c_1 N/2g \le L_1 \le (N + 2K)/2g. \tag{3.7}$$

Let $p_1 < \cdots < p_s$ be all the primes which are equal to 1 modulo $h$ and are greater than $L_1$ and smaller than $L_2$. Then, by (3.6), we have $p_s < L_2 \le N/g$ and, by (3.7), $p_1 > L_1 \ge c_1 N/2g$. Hence

$$c_1 N/2 < p_1 g < \cdots < p_s g < N. \tag{3.8}$$

Note that $p_1 > g$, by (3.3) and (3.8), so the formula (2.8) holds for the primes $p_1, \ldots, p_s$.

Now, for each $p \in \{p_1, \ldots, p_s\}$ using (2.7), (2.9) and (3.5) we find that

$$y(p) \ge (pg)^{-1}\Big(-K + \sum_{j=1}^{t} a_j S_{jg}\Big) = L_2/p \ge L_2/p_s > 1.$$

Similarly,

$$y(p) \le (pg)^{-1}\Big(K + \sum_{j=1}^{t} a_j S_{jg}\Big) = 2L_1/p \le 2L_1/p_1 < 2.$$

Hence (2.8) yields

$$\{Q(\alpha^{pg})/pg\} = y(p) - 1$$

for each $p \in \{p_1, \ldots, p_s\}$.

By (3.8), all the integers $p_1 g, \ldots, p_s g$ are smaller than $N$. We will show that for any $c > 0$ and any sufficiently large integer $N$ every interval $J \subseteq [0,1]$ of length $|J| \ge cN^{-0.475}$ contains at least one number $\{Q(\alpha^{pg})/pg\} = y(p) - 1$ with $p \in \{p_1, \ldots, p_s\}$. For a contradiction, suppose that there is an interval $J \subseteq [0,1]$ of length $cN^{-0.475}$ which contains no numbers of the form $y(p) - 1$ with $p \in \{p_1, \ldots, p_s\}$. Our aim is to show that the number $y(p_s) - 1$ is 'very close' to 0, the number $y(p_1) - 1$ is 'very close' to 1 and, moreover, the difference between two consecutive values $y(p_i) - 1$ and $y(p_{i+1}) - 1$ is 'very small' too. If this is the case then moving from $i = 1$ (with $y(p_1) - 1$ being almost the right endpoint of the interval $[0,1]$) to $i = s$ (with $y(p_s) - 1$ being almost the left endpoint of the interval $[0,1]$) step by step we will get values all over the interval $[0,1]$ lying in every interval of length $cN^{-0.475}$.

Indeed, observe first that, by (2.7), (2.9) and (3.5),

$$\begin{aligned} y(p_s) &= (p_s g)^{-1}\Big(D_{p_s g} + \sum_{j=1}^{t} a_j S_{jg}\Big) \\ &\le (p_s g)^{-1}\Big(K + \sum_{j=1}^{t} a_j S_{jg}\Big) = 2L_1/p_s = L_2/p_s + 2K/p_s g. \end{aligned}$$

By Lemma 1.1, we have $L_2 - L_2^\theta < p_s < L_2$ with $\theta < 0.525$. Using (3.3) and (3.6) we find that

$$0 < y(p_s) - 1 < \frac{L_2 + 2K/g}{L_2 - L_2^\theta} - 1 = \frac{L_2^\theta + 2K/g}{L_2 - L_2^\theta} < cN^{-0.475} \tag{3.9}$$

in view of $\theta < 0.525$. Similarly, as

$$y(p_1) = (p_1 g)^{-1}\Big(D_{p_1 g} + \sum_{j=1}^{t} a_j S_{jg}\Big) \ge (p_1 g)^{-1}\Big(-K + \sum_{j=1}^{t} a_j S_{jg}\Big) = L_2/p_1,$$

and, by Lemma 1.1, $L_1 < p_1 < L_1 + L_1^\theta$, applying (3.3) and (3.7) we find that

$$2 - y(p_1) < 2 - \frac{L_2}{L_1 + L_1^\theta} = 2 - \frac{2L_1 - 2K/g}{L_1 + L_1^\theta} = \frac{2L_1^\theta + 2K/g}{L_1 + L_1^\theta} < cN^{-0.475}.$$

Thus

$$1 - cN^{-0.475} < y(p_1) - 1 < 1. \tag{3.10}$$

From (3.9) and (3.10) it follows that if such an interval $J$ of length $cN^{-0.475}$ (which contains no numbers of the form $y(p) - 1$, where $p \in \{p_1, \dots, p_s\}$) exists then $J = [u, v]$ with $y(p_s) - 1 < u$ and $v < y(p_1) - 1$. Moreover, for some $i \in \{1, \dots, s-1\}$ the distance between two consecutive points $y(p_i) - 1$ and $y(p_{i+1}) - 1$ must be greater than $cN^{-0.475}$. So for a contradiction it suffices to show that

$$|y(p_{i+1}) - y(p_i)| < cN^{-0.475}$$

for each $i \in \{1, \dots, s-1\}$.

Since, by (2.9),

$$y(p_{i+1}) - y(p_i) = (p_{i+1} g)^{-1}\Big(D_{p_{i+1} g} + \sum_{j=1}^{t} a_j S_{jg}\Big) - (p_i g)^{-1}\Big(D_{p_i g} + \sum_{j=1}^{t} a_j S_{jg}\Big),$$

from $|D_{p_{i+1}g}|, |D_{p_i g}| \le K$ it follows that

$$|y(p_{i+1}) - y(p_i)| \le \frac{K}{p_{i+1}g} + \frac{K}{p_i g} + \frac{(p_{i+1} - p_i)|\sum_{j=1}^{t} a_j S_{jg}|}{p_{i+1}p_i g}$$
$$< \frac{2K}{p_1 g} + \frac{(p_{i+1} - p_i)|\sum_{j=1}^{t} a_j S_{jg}|}{p_i^2 g}.$$

From (3.8) we see that the first term, $2K/p_1 g$, is less than $c_3/N$. Using $p_{i+1} - p_i < p_i^{\theta}$ (see Lemma 1.1) and (3.1), (3.2) we can bound the second term:

$$\frac{(p_{i+1} - p_i)|\sum_{j=1}^{t} a_j S_{jg}|}{p_i^2 g} < \frac{p_i^{\theta}(N+K)}{p_i^2} = \frac{N+K}{p_i^{2-\theta}} \le \frac{N+K}{p_1^{2-\theta}}.$$

In view of (3.3) and (3.8) this second term is less than

$$\frac{N+K}{(c_1 N/2g)^{2-\theta}} < \frac{(\log N)^2}{N^{1-\theta}}.$$

Therefore, as $\theta < 0.525$, we conclude that for $N$ large enough

$$|y(p_{i+1}) - y(p_i)| < \frac{c_3}{N} + \frac{(\log N)^2}{N^{1-\theta}} < cN^{-0.475},$$

as claimed. This completes the proof of Theorem 1.2.

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