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LIMIT THEOREMS FOR STOCHASTIC RECURSIONS WITH MARKOV DEPENDENT COEFFICIENTS

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Abstract. We consider the stochastic recursion $X_n = A_n X_{n-1} + B_n$ for Markov dependent coefficients $(A_n, B_n) \in \mathbb{R}^+ \times \mathbb{R}$. We prove the central limit theorem, the local limit theorem and the renewal theorem for the partial sums $S_n = X_1 + \cdots + X_n$.

1. Introduction. In this paper we consider the affine stochastic recursion on $\mathbb{R}$ defined by the action of the affine group on the real line:

$$\begin{aligned} X_0^x &= x, \\ X_n^x &= A_n X_{n-1}^x + B_n, \qquad n \in \mathbb{Z}_+, \end{aligned} \tag{1.1}$$

where $\{(A_n, B_n)\}_{n\geq 1}$ is a stationary and ergodic sequence of random variables valued in $\mathbb{R}^+ \times \mathbb{R}$. It was proved by Brandt [2] that if $\mathbb{E}[\log A_1] < 0$ and $\mathbb{E}[\log|B_1|] < \infty$, then the recursion has a unique stationary measure ν. Moreover X_n^x converges in distribution to a random variable R with the law ν and the limit R does not depend on the starting point x.

One can also consider the two-sided infinite affine recursion

$$X_n = A_n X_{n-1} + B_n, \qquad n \in \mathbb{Z}, \tag{1.2}$$

where $\{(A_n, B_n)\}_{n\geq 1}$ is a stationary and ergodic sequence. Then under the assumptions stated above the extended recursion has a unique solution

$$X_n = \sum_{j=0}^{\infty} \Big(\prod_{i=n+1-j}^{n} A_i \Big) B_{n-j} \tag{1.3}$$

and all X_n's are distributed according to ν.

The recursion (1.1) has been studied for almost forty years mainly under the assumption of independence of the random coefficients (A_n, B_n). The most significant result is due to Kesten [13] (see also Goldie [6]) who proved that if $\mathbb{E}A^\alpha = 1$ for some $\alpha > 0$ (and of course a number of further assumptions are satisfied), then the stationary measure ν is α-regularly varying. After this result an enormous number of further properties of the

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process $\{X_n\}$ has been proved, including limit theorems for the partial sums $S_n = X_1 + \cdots + X_n$ (see [1, 4, 8, 18]).

During the last several years substantial progress has been made in understanding the case of dependent coefficients (A_n, B_n). Here one should mention the results of Collamore [5] and Roitershtein [17] who, under different sets of assumptions, independently proved the Kesten theorem for (A_n, B_n) depending on an underlying Markov chain. Collamore [5], indeed, considered more general recursions.

The main purpose of the present paper is to prove limit theorems related to the partial sums S_n. We study here the case when $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is a stationary sequence of random variables modulated by some Markov process. More precisely, let $(S, \mathcal{S})$ be a measurable space with countably generated σ-field $\mathcal{S}$ and $\{s_n\}_{n\in\mathbb{Z}}$ be a stationary Markov chain with transition kernel H and stationary measure π. For any measurable function f on S we denote by $\|f\|$ the essential supremum of f with respect to the measure π. We assume that there exists a kernel G on $S \times S \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}^+ \times \mathbb{R}$, such that the transition kernel $\widetilde{H}$ on $(S \times \mathbb{R}^+ \times \mathbb{R}, \mathcal{S} \times \mathcal{B})$ given by

$$\widetilde{H}((s,\zeta), U \times W) = \int_U H(s,dt)\, G(s,t,W)$$

defines a Markov modulated process (MMP) associated with $\{s_n\}$, i.e. a stationary Markov chain $\{(s_n, \zeta_n)\}_{n\in\mathbb{Z}}$ on the product space $S \times (\mathbb{R}^+ \times \mathbb{R})$, where $\zeta_n = (A_n, B_n)$ (cf. [17]). Notice that (s_n, ζ_n) depends only on s_{n-1}. The stationary measure $\widetilde{\pi}$ of this Markov chain is given by

$$\widetilde{\pi}(U \times W) = \int_S \widetilde{H}((s,\xi), U \times W)\, \pi(ds) = \int_S \int_U H(s,dt)\, G(s,t,W)\, \pi(ds)$$

Given such a sequence we consider the real valued stationary process $\{X_n^x\}$ or $\{X_n\}$ defined by (1.1) or (1.2), respectively.

Our main results do not deeply depend on the structure of the underlying Markov chain s_n. The only property of the process we need is convergence of the powers of the Markov operator H to the stationary measure π, i.e. we assume that

$$\lim_{n\to\infty} H^n(f)(s) = \pi(f) \qquad \pi\text{-a.s.,} \tag{1.4}$$

for any bounded function f. This is satisfied for a large class of Markov chains, e.g. when s_n is an aperiodic Harris recurrent Markov chain (see [16, Corollary 6.7]).

