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GENERIC EXTENSIONS OF NILPOTENT $k[T]$-MODULES, MONOIDS OF PARTITIONS AND CONSTANT TERMS OF HALL POLYNOMIALS

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Abstract. We prove that the monoid of generic extensions of finite-dimensional nilpotent $k[T]$-modules is isomorphic to the monoid of partitions (with addition of partitions). This gives us a simple method for computing generic extensions, by addition of partitions. Moreover we give a combinatorial algorithm that calculates the constant terms of classical Hall polynomials.

1. Introduction. Let k be a field and let $k[T]$ be the k-algebra of polynomials in one variable T. We consider nilpotent $k[T]$-modules M, N and the generic extension $M*N$ of M by N, i.e. an extension of M by N with the minimal dimension of its endomorphism ring (see Section 2 for definitions). By results presented in [B, DD, DDM, Rei1] generic extensions of nilpotent $k[T]$-modules exist and the operation of taking the generic extension provides the set $\mathcal{M}^*$ of all isomorphism classes of nilpotent $k[T]$-modules with a monoid structure. There are many results concerning this monoid and its properties (see [DD, DDM, Rei1, Hu, W]). In this paper we study connections of the monoid $\mathcal{M}^*$ with the monoid $\mathcal{P}^+$ of all partitions with addition of partitions as operation. More precisely, we prove in Theorem 3.1 that these two monoids are isomorphic. This isomorphism gives us a combinatorial description of generic extensions that have a geometric nature. For a geometric interpretation of generic extensions the reader is referred to [Rei1, Rei2].

On the other hand, there is a $\mathbb{C}$-algebra isomorphism $\mathbb{C}\mathcal{M}^* \simeq \mathcal{H}_0$, where $\mathcal{H}_0$ is the specialisation of the Hall algebra $\mathcal{H}_q$ to $q = 0$ and $\mathbb{C}\mathcal{M}^*$ is the monoid algebra of $\mathcal{M}^*$ (see [DD, Hu, W] and Section 3). There are many results that show connections between generic extensions, degenerations, Lie algebras, Hall polynomials and Ringel–Hall algebras (see [Rei1, Rei2, Hu, W] for Dynkin and extended Dynkin quivers, [DD, DDM] for cyclic quivers,

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[Ri, Ried, KK, K4] for representation finite and representation directed algebras, [K1, K2, K3] for poset representations).

In Section 4, exploring the isomorphism $\mathbb{C}\mathcal{M}^* \simeq \mathcal{H}_0$ (explicitly given in [W]), we describe a combinatorial algorithm of calculating the constant terms of classical Hall polynomials.

2. Notation and definitions. Throughout this paper k is a fixed field.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition (i.e. a sequence of non-negative integers containing only finitely many non-zero terms with $\lambda_1 \geq \lambda_2 \geq \cdots$). Denote by $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots)$ the dual partition of λ, i.e.

$$\overline{\lambda}_i = \#\{j \,;\, \lambda_j \geq i\},$$

where $\#X$ denotes the cardinality of a finite set X. We identify partitions that differ only by a string of zeros at the end. Let $\mathcal{P}$ be the set of all partitions. Denote by $|\lambda|$ the *weight* of λ defined by

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots$$

and by $0 = (0)$ the unique partition of zero. Consider two associative monoids:

- $\mathcal{P}^+ = (\mathcal{P}, +, 0)$, where $(\lambda_1, \lambda_2, \ldots) + (\nu_1, \nu_2, \ldots) = (\lambda_1 + \nu_1, \lambda_2 + \nu_2, \ldots)$;
- $\mathcal{P}^\cup = (\mathcal{P}, \cup, 0)$, where $(\lambda_1, \lambda_2, \ldots) \cup (\nu_1, \nu_2, \ldots) = (\mu_1, \mu_2, \ldots)$ and $(\mu_1, \mu_2, \ldots)$ is the partition formed by the integers $\lambda_1, \lambda_2, \ldots, \nu_1, \nu_2, \ldots$ arranged in descending order (e.g. $(3,3,2,1) + (2,2) = (5,5,2,1)$ and $(3,3,2,1) \cup (2,2) = (3,3,2,2,2,1)$).

