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ON THE DIOPHANTINE EQUATION $x^y - y^x = c^z$

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Abstract. Applying results on linear forms in p-adic logarithms, we prove that if (x,y,z) is a positive integer solution to the equation $x^y - y^x = c^z$ with $\gcd(x,y)=1$ then $(x,y,z)=(2,1,k)$, $(3,2,k)$, $k\ge 1$ if $c=1$, and either $(x,y,z)=(c^k+1,1,k)$, $k\ge 1$ or $2\le x<y\le \max\{1.5\times 10^{10}, c\}$ if $c\ge 2$.

1. Introduction. Kenichiro Kashihara [Ka] asked to solve the equation $x^y+y^z+z^x=0$, i.e., to find its integer solutions. Recently, Yanni Liu and Xiaoyan Guo [LG] answered this question by showing that $(x,y,z)=(-2,1,1)$, $(1,-2,1)$, $(1,1,-2)$, $(1,-1,-2)$, $(-1,-2,1)$, $(-2,1,-1)$ are the only integer solutions.

Let a,b,c be odd positive integers and $H=\max\{a,b,c\}$. The first and the third authors [ZY] proved that all integer solutions to the equation $ax^y+by^z+cz^x=0$ with $xyz\ne 0$ satisfy $\max\{|x|,|y|,|z|\}\le 2H$. In that paper, they also considered the equation $x^y+y^z=z^x$ and using a result of Stewart and Kunrui Yu [SY] on the ABC conjecture, showed that all positive integer solutions satisfy $\max\{x,y,z\}<\exp(\exp(\exp(5)))$.

The aim of this paper is to consider the equation $x^y-y^x=c^z$, where the positive integer c is given, and to prove the following result.

THEOREM 1.1. *Let c be a positive integer, and (x,y,z) a positive integer solution to the equation*

$$x^y - y^x = c^z,\quad \gcd(x,y)=1. \tag{1.1}$$

Then either

(i) $c=1$, $(x,y,z)=(2,1,k),(3,2,k)$, $k\ge 1$; *or*
(ii) $c\ge 2$, $(x,y,z)=(c^k+1,1,k)$, $k\ge 1$, *or* $2\le x<y\le \max\{1.5\times 10^{10}, c\}$.

In the case where $c\ge 2$ and $y>x\ge 2$, from $c^z<x^y$ we get $z<y\log x/\log c<y\log y/\log c$, which is a bound for z depending on c. We deduce the following corollary.

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COROLLARY 1.2. *Let (x, y, z) be a positive integer solution to the equation*

$$x^y - y^x = z^z. \tag{1.2}$$

Then $\gcd(x, y, z) = 1$ *and* $(x, y, z) = (3, 2, 1)$ *or* $(x, y, z) = (k^k + 1, 1, k)$ *for some* $k \geq 1$ *or* $2 \leq x < z < y < 1.5 \times 10^{10}$.

This paper is organized as follows. In Section 2, we recall or prove some lemmas needed for the proofs of Theorem 1.1 and Corollary 1.2. In Section 3, we prove Theorem 1.1 and Corollary 1.2.

2. Some preliminary results. In this section, we recall two lemmas and prove one. They will help us to prove Theorem 1.1 and Corollary 1.2 in the next section.

Let x_1/y_1 and x_2/y_2 be two nonzero rational numbers and p a prime satisfying $v_p(x_1/y_1) = v_p(x_2/y_2) = 0$. We denote by g the smallest positive integer such that $v_p((x_1/y_1)^g - 1) > 0$ and $v_p((x_2/y_2)^g - 1) > 0$.

Let E be a real number such that

$$v_p\left(\left(\frac{x_1}{y_1}\right)^g - 1\right) \geq E > \frac{1}{p-1}.$$

We need an explicit upper bound for the p-adic valuation of

$$\Lambda = \left(\frac{x_1}{y_1}\right)^{b_1} - \left(\frac{x_2}{y_2}\right)^{b_2},$$

where b_1, b_2 are positive integers. Let $A_1, A_2 > 1$ be real numbers such that

$$\log A_i \geq \max\{\log|x_i|, \log|y_i|, E\log p\}, \qquad i = 1, 2,$$

and put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1},$$
$$f = \log b' + \log(E\log p) + 0.4.$$

Then we have the following lemma (see [B, Theorem 2]).