To avoid considerations of the degenerate case, when $X_n = x$ a.s. for some $x \in \mathbb{R}$, we assume that

$$\mathbb{P}[A_0 x + B_0 = x] < 1 \qquad \text{for every } x \in \mathbb{R}. \tag{1.5}$$

Our main results are the following:

THEOREM 1.6 (Central Limit Theorem). *Assume that* (1.4) *and* (1.5) *are satisfied and there exists* $\gamma > 2$ *such that*

$$\Lambda(\gamma) = \|\mathbb{E}_s[A_1^\gamma \dots A_n^\gamma]\| < 1 \quad \textit{and} \quad \|\mathbb{E}_s[B_1^\gamma]\| < \infty. \tag{1.7}$$

Then $\frac{1}{\sqrt{n}}(S_n - nm)$ *converges in distribution to the normal law* $N(0, \sigma^2)$, *for some* $\sigma^2 > 0$ *and* $m = \mathbb{E}X_0$.

REMARK. The parameter σ^2 can be explicitly computed by applying the methods described in [4, Lemma 6.2].

THEOREM 1.8 (Local limit theorem). *Under the hypotheses of the theorem above, for every compact set* $I \subset \mathbb{R}$ *with negligible boundary,*

$$\lim_{n\to\infty} \sqrt{n}\,\mathbb{P}[S_n - nm \in I] = C_0 l(I),$$

where l *denotes the Lebesgue measure and* $C_0 = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t^2\sigma^2/2}\,dt$.

THEOREM 1.9 (Renewal theorem). *Let* $U(I) = \sum_{n\ge 1} \mathbb{E}[\mathbf{1}_I(S_n)]$. *Assume that* $m > 0$ *and that* (1.4), (1.5) *and* (1.7) *hold for some* $\gamma > 1$. *Then*

$$\lim_{y\to\infty} U(I + y) = \frac{l(I)}{m}$$

for any compact set I *with negligible boundary.*

Proofs of the results stated above are based on the spectral method introduced in the fifties by Nagaev to study limit theorems for certain classes of Markov chains and strongly developed during last years (see e.g. [8, 7, 9, 11]). We investigate the spectral properties of the Markov operator P related to the Markov chain (X_n^x, s_n) and of its perturbations P_t by Fourier characters. On appropriately defined Banach spaces the operator P is quasi-compact and has a unique peripheral eigenvalue (i.e. the eigenvalue whose modulus is equal to the spectral radius) equal to 1. It turns out that for small t the peripheral eigenvalue $k(t)$ of P_t is also unique and the corresponding eigenspace is one-dimensional. Thus, the study of the characteristic function $\mathbb{E}_s[e^{itS_n^x}] = P_t^n\mathbf{1}(x,s)$, which is asymptotically close to $k^n(t)$, can be reduced to studying the behavior of $k(t)$ close to 1 for small values of t.

2. Fourier operators and their spectral properties. Given $x \in \mathbb{R}$ we consider on $\mathbb{R} \times S$ the Markov chain (X_n^x, s_n), where X_n^x is defined in (1.1) and $s_0 = s$ for some $s \in S$. We denote by P the corresponding Markov operator

$$Pf(x,s) = \mathbb{E}_s[f(X_1^x, s_1)]$$

and by $\widetilde{\nu}(f) = \mathbb{E}[f(X_0, s_0)]$ its stationary measure, for X_0 as in (1.3), where f is an arbitrary bounded measurable function on $\mathbb{R} \times S$.

On functions on $\mathbb{R} \times S$ we introduce the seminorm

$$[f]_{\varepsilon,\lambda} = \left\| \sup_{x \neq y} \frac{|f(x,s) - f(y,s)|}{|x-y|^{\varepsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}} \right\|$$

and the two norms

$$|f|_{\theta} = \left\| \sup_{x} \frac{|f(x,s)|}{(1+|x|)^{\theta}} \right\|, \qquad \|f\|_{\theta,\varepsilon,\lambda} = |f|_{\theta} + [f]_{\varepsilon,\lambda}.$$

Given γ, we may assume that the parameters $\varepsilon, \lambda, \gamma$ satisfy

$$\varepsilon < 1, \qquad \lambda + 3\varepsilon < \theta < 2\lambda < \gamma - 3\varepsilon, \qquad 1 + \varepsilon < \gamma. \tag{2.1}$$

Moreover we assume that all the assumptions of the limit theorems are satisfied, and from now we will refer to them without any further mention.

The norms define the Banach spaces

$$\mathbb{C}_{\theta} = \{f \colon |f|_{\theta} < \infty\}, \qquad \mathcal{B}_{\theta,\varepsilon,\lambda} = \{f \colon \|f\|_{\theta,\varepsilon,\lambda} < \infty\}.$$

On these Banach spaces we consider a family of Fourier operators related to the Markov kernel P. For $t \in \mathbb{R}$ we define

$$P_t f(x,s) = \mathbb{E}_s[e^{itX_1^x} f(X_1^x, s_1)].$$

We notice that by the Markov property,

$$(P_t^n f)(x,s) = \mathbb{E}_s[e^{itS_n^x} f(X_n^x, s_n)], \tag{2.2}$$

where $S_n^x = \sum_{k=1}^n X_k^x$. In particular if $\mathbf{1}$ is the function on $\mathbb{R} \times S$ identically equal to 1, then

$$(P_t^n \mathbf{1})(x,s) = \mathbb{E}_s[e^{itS_n^x}],$$

i.e. $P_t^n \mathbf{1}$ is just the characteristic function of S_n^x, which explains the role of the operator P_t in studying limit theorems related to S_n^x.