By [M, 1.8] the operations $+$ and $\cup$ are dual to each other (i.e. $\overline{\lambda \cup \nu} = \overline{\lambda} + \overline{\nu}$). One of the main aims of the paper is to describe connections of these monoids with extensions of nilpotent $k[T]$-modules.

Let $k[T]$ be the k-algebra of polynomials in the variable T. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$ for some n, denote by

$$M(\lambda) = M(\lambda, k) \cong k[T]/(T^{\lambda_1}) \oplus \cdots \oplus k[T]/(T^{\lambda_n})$$

the corresponding $k[T]$-module. It is obvious that the function $\lambda \mapsto M(\lambda)$ gives a bijection between the set $\mathcal{P}$ of all partitions and the set of all isomorphism classes of nilpotent $k[T]$-modules (i.e. finitely generated $k[T]$-modules M such that $T^a M = 0$ for some $a \geq 0$). Denote by $\mathcal{M}$ a set of representatives of all isomorphism classes of nilpotent $k[T]$-modules.

Let $M, N \in \mathcal{M}$. By [B], [DD] and [Rei1], there is a unique (up to isomorphism) extension X of M by N with the minimal dimension of the endomorphism ring $\mathrm{End}_{k[T]}(X)$, i.e. a nilpotent $k[T]$-module X such that there exists a short exact sequence of the form

$$0 \to N \to X \to M \to 0.$$

The module X is called the *generic extension* of M by N and is denoted by $X = M * N$. Denote by $M \oplus N$ the direct sum of the modules M and N and by 0 the unique zero module. Consider two monoids:

- $\mathcal{M}^* = (\mathcal{M}, *, 0)$ (the *monoid of generic extensions*),
- $\mathcal{M}^{\oplus} = (\mathcal{M}, \oplus, 0)$.

The associativity of the monoid $\mathcal{M}^* = (\mathcal{M}, *, 0)$ follows by [DD], whereas that of the monoid $\mathcal{M}^{\oplus} = (\mathcal{M}, \oplus, 0)$ is obvious.

3. Generic extensions and partitions. The following is one of the main results of the paper.

THEOREM 3.1. *The function $\Phi : \mathcal{P} \to \mathcal{M}$ such that $\Phi(\lambda) = M(\lambda)$ for any partition λ induces isomorphisms of monoids:*

$$\Phi : \mathcal{P}^{+} \to \mathcal{M}^{*} \quad \textit{and} \quad \Phi : \mathcal{P}^{\cup} \to \mathcal{M}^{\oplus}.$$

Moreover

$$M(\overline{\alpha}) * M(\overline{\beta}) = M(\overline{\alpha \cup \beta}).$$

To prove Theorem 3.1 we need a geometric interpretation of generic extensions.

We identify $k[T]$-modules of the form $M(\lambda, k)$ with systems $M(\lambda, k) = (V, \varphi)$, where V is a finite-dimensional k-vector space and $\varphi : V \to V$ is a nilpotent linear endomorphism of Jordan type λ (i.e. a nilpotent representation of a loop quiver). We denote by $\mathcal{N}(k)$ the category of all such systems. If (V, φ), (V_1, φ_1) are objects in $\mathcal{N}(k)$, then a morphism $f : (V, \varphi) \to (V_1, \varphi_1)$ is a linear map $f : V \to V$ such that $\varphi_1 f = f\varphi$. It is easy to see that the category $\mathcal{N}(k)$ is equivalent to the category of all finite-dimensional nilpotent $k[T]$-modules. For an account of the theory of modules and quiver representations we refer the reader to [ASS] and [ARS].