LEMMA 2.1. *With the above notation, if x_1/y_1 and x_2/y_2 are multiplicatively independent, then*

$$v_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}(\max\{f, 6E\log p, 5\})^2 \log A_1 \log A_2 \tag{2.1}$$

and

$$v_p(\Lambda) \leq \frac{53.8g}{E^3(\log p)^4}(\max\{f, 4E\log p, 5\})^2 \log A_1 \log A_2, \tag{2.2}$$

whenever p is odd or when $p = 2$ and $v_2(x_2/y_2 - 1) \geq 2$. Else, we have

$$v_2(\Lambda) \leq 208(\max\{\log b' + 0.04, 10\})^2 \log A_1 \log A_2.$$

The following result can be found in many calculus textbooks. Nevertheless, we give its proof.

LEMMA 2.2. *Let $b \geq 3$ be an integer nd $f(x) = b^x - x^b$. Then $f(x)$ is an increasing function when $x > b$. In particular, $f(x) > 0$ when $x \in (b, \infty)$.*

Proof. We have $f^{(b)}(x) = b^x(\log b)^b - b! > 0$ for $x > b$. In the same way, $f^{(b-1)}(b) = b^b(\log b)^{b-1} - b \cdot b! > 0$. So we deduce that $f^{(b-1)}(x) > 0$ when $x > b$. Continuing this process we get $f'(x) > 0$ for $x > b$. This means that $f(x)$ is an increasing function when $x > b$. As $f(b) = 0$ one can see that $f(x) > 0$ when $x > b \geq 3$. ∎

3. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Let (x, y, z) be a positive integer solution to equation (1.1).

If $c = 1$, then $2 \mid xy$ and by the results of V. Lebesgue [L] and Chao Ko [Ko] on the Catalan equation, one has $(x, y, z) = (3, 2, k)$ with $k \geq 1$ when $y \geq 2$ or $(x, y, z) = (2, 1, k), k \geq 1$ when $y = 1$.

So we consider $c \geq 2$. Moreover, if $y = 1$, one gets $(x, y, z) = (c^k+1, 1, k)$, $k \geq 1$. It is obvious that $x \geq 2$. Therefore, we assume $x, y, c \geq 2$.

CLAIM 1. $x < y$.

Assume the contrary, i.e. $x \geq y$. Since $\gcd(x, y) = 1$, one has $x > y \geq 2$. Then $y \geq 3$ is impossible by Lemma 2.2. Hence $y = 2$ and (1.1) becomes $x^2 = 2^x + c^z \geq 2^x + 2$. Let $g(x) = 2^x + 2 - x^2$. Then $g(3) = 1 > 0$, $g(4) = 2 > 0$, and $g'(x) = 2^x \log 2 - 2x > 0$ when $x \geq 4$. Therefore, $g(x) > 0$ when $x \geq 4$, which is a contradiction.

CLAIM 2. $y \leq \max\{1.5 \times 10^{10}, c\}$.

Suppose that $y > \max\{1.5 \times 10^{10}, c\}$. We will deduce a contradiction from Lemma 2.1. Equation (1.1) implies $c^z < x^y$. Then $z < y \log x/\log c < y \log y/\log c$. There are three cases covering all possibilities.

CASE 2.1: $2 \mid x$, $2 \nmid cy$. In this case, we have $c \geq 3$. If $2 \mid z$, then $y^x + c^z \equiv 1 + 1 \equiv 2 \pmod 8$, which is impossible since $v_2(x^y) \geq y \geq x + 1 \geq 3$. Therefore, $2 \nmid z$.

Let
$$\Lambda = y'^x - (-c)^z,$$
where $y' = \pm y$ with $y' \equiv 1 \pmod 4$. We will apply Lemma 2.1 to get an upper bound for the 2-adic valuation of Λ. We take
$$b_1 = x,\ b_2 = z,\ g = 1,\ x_1 = y',\ y_1 = 1,\ x_2 = -c,\ y_2 = 1$$
and $E = 2$ as $v_2(y' - 1) \geq 2 > 1/(2 - 1) = 1$. Also we choose $A_1 = A_2 = y$ since $y > \max\{1.5 \times 10^{10}, c\}$. So

$$b' = \frac{x}{\log y} + \frac{z}{\log y}.$$

As $-c \equiv 1 \pmod 8$ we have $v_2(-c-1) \geq 3$. We can use (2.1) to obtain a bound. The assumption $y \geq 1.5 \times 10^{10}$ implies $6E \log p = 6 \times 2 \log 2 < \log y$, $5 < \log y$ and

$$\log b' + \log(E \log p) + 0.4 < \log\left(\frac{x}{\log y} + \frac{y \log y}{\log y \log c}\right) + \log(2 \log 2) + 0.4$$
$$< \log 2y - \log(\log 3) + \log(2 \log 2) + 0.4 < 1.057 \log y.$$

By inequality (2.1) and $x^y = \Lambda$, we get

$$y \leq v_2(\Lambda) \leq \frac{36.1 \times 1}{2^3 \times (\log 2)^4}(1.057 \log y)^2 \log y \log y < 21.841 \log^4 y,$$

which is impossible for $y \geq 1.5 \times 10^{10}$. In fact, the monotonicity of the function $h(y) = 21.841 \log^4 y - y$ yields $h(y) < 0$ when $y \geq 1.5 \times 10^{10}$.