Notice that by the Jensen inequality, (1.7) implies $\mathbb{E}[\log A_0] < 0$. Therefore the sequence $\{(A_n, B_n)\}$ satisfies the assumptions of Brandt's theorem [2]. Hence (1.2) has a unique solution given by (1.3).

Lemma 2.3. *For every $\theta < \gamma$,*

$$\left\| \sup_{n \in \mathbb{N}} \mathbb{E}_s[|X_n^0|^{\theta}] \right\| < C(\theta) < \infty.$$

Proof. We consider two cases. If $\theta > 1$, then we use the Minkowski inequality. By the Markov property and (1.7) we have

$$\begin{aligned} \|(\mathbb{E}_s[|X_n^0|^{\theta}])^{1/\theta}\| &\leq \Big\| \sum_{j=1}^n \Big(\mathbb{E}_s \Big[|B_j|^{\theta} \prod_{i=j+1}^n A_i^{\theta} \Big] \Big)^{1/\theta} \Big\| \\ &\leq \sum_{j=1}^n \Big\| \mathbb{E}_s \Big[|B_j|^{\theta} \mathbb{E}_s \Big[\mathbb{E}_{s'} \Big[\prod_{i=1}^{n-i} A_i^{\theta} \Big] \Bigm| s_j = s' \Big] \Big] \Big\| \\ &\leq C_{\theta} \sum_{j=1}^{\infty} \rho_{\theta}^{j/\theta} \leq C(\theta). \end{aligned}$$

For $\theta \le 1$ we can repeat the calculations above, but we use the inequality $(a+b)^\theta \le a^\theta + b^\theta$ valid for any $a, b > 0$, instead of the Minkowski inequality. ∎

LEMMA 2.4. *For every $t \in \mathbb{R}$,*

$$|P_t^n f|_\theta \le C_1 |f|_\theta.$$

Moreover there exists $\rho < 1$ such that

$$[P_t^n f]_{\varepsilon,\lambda} \le C_2 \rho^n [f]_{\varepsilon,\lambda} + C_3 |t|^\varepsilon |f|_\theta.$$

Finally, for every η satisfying $\lambda + 2\varepsilon \le \eta \le \theta$, $\delta < \varepsilon$ and $s, t \in \mathbb{R}$,

$$|(P_s - P_t) f|_\eta \le C_4 |s-t|^\delta \|f\|_{\theta,\varepsilon,\lambda}.$$

We omit the proof since the arguments are exactly as in the i.i.d. case. See e.g. [4] for more details. One just has to use (1.7) and Lemma 2.3.

LEMMA 2.5. *The unique eigenvalue of modulus* 1 *of P acting on $\mathcal{B}_{\theta,\varepsilon,\lambda}$ is* 1 *and the corresponding eigenspace is* $\mathbb{C}\mathbf{1}$.

Proof. Assume that $Pf = zf$ for some nonzero $f \in \mathcal{B}_{\theta,\varepsilon,\lambda}$ and $z \in \mathbb{C}$, $|z| = 1$. Then for π-almost every s and every x by Lemma 2.4 we have

$$|f(x,s) - f(0,s)| = |P^n f(x,s) - P^n f(0,s)| \le C\rho^n [f]_{\varepsilon,\lambda} \cdot |x|^\varepsilon (1+|x|)^\lambda.$$

Since $\rho < 1$, the value above tends to 0 as n tends to ∞. Therefore $f(0,s) = f(x,s)$ for s, π-a.s., and f depends only on the second coordinate. The operator P acting on functions defined on S coincides with the transition probability H. Therefore, by (1.4), we have

$$z^n f(s) = P^n f(s) = H^n f(s) \to \pi(f).$$

So, z must be equal to 1 and then

$$f(s) = H^n f(s) \to \pi(f),$$

hence $f(s) = \pi(f)$ π-a.s. ∎

In view of the last lemma we may use the Ionescu Tulcea–Marinescu theorem [12] (see also [9]) for the operator P. It says that the operator P on $\mathcal{B}_{\theta,\varepsilon,\lambda}$ is quasi-compact, i.e. in our case the Banach space $\mathcal{B}_{\theta,\varepsilon,\lambda}$ can be decomposed into a sum of two closed P-invariant subspaces: $\mathcal{B}_{\theta,\varepsilon,\lambda} = \{\mathbb{C}\mathbf{1}\} \oplus \mathcal{H}$, where $\mathcal{H} = \{f \in \mathcal{B}_{\theta,\varepsilon,\lambda} : \widetilde{\nu}(f) = 0\}$, and moreover $r(P_{|\mathcal{H}}) < 1$. For our purpose we need a uniform control of the spectra $\sigma(P_t)$ for small t, and this is provided by the Keller–Liverani theorem [14].