Consider the affine k-scheme $\mathbb{M}_n(k)$ of all $n \times n$-matrices with coefficients in k. The general linear group $\mathrm{Gl}_n(k)$ acts on $\mathbb{M}_n(k)$ via conjugations, i.e. for $g \in \mathrm{Gl}_n(k)$ and $M \in \mathbb{M}_n(k)$, we put $g \cdot M = gMg^{-1}$. Let $\mathbb{M}_n^{\mathrm{nil}}(k)$ be the subset of $\mathbb{M}_n(k)$ consisting of all nilpotent matrices. The subset $\mathbb{M}_n^{\mathrm{nil}}(k)$ is closed in $\mathbb{M}_n(k)$ (in the Zariski topology) and it is closed under the action of $\mathrm{Gl}_n(k)$. It is easy to observe that the points of $\mathbb{M}_n^{\mathrm{nil}}(k)$ correspond bijectively to the objects (V, φ) of $\mathcal{N}(k)$ with $\dim_k V = n$. Moreover the orbits of the action of $\mathrm{Gl}_n(k)$ on $\mathbb{M}_n^{\mathrm{nil}}(k)$ correspond bijectively to the isomorphism classes of the objects V in $\mathcal{N}(k)$ (with $\dim_k V = n$) and hence to the isomorphism classes of nilpotent $k[T]$-modules V (with $\dim_k V = n$). If $M(\lambda) \equiv (V, \varphi)$ is a nilpotent $k[T]$-module with $\dim_k M(\lambda) = n$, then we denote by $\mathcal{O}_\lambda$ (resp. $\overline{\mathcal{O}_\lambda}$) the orbit (resp. the Zariski-closure of the orbit) of $\varphi \in \mathbb{M}_n^{\mathrm{nil}}(k)$ under the $\mathrm{Gl}_n(k)$-action (see [G] and [H]).

Let λ, ν be partitions with weights $|\lambda| = |\nu| = n$. We say that a module $M(\lambda)$ *degenerates* to the module $M(\nu)$ if $\mathcal{O}_\nu \subset \overline{\mathcal{O}_\lambda}$. If $M(\lambda)$ degenerates to $M(\nu)$ we write $M(\lambda) \leq_{\deg} M(\nu)$. The relation $\leq_{\deg}$ is a partial order on isomorphism classes of finite-dimensional nilpotent $k[T]$-modules. Geometrically, the generic extension $M(\lambda) * M(\nu)$ (resp. the direct sum $M(\lambda) \oplus M(\nu)$) is the $\leq_{\deg}$-minimal (resp. $\leq_{\deg}$-maximal) extension of $M(\nu)$ by $M(\mu)$, i.e. if X is an extension of $M(\nu)$ by $M(\mu)$, then $M(\lambda) * M(\nu) \leq_{\deg} X$ (resp. $X \leq_{\deg} M(\lambda) \oplus M(\nu)$) (see [B], [DD] and [Rei1]). For an introduction to geometric methods in representation theory the reader is referred to [Kr1] and [B].

The following fact is proved in [G] and [H] (see also [Kr2, I.3]).

THEOREM 3.2. *Let λ, ν be partitions with $|\lambda| = |\nu|$. Then we have $M(\lambda) \leq_{\deg} M(\nu)$ if and only if*

$$\sum_{i=1}^{m} \overline{\lambda}_i \leq \sum_{i=1}^{m} \overline{\nu}_i \quad \textit{for any } m \geq 1.$$

The following lemma is used in the proof of Theorem 3.1.

LEMMA 3.3. *Let σ, ν, μ be partitions. If there exists a short exact sequence*

$$0 \to M(\nu) \xrightarrow{a} M(\sigma) \xrightarrow{b} M(\mu) \to 0,$$

then

$$\sum_{i=1}^{m} \sigma_i \leq \sum_{i=1}^{m} \lambda_i \quad \textit{for any } m \geq 1,$$

where $\lambda = \mu + \nu$.