Case 2.2: $2 \mid y, 2 \nmid cx$. Again here we have $c, x \geq 3$. If $2 \nmid z$, then $0 \equiv y^x = x^y - c^z \equiv 1 - c \pmod 8$, i.e. $c \equiv 1 \pmod 8$.

Let

$$\Lambda = x'^y - c'^z,$$

where $x' = \pm x$ with $x' \equiv 1 \pmod 4$ and

$$c' = \begin{cases} c, & 2 \nmid z, \\ \pm c, & 2 \mid z, \end{cases}$$

with $c' \equiv 1 \pmod 4$. Using an argument similar to that of Case 2.1 we get $x < 21.841 \log^4 y$.

Let p be a prime factor of x. One has $3 \leq p \leq x$, $p \nmid cy$. Let

$$\Lambda = c^z - (-y)^x.$$

Then $g \leq p - 1 \leq x - 1 < x$. We take

$$b_1 = z,\ b_2 = x,\ E = 1,\ x_1 = c,\ y_1 = 1,\ x_2 = -y,\ y_2 = 1.$$

As $y > \max\{1.5 \times 10^{10}, c\}$, we put $A_1 = A_2 = y$. Thus

$$b' = \frac{z}{\log y} + \frac{x}{\log y}.$$

Since $p \geq 3$, we will use (2.2) to get a bound. In fact, as $y > x \geq p$ and $y > \max\{1.5 \times 10^{10}, c\}$, we have

$$\log b' + \log(\log p) + 0.4 < \log\left(\frac{y \log y}{\log y \log c} + \frac{y}{\log y}\right) + \log(\log y) + 0.4$$
$$< \log 2y + \log(\log y) - \log(\log 3) + 0.4 < 1.038 \log(y \log y).$$

We consider two subcases.

Subcase 2.2.1: $1.038\log(y\log y) \geq 4\log p$. By inequality (2.2) and $\Lambda = x^y$, one has

$$y \leq v_p(\Lambda) \leq \frac{53.8 \times g}{1^3 \times \log^4 p} 1.038^2 \log^2(y\log y)\log y \log y$$
$$< \frac{57.97p}{\log^4 p}\log^2(y\log y)\log^2 y.$$

We claim

$$\frac{57.97p}{\log^4 p} < 0.03\log^4 y.$$

If $p > 1.7 \times 10^6$, then as

$$\frac{57.97 \times 21.841}{\log^4(1.7 \times 10^6)} < 0.0299 < 0.03,$$

one has

$$\frac{57.97p}{\log^4 p} < \frac{57.97x}{\log^4(1.7 \times 10^6)} < \frac{57.97 \times 21.841}{\log^4(1.7 \times 10^6)}\log^4 y < 0.03\log^4 y.$$

Now we suppose that $3 \leq p < 1.7 \times 10^6$. As

$$0.03\log^4 y \geq 0.03\log^4(1.5 \times 10^{10}) > 9042$$

and the function $k(p) = p/\log^4 p$ is increasing for $p \geq 59$ and decreasing for $3 \leq p \leq 53$, it follows that

$$\frac{57.97 \times 3}{\log^4 3} < 120, \qquad \frac{57.97 \times 1.7 \times 10^6}{\log^4(1.7 \times 10^6)} < 2327.$$

Therefore, one also has

$$\frac{57.97p}{\log^4 p} < 0.03\log^4 y.$$

From the above, we have

$$y < 0.03\log^4 y \log^2(y\log y)\log^2 y = 0.03\log^2(y\log y)\log^6 y.$$

This is impossible when $y \geq 1.5 \times 10^{10}$.