PROPOSITION 2.6. *There exist $t_0 > 0$, $\delta > 0$ and $\rho < 1 - \delta$ such that for every $|t| < t_0$:*

- *The spectrum of P_t acting on $\mathcal{B}_{\theta,\varepsilon,\lambda}$ is contained in $D = \{z : |z| \le \rho\} \cup \{z : |z-1| < \delta\}$.*
- *The set $\sigma(P_t) \cap \{z : |z-1| < \delta\}$ consists of exactly one eigenvalue $k(t)$, the corresponding eigenspace is one-dimensional, and* $\lim_{t\to 0} k(t) = 1$.

- *If π_t is the projection of P_t onto the eigenspace mentioned above, then there exists an operator Q_t with spectral radius at most ρ, $\pi_t Q_t = Q_t \pi_t = 0$ and for every n,*

$$P_t^n f = k(t)^n \pi_t(f) + Q_t^n(f), \qquad f \in \mathcal{B}_{\theta,\varepsilon,\lambda}.$$

For any z belonging to the complement of D,

$$\|(z - P_t)^{-1} f\|_{\theta,\varepsilon,\lambda} \le C \|f\|_{\theta,\varepsilon,\lambda}$$

for some constant C independent of t.

The identity embedding of $\mathcal{B}_{\theta,\varepsilon,\lambda}$ into $\mathcal{B}_{\theta,\varepsilon,\lambda+\varepsilon}$ is continuous and the decomposition $P_t = k(t)\pi_t + Q_t$ coincides on both spaces and

$$\|(\pi_t - \pi_0) f\|_{\theta,\varepsilon,\lambda+\varepsilon} \le C|t|^{\varepsilon} \|f\|_{\theta,\varepsilon,\lambda}.$$

Finally

$$|k(t) - 1| \le C|t|^{\varepsilon}.$$

We omit the proof of the proposition; see [8] and [4] for details.

LEMMA 2.7. *For every $t \neq 0$, the spectral radius of P_t is strictly smaller than* 1.

Proof. Assume that there exists a nonzero function f belonging to $\mathcal{B}_{\theta,\varepsilon,\lambda}$ and z of modulus 1 such that

$$P_t f = z f.$$

The function f is bounded, since by Lemma 2.3,

$$|f(x,s)| = |P_t f(x,s)| \le P(|f|)(x,s) \le |f|_\theta \mathbb{E}_s[(1 + |X_1^x|^\theta)] < \infty.$$

Therefore for every n,

$$\widetilde{\nu}(P^n|f| - |f|) = 0.$$

However, the integrated function is positive:

$$|f(x,s)| = |z^n f(x,s)| = |P_t^n f(x,s)| \le P^n(|f|)(x,s).$$

Therefore for every n,

$$|f(x,s)| = P^n(|f|)(x,s) \qquad \widetilde{\nu}\text{-a.s.}$$

In view of Lemma 2.5, $|f|$ must be constant.

Next a convexity argument implies that for every n, s π-a.s. and for every x,

$$z^n f(x,s) = e^{itS_n^x} f(X_n^x, s_n) \qquad \mathbb{P}\text{-a.s.}$$

Hence

$$\frac{f(x,s)}{f(y,s)} e^{it(y-x)\sum_{j=0}^n A_1 \ldots A_j} = \frac{f(X_n^x, s_n)}{f(X_n^y, s_n)} \qquad \mathbb{P}\text{-a.s.}$$

Notice that the left hand side has a limit $\mathbb{P}$-a.s. as n tends to infinity. Since

$$\lim_{n\to\infty} \mathbb{E}\left| \frac{f(X_n^x, s_n)}{f(X_n^y, s_n)} - 1 \right| = 0$$

(this can be proved exactly as in [4, Lemma 3.14]), the limit of the right hand side is 1, $\mathbb{P}$-a.s. Therefore

$$\frac{f(x,s)}{f(y,s)} = e^{it(x-y)\sum_{j=0}^{\infty} A_1 \ldots A_j}.$$

Since the left hand side is nonrandom and the right one is random and nonconstant, t must be 0. Thus, $zf = Pf$ and in view of Lemma 2.5, z must be 1, and f must be constant. ∎

Now our aim is to prove the following proposition, which will be crucial in proving limit theorems.

PROPOSITION 2.8. *If $\gamma > 1$, then there exists $\delta > 0$ such that*

$$k(t) = 1 + itm + o(t^{1+\delta})$$

for $m = \int_{\mathbb{R}} x\,\nu(dx) = \mathbb{E}X_0$ and $\varepsilon < \delta < \gamma - 1$. If $\gamma > 2$, then

$$k(t) = 1 + itm - t^2\sigma^2/2 + o(t^2)$$

for some $\sigma^2 > 0$.

Proof. First applying rather standard arguments (see [9, 10]) we will prove the second part of the proposition.