Proof. The proof is by induction on $|\nu|$. If $|\nu| = 0$, then $M(\sigma) \cong M(\mu)$, $\sigma = \mu$ and we are done.

Assume that $|\nu| > 0$. We have $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_n \neq 0$ and

$$M(\nu) = M(\nu_1) \oplus \cdots \oplus M(\nu_n).$$

Consider the monomorphism

$$f = [\iota, 0, \ldots, 0] : M(1) \to M(\nu_1) \oplus \cdots \oplus M(\nu_n),$$

where $\iota : M(1) \to M(\nu_1)$ is the inclusion. By the Snake Lemma, we get the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M(\nu') & \longrightarrow & M(\sigma') & \longrightarrow & M(\mu) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M(\nu) & \xrightarrow{a} & M(\sigma) & \xrightarrow{b} & M(\mu) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle a\cdot f} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & M(1) & \xrightarrow{\mathrm{id}} & M(1) & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\nu' = (\nu_1 - 1, \nu_2, \ldots, \nu_{n-1}, \nu_n)$ and there exists i_0 such that $\sigma'_{i_0} = \sigma_{i_0} - 1$ and $\sigma'_j = \sigma_j$ for $j \neq i_0$. By the induction hypothesis we get

$$\sum_{i=1}^{m} \sigma'_i \leq \sum_{i=1}^{m} \lambda'_i,$$

where $\lambda' = \mu + \nu'$. Therefore, for $m < i_0$,

$$\sum_{i=1}^{m} \sigma_i = \sum_{i=1}^{m} \sigma'_i \leq \sum_{i=1}^{m} \lambda'_i = \sum_{i=1}^{m} \lambda_i - 1 \leq \sum_{i=1}^{m} \lambda_i,$$

while for $m \geq i_0$,

$$\sum_{i=1}^{m} \sigma_i = \sum_{i=1}^{m} \sigma'_i + 1 \leq \sum_{i=1}^{m} \lambda'_i + 1 = \sum_{i=1}^{m} \lambda_i - 1 + 1 = \sum_{i=1}^{m} \lambda_i,$$

where $\lambda = \mu + \nu$, and we are done. ■

LEMMA 3.4. *Let ν, μ be partitions. We have*

$$M(\nu) * M(\mu) = M(\nu + \mu),$$

where $M(\nu) * M(\mu)$ is the generic extension of $M(\nu)$ by $M(\mu)$.

Proof. It is easy to see that $M(\nu+\mu)$ is an extension of $M(\nu)$ by $M(\mu)$. If $M(\sigma)$ is any extension of $M(\mu)$ by $M(\nu)$, then by Lemma 3.3,

$$\sum_{i=1}^{m} \sigma_i \leq \sum_{i=1}^{m} \lambda_i \quad \text{for any } m \geq 1,$$

where $\lambda = \mu + \nu$. By [M, 1.11],

$$\sum_{i=1}^{m} \overline{\sigma}_i \geq \sum_{i=1}^{m} \overline{\lambda}_i \quad \text{for any } m \geq 1.$$

Theorem 3.2 yields

$$M(\nu+\mu) = M(\lambda) \leq_{\deg} M(\sigma).$$

Since $M(\lambda) * M(\nu)$ is the $\leq_{\deg}$-minimal extension of $M(\nu)$ by $M(\mu)$, we are done. ■

Proof of Theorem 3.1. Let $\Phi : \mathcal{P} \to \mathcal{M}$ be such that $\Phi(\lambda) = M(\lambda)$ for any partition λ. By Lemma 3.4, the induced function $\Phi : \mathcal{P}^{+} \to \mathcal{M}^{*}$ is an isomorphism of monoids. It is easy to see that so too is $\Phi : \mathcal{P}^{\cup} \to \mathcal{M}^{\oplus}$. Moreover

$$M(\overline{\alpha}) * M(\overline{\beta}) = M(\overline{\alpha}+\overline{\beta}) = M(\overline{\alpha \cup \beta}),$$

because $\overline{\alpha}+\overline{\beta} = \overline{\alpha \cup \beta}$. ■

4. Constant terms of Hall polynomials. In this section we describe a combinatorial algorithm for calculating the constant term of a given Hall polynomial.