Subcase 2.2.2: $1.038\log(y\log y) < 4\log p$. Inequality (2.2) and $\Lambda = x^y$ imply

$$y \leq v_p(\Lambda) \leq \frac{53.8 \times g}{1^3 \times \log^4 p} 4^2 \log^2 p \log y \log y < \frac{860.8p}{\log^2 p}\log^2 y.$$

We claim that

$$\frac{860.8p}{\log^2 p} < 76.28\log^4 y.$$

If $p > 6.58 \times 10^6$, then

$$\frac{860.8 \times 21.841}{\log^2(6.58 \times 10^6)} < 76.28.$$

Hence

$$\frac{860.8p}{\log^2 p} < \frac{860.8x}{\log^2(6.58 \times 10^6)} < \frac{860.8 \times 21.841}{\log^2(6.58 \times 10^6)} \log^4 y < 76.28 \log^4 y.$$

In the same way, if $3 \le p < 6.58 \times 10^7$, one gets

$$76.28 \log^4 y > 76.28 \log^4(1.5 \times 10^{10}) > 22993093.$$

As the function $j(p) = p/\log^2 p$ is increasing for $p \ge 11$ and decreasing for $3 \le p \le 7$, we obtain

$$\frac{860.8 \times 3}{\log^2 3} < 2140, \qquad \frac{860.8 \times 6.58 \times 10^6}{\log^2(6.58 \times 10^6)} < 22980211.$$

So again we get

$$\frac{860.8p}{\log^2 p} < 76.28 \log^4 y.$$

Therefore, we have $y < 76.28 \log^4 y \log^2 y < 76.28 \log^6 y$, which contradicts the monotonicity of the function $m(y) = 76.28 \log^6 y - y$ when $y \ge 1.5 \times 10^{10}$.

Case 2.3: $2 \mid c$, $2 \nmid xy$. We begin by proving the following assertion:

Assertion. $z < 19.6 \log^4 y$.

Without loss of generality, we assume $z \ge 2$. Then we have $x^y - y^x \equiv x - y \equiv 0 \pmod 4$.

First, suppose that $x \equiv y \equiv 1 \pmod 4$. Let

$$\Lambda = x^y - y^x.$$

We take

$$b_1 = y,\ b_2 = x,\ g = 1,\ x_1 = x,\ y_1 = 1,\ x_2 = y,\ y_2 = 1,\ E = 2.$$

Then we can choose $A_1 = A_2 = y$, since $x < y$, $y > 1.5 \times 10^{10}$. So we have

$$b' = \frac{y}{\log y} + \frac{x}{\log y}.$$

As $v_2(y-1) \ge 2$, we will use (2.1) of Lemma 2.1 to obtain a bound. The assumption $y \ge 1.5 \times 10^{10}$ implies $6E \log p = 6 \times 2 \log 2 < \log y$, $5 < \log y$ and

$$\log b' + \log(E \log p) + 0.4 < \log y.$$

From inequality (2.1) and the equation $c^z = \Lambda$, we have

$$z \le v_2(\Lambda) \le \frac{36.1 \times 1}{2^3 \times (\log 2)^4} (\log y)^2 \log y \log y < 19.6 \log^4 y.$$

Second, we suppose that $x \equiv y \equiv 3 \pmod 4$. In this case, we take

$$\Lambda = (-y)^x - (-x)^y$$

and we use an argument similar to that of the case $x \equiv y \equiv 1 \pmod 4$ to obtain $z < 19.6 \log^4 y$.

The remaining proof is divided into two subcases: $z < x$ or $z \geq x$.

SUBCASE 2.3.1: $z < x$. If $z < x \leq c$, then we obtain $z < x < c$ and $x \geq 3$, $c \geq 4$ since $2 \nmid x$, $2 \mid c$.

Let $n(y) = x^y - y^x - c^{x-1}$. Then $n(y) \leq x^y - y^x - c^z = 0$. On the other hand, by Lemma 2.2, one has $n'(y) = x^y \log x - xy^{x-1} > x^y - y^x > 0$ when $y \geq c+1 > x \geq 3$ and

$$\begin{aligned} n(c+1) &= x^{c+1} - (c+1)^x - c^{x-1} = x^{c+1} - c^x\left(\left(1 + \frac{1}{c}\right)^x + \frac{1}{c}\right) \\ &> x^{c+1} - c^x\left(\left(1 + \frac{1}{c}\right)^c + \frac{1}{c}\right) > x^{c+1} - c^x(2.72 + 0.25) \\ &> x^{c+1} - 3c^x \geq x(x^c - c^x) > 0 \end{aligned}$$

when $c > x \geq 3$. This means that $n(y) > 0$ when $y \geq c+1$. Therefore, we get $y \leq c$, which contradicts the assumption $y > \max\{1.5 \times 10^{10}, c\}$.