Assume $\gamma > 2$. Fix two triples $(\theta, \varepsilon, \lambda)$ and $(\theta', \varepsilon, \lambda')$ satisfying (2.1) and additionally such that $\theta > \theta' + 2$ and $\lambda > \lambda' + 2$. For $k = 1, 2$ define

$$L_{k,t} f(x,s) = \mathbb{E}_s[(iX_1^x)^k e^{itX_1^x} f(X_1^x, s_1)].$$

Then $L_{k,t}$ is a bounded operator from $\mathcal{B}_{\theta',\varepsilon,\lambda'}$ to $\mathcal{B}_{\theta,\varepsilon,\lambda}$ and (cf. [10, Proposition 6.3])

$$\lim_{h\to 0} \frac{1}{|h|^n} \Big\| P_{t+h} - P_t - \sum_{k=1}^{n} \frac{h^k}{k!} L_{k,t} \Big\|_{\mathcal{B}_{\theta',\varepsilon,\lambda'}, \mathcal{B}_{\theta,\varepsilon,\lambda}} = 0. \tag{2.9}$$

Let $h_t \in \mathcal{B}_{\theta',\varepsilon,\lambda'} \subset \mathcal{B}_{\theta,\varepsilon,\lambda}$ be the eigenfunction of P_t,

$$P_t(h_t) = k(t) h_t,$$

such that $\widetilde{\nu}(h_t) = 1$. Then $h_t = \pi_t(1)/\widetilde{\nu}(\pi_t(1))$. Notice that

$$\widetilde{\nu}(\chi_t h_t) = k(t), \tag{2.10}$$

where $\chi_t(x,s) = e^{itx}$. We will prove that $t \mapsto h_t$ is twice differentiable. Indeed, it is enough to prove that π_t has, second derivative.

To compute the first derivative of π_t we will use the formula (see [14])

$$\pi_t = \frac{1}{2\pi i} \int_{|z-1|=\delta} (z - P_t)^{-1}\, dz.$$

Then for $f \in \mathcal{B}_{\theta',\varepsilon,\lambda'}$ we write

$$
\begin{aligned}
\frac{1}{h}(\pi_{t+h}-\pi_t)f &= \frac{1}{h}\,\frac{1}{2\pi i}\int\limits_{|z-1|=\delta}((z-P_{t+h})^{-1}-(z-P_t)^{-1})f\,dz\\
&= \frac{1}{h}\,\frac{1}{2\pi i}\int\limits_{|z-1|=\delta}(z-P_{t+h})^{-1}(P_{t+h}-P_t)(z-P_t)^{-1}f\,dz\\
&= \frac{1}{2\pi i}\int\limits_{|z-1|=\delta}(z-P_t)^{-1}L_{1,t}(z-P_t)^{-1}f\,dz\\
&\quad+\frac{1}{2\pi i}\int\limits_{|z-1|=\delta}((z-P_{t+h})^{-1}-(z-P_t)^{-1})L_{1,t}(z-P_t)^{-1}f\,dz\\
&\quad+\frac{1}{2\pi i}\int\limits_{|z-1|=\delta}(z-P_{t+h})^{-1}\left(\frac{P_{t+h}-P_t-hL_{1,t}}{h}\right)(z-P_t)^{-1}f\,dz.
\end{aligned}
$$

By Proposition 2.6 and (2.9) the second and the third integrals go to 0. So, the derivative $\pi_t^{(1)}$ of π_t is a bounded operator from $\mathcal{B}_{\theta',\varepsilon,\lambda'}$ to $\mathcal{B}_{\theta'+1,\varepsilon,\lambda'+1}$ and

$$\pi_t^{(1)}f = \frac{1}{2\pi i}\int\limits_{|z-1|=\delta}(z-P_t)^{-1}L_{1,t}(z-P_t)^{-1}f\,dz.$$

In the same we may compute the second derivative of π_t:

$$\pi_t^{(2)}f = \frac{1}{2\pi i}\int\limits_{|z-1|=\delta}(z-P_t)^{-1}L_{2,t}(z-P_t)^{-1}f\,dz.$$

Therefore h_t and hence $k(t)$ are twice differentiable at 0. In particular $k(t)$ can be expanded in a Taylor series

$$k(t) = 1 + k'(0)t + \frac{k''(0)}{2}t^2 + o(t^2).$$

To compute the derivative of k denote by ζ the derivative of $t \mapsto h_t$ at 0. Then differentiating (2.10) we obtain

$$k'(0) = im + \widetilde{\nu}(\zeta).$$

Since differentiating the equation $\widetilde{\nu}(h_t) = 1$ at zero we obtain $\widetilde{\nu}(\zeta) = 0$, it follows that

$$k'(0) = im.$$

Proceeding as in [4, Lemmas 6.2 and 6.7] one can explicitly compute the value of $k''(0)$, i.e. of $\sigma^2 = -k''(0)$ in terms of the function ζ, and then to prove that Lemma 2.7 implies $\sigma^2 > 0$.

Assume now $\gamma > 1$. We proceed as in [15]. Although that paper uses much stronger assumptions, including a heavy tail of ν, the method can be adapted to our situation.