Let α, β, γ be partitions and let k be a finite field. Denote by $F^{\gamma}_{\alpha,\beta}(k)$ the number of submodules U of $M(\gamma, k)$ such that U is isomorphic to $M(\beta, k)$ and the factor module $M(\gamma, k)/U$ is isomorphic to $M(\alpha, k)$. By the result of Hall (see [M, II.4.3]), there exists a polynomial $\varphi^{\gamma}_{\alpha\beta}$ with integral coefficients such that

$$\varphi^{\gamma}_{\alpha\beta}(\#k) = F^{\gamma}_{\alpha,\beta}(k)$$

for any finite field k. We call $\varphi^{\gamma}_{\alpha,\beta}$ the *Hall polynomial* associated with the partitions α, β, γ.

By [DD], [Hu] and [W], the complex algebra $\mathbb{C}\mathcal{M}^{*}$ generated by the monoid $\mathcal{M}^{*}$ of generic extensions is isomorphic to the degenerate complex Hall algebra $\mathcal{H}_0$, where $\mathcal{H}_0$ has a basis $\{u_\alpha \ ; \ \alpha \in \mathcal{P}\}$ as a $\mathbb{C}$-vector space and multiplication is given by the formula

$$u_\alpha u_\beta = \sum_{\gamma} \varphi^{\gamma}_{\alpha\beta}(0) u_\gamma.$$

By [W], the isomorphism $F : \mathbb{C}\mathcal{M}^{*} \to \mathcal{H}_0$ is given by the formula

$$F(M(\alpha)) = \sum_{\beta\,:\, M(\alpha) \leq_{\deg} M(\beta)} u_\beta.$$

We use the following notation. A partition $\alpha = (\alpha_1, \alpha_2, \ldots)$ will be written as

$$(\ldots, r^{m_r}, \ldots, 2^{m_2}, 1^{m_1}),$$

where m_r indicates the number of times the integer r occurs in α, e.g.

$$(3,3,2,2,2,1,1,1,1) = (3^2, 2^3, 1^4).$$

LEMMA 4.1. *Let γ be an arbitrary partition and let $\alpha=(1^n)$ and $\beta=(1^m)$ be partitions with $\varphi^{\gamma}_{\alpha,\beta}\neq 0$. Then*

$$\varphi^{\gamma}_{\alpha,\beta}(0)=1.$$

Proof. Note that $F(M(\alpha))=u_\alpha$ and $F(M(\beta))=u_\beta$ since $\alpha=(1^n)$ and $\beta=(1^m)$, i.e. $M(\alpha)$ and $M(\beta)$ are semisimple. Then

$$F(M(\alpha))F(M(\beta))=u_\alpha u_\beta=\sum_{\delta}\varphi^{\delta}_{\alpha\beta}(0)u_\delta$$

and

$$F(M(\alpha))F(M(\beta))=F(M(\alpha)*M(\beta))=F(M(\alpha+\beta))=\sum_{M(\alpha+\beta)\leq_{\deg}M(\delta)}u_\delta.$$

Comparing these sums we get $\varphi^{\gamma}_{\alpha,\beta}(0)=1$ if $\varphi^{\gamma}_{\alpha,\beta}\neq 0$. ∎

Applying recursively (following the $\leq_{\deg}$-order) the methods used in the proof of Lemma 4.1 one can calculate the constant terms of Hall polynomials. We illustrate this by the following example.

EXAMPLE 4.2. We calculate the constant term of the Hall polynomial $\varphi^{(4,1)}_{(2,1)(2)}$. We apply Theorem 3.2 and the definition of F.