Now we assume that $x > c, x > z$. Then

$$\begin{aligned} n(x+1) &= x^{x+1} - (x+1)^x - c^{x-1} = x^x\left(x - \left(1 + \frac{1}{x}\right)^x\right) - c^{x-1} \\ &> x^x(x - 2.72) - c^{x-1} > 0. \end{aligned}$$

This inequality is obvious if $x \geq 4$. If $x = 3$, $c = 2$, then the inequality can be easily verified by direct calculation. From Lemma 2.2, we know that $n(y) \geq n(x+1) > 0$ as $3 \leq x < y$, which also contradicts $n(y) \leq x^y - y^x - c^z = 0$. Therefore, $x \leq z$ when $x > c$.

SUBCASE 2.3.2: $z \geq x$. In this case, we have $x \leq z < 19.6 \log^4 y < 21.841 \log^4 y$. From the proof of Case 2.2, we know that this is impossible when $y > \max\{1.5 \times 10^{10}, c\}$. This completes the proof of Theorem 1.1.

Proof of Corollary 1.2. From the proof of Theorem 1.1, we know that $x < y$ when $x, y, z \geq 2$. Then the inequality $z < y$ is obvious. Let $n_1(y) = x^y - y^x - (x-1)^{x-1}$. One has

$$\begin{aligned} n_1(x+1) &= x^{x+1} - (x+1)^x - (x-1)^{x-1} \\ &= x^x\left(x - \left(1 + \frac{1}{x}\right)^x\right) - (x-1)^{x-1} \\ &> x^x(x - 2.72) - (x-1)^{x-1} > 0, \end{aligned}$$

when $x \geq 3$. By Lemma 2.2, we have $n_1(y) \geq n_1(x+1) > 0$ since $3 \leq x < y$. This means that $x \leq z$. We will prove that $\gcd(x, y, z) = 1$. This implies $x < z < y$, and Corollary 1.2 will follow from Theorem 1.1.

Suppose $p \mid \gcd(x, y, z)$ where p is a prime. Then $p = 2$ by the result of Wiles [W] on Fermat's Last Theorem. Therefore, we put

$$x = 2^\alpha u,\ y = 2^\beta v,\ z = 2^\gamma w,\ \alpha, \beta, \gamma \geq 1,\ 2 \nmid uvw,\ \gcd(u, v, w) = 1.$$

Substitution in equation (1.2) gives

$$2^{\alpha y} u^y - 2^{\beta x} v^x = 2^{\gamma z} w^z. \tag{3.1}$$

By the well known results of Fermat on the equations $x^4 \pm y^4 = z^2$, we know that at least one of α, β and at least one of α, γ must equal to one.

Case 1: $\alpha \geq 2$. Then $\beta = \gamma = 1$ and (3.1) becomes $2^{\alpha y} u^y - 2^x v^x = 2^z w^z$. Since $x < y$, $z < y$, we compare the exponents of 2 in the equation to get $x = z$. Thus $2^\alpha u = 2^\gamma w = 2w$. This implies $\alpha = 1$, a contradiction.

Case 2: $\alpha = 1$. Then (3.1) becomes

$$2^y u^y - 2^{\beta x} v^x = 2^{\gamma z} w^z. \tag{3.2}$$

So two of $y, \beta x, \gamma z$ should be equal. Therefore, we consider three subcases.

Subcase 2.1: $y = \beta x$. In this case, $\beta > 1$ and $\gamma z > y$. The condition $y = \beta x$ can be rewritten as $2^\beta v = \beta 2^\alpha u = 2\beta u$. Then we get $2^\beta \mid 2\beta$. Hence $\beta > 1$ leads to $\beta = 2$. Thus we have $y = 2x$, $u = v$. Since $\gcd(u, v) = 1$, one has $u = v = 1$. Equation (3.2) becomes $2^{\gamma z - 2x} w^z = u^{2x} - u^x = 0$, which is impossible.

Subcase 2.2: $y = \gamma z$. Then $\beta x > y$ and $z \mid y$. So $w \mid v$. Thus we get $w = 1$. Equation (3.2) becomes $u^y - 2^{\beta x} v^x = 1$. From the condition $2 \mid x$, $2 \mid y$ we know that this is impossible since $v \neq 0$.

Subcase 2.3: $\beta x = \gamma z$. In this case, we have $y > \beta x$ and equation (3.2) becomes $2^{y - \beta x} u^y = v^x + w^z$. Hence $2^{y - \beta x} u^y = v^x + w^z \equiv 2 \pmod 8$. Therefore, $y - \beta x = 1$, which contradicts the assumption $2 \mid x$ and $2 \mid y$. From the above discussion, we get $\gcd(x, y, z) = 1$. This completes the proof of Corollary 1.2.

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