Let $\delta_t f(x,s) = f(tx,s)$. We define another family of Fourier operators on $\mathcal{B}_{\theta,\varepsilon,\lambda}$:

$$T_t f(x,s) = \delta_t^{-1} P_t \delta_t f(x,s) = \mathbb{E}_s[e^{i(A_1x+tB_1)} f(A_1x + tB_1, s_1)], \quad t \neq 0,$$
$$Tf(x,s) = T_0 f(x,s) = \mathbb{E}_s[e^{i(A_1x)} f(A_1x, s_1)].$$

Notice that for $t \neq 0$, a function f is an eigenvalue of P_t if and only if $\delta_t^{-1} f$ is an eigenvalue of T_t. Moreover the function

$$h(x,s) = \mathbb{E}_s[e^{ix\sum_{j=1}^{\infty} A_1\ldots A_j}]$$

is the unique eigenfunction of T corresponding to eigenvalue 1 and there are no other eigenvalues of modulus 1. Indeed, by the Markov property,

$$\begin{aligned} Th(x,s) &= \mathbb{E}_s[e^{iA_1x} h(A_1x, s_1)] = \mathbb{E}_s\big[e^{iA_1x}\mathbb{E}_{s_1}\big[e^{iA_1x\sum_{j=2}^{\infty} A_2\ldots A_j}\big]\big] \\ &= \mathbb{E}_s\big[e^{ix\sum_{j=1}^{\infty} A_1\ldots A_j}\big] = h(x,s). \end{aligned}$$

To prove uniqueness one has to argue as in the proof of Lemma 2.5.

One can prove that for $\theta, \varepsilon, \lambda$ satisfying (2.1) the family T_t satisfies the conclusion Proposition 2.6 (in place of P_t), and in particular for small values of t, $T_t f = k(t)\widetilde{\pi}(t) f + \widetilde{Q}(t) f$ with the same eigenvalue $k(t)$ as for P_t.

Notice that $h_t = \delta_t \widetilde{\pi}_t h$ is an eigenfunction of P_t and since $\widetilde{\nu} P = \widetilde{\nu}$ we have

$$(k(t)-1)\widetilde{\nu}(h_t) = \widetilde{\nu} P_t(h_t) - \widetilde{\nu}(h_t) = \widetilde{\nu}((\chi_t - 1)h_t).$$

Therefore, denoting by g the function $g(x,s) = x$, we obtain

$$\begin{aligned} k(t) - 1 - itm &= \frac{1}{\widetilde{\nu}(h_t)}[\widetilde{\nu}((\chi_t - 1)h_t) - itm + itm(1 - \widetilde{\nu}(h_t))] \\ &= \frac{1}{\widetilde{\nu}(h_t)}[\widetilde{\nu}(\chi_t - 1 - itx) + \widetilde{\nu}((\chi_t - 1)(h_t - 1)) + itm\widetilde{\nu}(1 - h_t)]. \end{aligned}$$

Now, reasoning exactly as in [15] (see the proof of Theorem 6.3 there and the last inequality in it) we find that for any ρ, δ such that $0 < \rho < \gamma - 1$ and $\varepsilon < \delta < \gamma - 1$ there exists C such that

$$|h_t(x,s) - 1| \leq Ct^{\delta}(1 + |x|)^{\rho}.$$

Then, since $\widetilde{\nu}(h_t)$ converges to 1 as t tends to 0 and $\widetilde{\nu}(x^{1+\rho}) < \infty$, we obtain

$$\begin{aligned} |k(t) - 1 - itm| &\leq C[\widetilde{\nu}((tx)^{1+\delta}) + \widetilde{\nu}(tx \cdot t^{\delta}(1+|x|)^{\rho}) + tm\widetilde{\nu}(t^{\delta}(1+|x|)^{\rho})] \\ &\leq Ct^{1+\delta}. \ \blacksquare \end{aligned}$$

3. Limit theorems. Now we are going to prove the limit theorems related to the partial sums $S_n = X_1 + \cdots + X_n$. In fact the main work has been done in the previous section and in view of Proposition 2.8 the proof is rather classical (see e.g. [9]). However for the reader's convenience we present some details.

Proof of Theorem 1.6. By Proposition 2.6, for fixed t we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big[e^{it\frac{S_n-mn}{\sqrt{n}}}\big] &= \lim_{n\to\infty}\int\mathbb{E}_s\big[e^{it\frac{S_n^x-mn}{\sqrt{n}}}\big]\,\widetilde{\nu}(dx\,ds)\\ &= \lim_{n\to\infty}\int e^{-itm\sqrt{n}}P^n_{t/\sqrt{n}}(1)(x,s)\,\widetilde{\nu}(dx\,ds)\\ &= \lim_{n\to\infty}\big(e^{-itm/\sqrt{n}}k(t/\sqrt{n})\big)^n\cdot\lim_{n\to\infty}\int\pi_{t/\sqrt{n}}(1)(x,s)\,\widetilde{\nu}(dx\,ds)\\ &\quad+\lim_{n\to\infty}\int Q^n_{t/\sqrt{n}}(1)(x,s)\,\widetilde{\nu}(dx\,ds)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\Big|\int\pi_{t/\sqrt{n}}(1)(x,s)\,\widetilde{\nu}(dx\,ds)-1\Big|\\ &= \lim_{n\to\infty}\Big|\int(\pi_{t/\sqrt{n}}(1)(x,s)-\pi(1)(x,s))\,\widetilde{\nu}(dx\,ds)\Big|\\ &\le C\lim_{n\to\infty}\Big|\int|t/\sqrt{n}|^{\varepsilon}(1+|x|^{\theta})\,\widetilde{\nu}(dx\,ds)\Big|=0,\end{aligned}$$

$$\lim_{n\to\infty}\Big|\int Q^n_{t/\sqrt{n}}(1)(x,s)\,\widetilde{\nu}(dx\,ds)\Big|\le C\lim_{n\to\infty}\rho^n\int(1+|x|^{\theta})\,\nu(dx)=0.$$