STEP 1. By Lemma 4.1, we have

$$\varphi^{(1^5)}_{(1^3)(1^2)}(0)=\varphi^{(2,1^3)}_{(1^3)(1^2)}(0)=\varphi^{(2^2,1)}_{(1^3)(1^2)}(0)=1.$$

STEP 2. Note that

$$\begin{aligned}F(M(1^3))F(M(2))&=u_{(1^3)}(u_{(1^2)}+u_{(2)})\\&=u_{(1^5)}+u_{(2,1^3)}+u_{(2^2,1)}+\varphi^{(2,1^3)}_{(1^3)(2)}(0)u_{(2,1^3)}\\&\quad+\varphi^{(3,1^2)}_{(1^3)(2)}(0)u_{(3,1^2)}.\end{aligned}$$

On the other hand

$$F(M(1^3)*M((2)))=F(M(3,1^2))=u_{(1^5)}+u_{(2,1^3)}+u_{(2^2,1)}+u_{(3,1^2)}.$$

Therefore

$$\varphi^{(3,1^2)}_{(1^3)(2)}(0)=1\quad\text{and}\quad\varphi^{(2,1^3)}_{(1^3)(2)}(0)=0.$$

STEP 3. We have

$$\begin{aligned}F(M(2,1))F(M(1^2))&=(u_{(2,1)}+u_{(1^3)})u_{(1^2)}\\&=\varphi^{(2,1^3)}_{(2,1)(1^2)}(0)u_{(2,1^3)}+\varphi^{(3,1^2)}_{(2,1)(1^2)}(0)u_{(3,1^2)}\\&\quad+\varphi^{(2^2,1)}_{(2,1)(1^2)}(0)u_{(2^2,1)}+\varphi^{(3,2)}_{(2,1)(1^2)}(0)u_{(3,2)}+u_{(1^5)}\\&\quad+u_{(2,1^3)}+u_{(2^2,1)}\end{aligned}$$

and

$$F(M(2,1)*M((1^2))) = F(M(3,2)) = u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)} + u_{(3,1^2)} + u_{(3,2)}.$$

Therefore

$$\varphi^{(2,1^3)}_{(2,1)(1^2)}(0) = \varphi^{(2^2,1)}_{(2,1)(1^2)}(0) = 0,$$
$$\varphi^{(3,1^2)}_{(2,1)(1^2)}(0) = \varphi^{(3,2)}_{(2,1)(1^2)}(0) = 1.$$

STEP 4. Finally

$$\begin{aligned}F(M(2,1))F(M(2)) &= (u_{(2,1)} + u_{(1^3)})(u_{(1^2)} + u_{(2)}) \\ &= \varphi^{(2^2,1)}_{(2,1)(2)}(0)u_{(2^2,1)} + \varphi^{(3,2)}_{(2,1)(2)}(0)u_{(3,2)} + \varphi^{(4,1)}_{(2,1)(2)}(0)u_{(4,1)} \\ &\quad + \varphi^{(3,1^2)}_{(2,1)(2)}(0)u_{(3,1^2)} + u_{(3,1^2)} + u_{(3,2)} + u_{(3,1^2)} + u_{(1^5)} \\ &\quad + u_{(2,1^3)} + u_{(2^2,1)}\end{aligned}$$

and

$$\begin{aligned}F(M(2,1)*M((2))) &= F(M(4,1)) \\ &= u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)} + u_{(3,1^2)} + u_{(3,2)} + u_{(4,1)}.\end{aligned}$$

Therefore

$$\varphi^{(2^2,1)}_{(2,1)(2)}(0) = \varphi^{(3,2)}_{(2,1)(2)}(0) = 0,$$
$$\varphi^{(4,1)}_{(2,1)(2)}(0) = 1,$$
$$\varphi^{(3,1^2)}_{(2,1)(2)}(0) = -1.$$

REMARK 4.3. In a similar way (exploring an isomorphism analogous to F given in [W]) one may calculate the constant terms of Hall polynomials for Dynkin quivers.

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