Therefore it is sufficient to prove

$$\begin{aligned}\lim_{n\to\infty}\big(e^{-i\frac{tm}{\sqrt{n}}}k(t/\sqrt{n})\big)^n &= e^{\lim_{n\to\infty}(n(e^{-itm/\sqrt{n}}k(t/\sqrt{n})-1))}\\ &= e^{-t^2/2\sigma^2}.\end{aligned}\tag{3.1}$$

For this purpose applying Proposition 2.8 we write

$$\begin{aligned}&\lim_{n\to\infty}\big(n\big(e^{-i\frac{tm}{\sqrt{n}}}k(t/\sqrt{n})-1\big)\big)\\ &= \lim_{n\to\infty}\bigg[n\bigg(1-\frac{itm}{\sqrt{n}}+\frac{t^2m^2}{n}+O(n^{-3/2})\bigg)\bigg(1+i\frac{tm}{\sqrt{n}}-\frac{t^2\sigma^2}{2n}+o(1/n)\bigg)-1\bigg]\\ &= -\frac{t^2}{2}\sigma^2.\ \blacksquare\end{aligned}$$

Proof of Theorem 1.8. In view of [3, Theorem 10.7] it is sufficient to prove

$$\lim_{n\to\infty}n^{1/2}\mathbb{E}[h(S_n-nm)]=C_0I(h)$$

for any $h\in L^1$ such that the Fourier transform of h is compactly supported. Take M such that $\operatorname{supp}\widehat{h}\subset[-M,M]$ and fix $0<\delta<t_0$. By the Fourier inversion formula and Proposition 2.6 we have

$$\begin{aligned}\sqrt{n}\,\mathbb{E}[h(S_n-nm)] &= \frac{\sqrt{n}}{2\pi}\int_{\mathbb{R}}\mathbb{E}[e^{it(S_n-nm)}\widehat{h}(t)]\,dt\\ &= \frac{\sqrt{n}}{2\pi}\int\int_{\mathbb{R}}e^{-itnm}P^n_t(1)(x,s)\widehat{h}(t)\,dt\,\widetilde{\nu}(dx\,ds)\end{aligned}$$

$$= \frac{\sqrt{n}}{2\pi}\int\int_{|t|<\delta} e^{-itnm}k(t)^n\pi_t(1)(x,s)\widehat{h}(t)\,dt\,\widetilde{\nu}(dx\,ds)$$

$$+ \frac{\sqrt{n}}{2\pi}\int\int_{|t|<\delta} e^{-itnm}Q_t^n(1)(x,s)\widehat{h}(t)\,dt\,\widetilde{\nu}(dx\,ds)$$

$$+ \frac{\sqrt{n}}{2\pi}\int\int_{\delta<|t|<M} e^{-itnm}P_t^n(1)(x,s)\widehat{h}(t)\,dt\,\widetilde{\nu}(dx\,ds);$$

since the function $t \mapsto \limsup_{n\to\infty}\|P_t^n 1\|^{1/n}$ is upper semicontinuous, $r(P_t) < 1$ (Lemma 2.7) and by Proposition 2.6, $r(Q_t) < 1$ for $t < \delta$, the second and the third expressions converge to 0 as n tends to infinity. Therefore, since $h(0) = l(h)$, by the Lebesgue theorem and (3.1) we have

$$\lim_{n\to\infty}\sqrt{n}\,\mathbb{E}[h(S_n - nm)]$$

$$= \lim_{n\to\infty}\frac{\sqrt{n}}{2\pi}\int\int_{|t|<\delta} e^{-itnm}k(t)^n\pi_t(1)(x,s)\widehat{h}(t)\,dt\,\widetilde{\nu}(dx\,ds)$$

$$= \lim_{n\to\infty}\frac{1}{2\pi}\int\int_{|t|<\delta\sqrt{n}} \big(e^{-itm/\sqrt{n}}k(t/\sqrt{n})\big)^n\pi_{t/\sqrt{n}}(1)(x,s)\widehat{h}(t/\sqrt{n})\,dt\,\widetilde{\nu}(dx\,ds)$$

$$= \frac{l(h)}{2\pi}\int_{\mathbb{R}} e^{-t^2\sigma^2/2}\,dt. \ \blacksquare$$

Proof of Theorem 1.9. It is sufficient to prove

$$\lim_{y\to\infty} U(h_y) = \frac{l(h)}{m}$$

where $h \in L^1$, the Fourier transform of h is differentiable and compactly supported and $h_y(x) = h(x-y)$. We assume that the support of $\widehat{h}$ is contained in the interval $[-M,M]$ for some $M > 0$. Since $\widehat{h}_y(t) = e^{-ity}\widehat{h}(t)$, also $\operatorname{supp}\widehat{h}_y \subset [-M,M]$. For $\eta < 1$ define

$$U_\eta(h) = \sum_{n\ge1}\eta^n\mathbb{E}[h(S_n)].$$

Then by the Fourier inversion theorem and Proposition 2.6 for $\delta < t_0$ we have

$$U_\eta(h_y) = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{h}_y(t)\sum_{n\ge1}\eta^n\mathbb{E}[e^{itS_n}]\,dt$$

$$= \frac{1}{2\pi}\int\int_{\mathbb{R}}\widehat{h}_y(t)\sum_{n\ge1}\eta^n P_t^n(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds)$$

$$= \frac{1}{2\pi}\int\int_{|t|<\delta}\widehat{h}_y(t)\sum_{n\ge1}\eta^n k^n(t)\pi_t(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds)$$

$$+\frac{1}{2\pi}\int\int_{|t|<\delta}\widehat{h}_y(t)\sum_{n\geq 1}\eta^n Q_t^n(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds)$$
$$+\frac{1}{2\pi}\int\int_{\delta<|t|<M}\widehat{h}_y(t)\sum_{n\geq 1}\eta^n P_t^n(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds).$$

The second and the third summand converge to 0 as $\eta\to 1$ and $y\to\infty$. Indeed, notice first that since

$$\int\Big|\widehat{h}(t)\sum_{n\geq 1}\eta^n Q_t^n(1)(x,s)\Big|\widetilde{\nu}(dx\,ds)\leq\Big|\widehat{h}(t)\cdot\frac{\rho\eta}{1-\rho\eta}\Big|,$$

for ρ being as in Proposition 2.6, the function on the left hand side is integrable on $[-\delta,\delta]$. Therefore, by the Riemann–Lebesgue theorem,

$$\lim_{y\to\infty}\lim_{\eta\to 1}\Big|\int\int_{|t|<\delta}e^{-iyt}\widehat{h}(t)\sum_{n\geq 1}\eta^n Q_t^n(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds)\Big|=0.$$

Similarly we deal with the third term. To estimate the first summand we decompose it further into three terms:

$$\frac{1}{2\pi}\int\int_{|t|<\delta}e^{-ity}\frac{\eta k(t)\widehat{h}(t)}{1-\eta k(t)}\pi_t(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds)$$
$$=\frac{\eta}{2\pi}\int\int_{|t|<\delta}e^{-ity}\bigg(\frac{\widehat{h}(t)k(t)}{1-\eta k(t)}-\frac{\widehat{h}(0)}{1-\eta(1+imt)}\bigg)\pi_t(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds)$$
$$+\frac{\eta}{2\pi}\int\int_{|t|<\delta}e^{-ity}\widehat{h}(0)\frac{\pi_t(1)(x,s)-1}{1-\eta(1+imt)}\,dt\,\widetilde{\nu}(dx\,ds)+\frac{\eta\widehat{h}(0)}{2\pi}\int_{|t|<\delta}\frac{e^{-ity}}{1-\eta(1+imt)}\,dt$$
$$=I(\eta,y)+II(\eta,y)+III(\eta,y).$$

We will prove that I and II converge to 0 as $\eta\to 1$ and $y\to\infty$. Indeed, notice that for t sufficiently small and η close to 1,

$$|1-\eta k(t)|\geq m|t|/2\quad\text{and}\quad|1-\eta(1+imt)|\geq m|t|/2.$$

Since both k and $\widehat{h}$ are differentiable there exists a continuous function ψ such that $\widehat{h}(t)k(t)=\widehat{h}(0)+t\psi(t)$. Therefore, writing

$$F(\eta,t)=\frac{\widehat{h}(t)k(t)}{1-\eta k(t)}-\frac{\widehat{h}(0)}{1-\eta(1+imt)}$$
$$=\frac{t\psi(t)}{1-\eta k(t)}+\widehat{h}(0)\cdot\frac{k(t)-1-imt}{(1-\eta k(t))(1-\eta(1+imt))},$$

we see that $\widetilde{F}(t)=\lim_{\eta\to 1}F(\eta,t)$ is an integrable function on the interval

$[-\delta, \delta]$. Then by the Riemann–Lebesgue theorem,

$$\lim_{y\to\infty}\lim_{\eta\to 1} I(\eta, y) = \lim_{y\to\infty}\iint e^{-iyt} 1_{\{|t|\le\delta\}}\widetilde{F}(t)\pi_t(1)(x,s)\,dt\,\widetilde{\nu}(dx\,ds) = 0.$$

Similarly we prove that II converges to 0. Indeed, Proposition 2.6 implies integrability of the function

$$t \mapsto \frac{\int (\pi_t(1)(x,s) - 1)\,\widetilde{\nu}(dx\,ds)}{imt}.$$

Finally

$$\begin{aligned}\lim_{y\to\infty} U(h_y) &= \lim_{y\to\infty}\lim_{\eta\to 1} U_\eta(h_y) = \lim_{y\to\infty}\lim_{\eta\to 1}\frac{1}{2\pi}\int\limits_{|t|<\delta}\frac{e^{-ity}}{1-\eta(1+imt)}\,dt\\ &= \lim_{y\to\infty}\frac{1}{2\pi m}\Bigg(\pi + \int\limits_{|t|<\delta y}\frac{\sin t}{dt}\Bigg)\cdot l(h) = \frac{l(h)}{m},\end{aligned}$$

where the last but one equality was proved in [9, p. 47]. ∎

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