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AFFINE LIFTINGS OF TORSION-FREE CONNECTIONS TO WEIL BUNDLES

BY

JACEK DĘBECKI (Kraków)

Abstract. This paper contains a classification of all affine liftings of torsion-free linear connections on n-dimensional manifolds to any linear connections on Weil bundles under the condition that $n \geq 3$.

A lifting (the so-called complete lifting) of connections to Weil bundles was constructed long ago in [5]. But the problem of finding all such liftings is still unsolved and seems to be hard. The recent result [2] on linear liftings of symmetric tensor fields of type $(1,2)$ to Weil bundles enables us to cope with a very special case of this problem, namely finding all affine liftings of torsion-free connections to Weil bundles.

Let A be a Weil algebra inducing the Weil functor T^A (see [4]) and let n be a non-negative integer. We will denote by $\operatorname{Co} M$ the set of all linear connections on a manifold M and by $\operatorname{ToFrCo} M$ the set of all torsion-free linear connections on M.

A *lifting of torsion-free linear connections to linear connections on* T^A is, by definition, a family of maps $L_M : \operatorname{ToFrCo} M \to \operatorname{Co} T^A M$ indexed by all n-dimensional manifolds and satisfying

$$L_M(\phi^*\nabla) = (T^A\phi)^*(L_N(\nabla)) \tag{1}$$

for all n-dimensional manifolds M, N, every embedding $\phi : M \to N$ and every $\nabla \in \operatorname{ToFrCo} N$ (obviously, the connection $\phi^*\nabla$ on M is defined by $(\phi^*\nabla)_{\phi^*V}(\phi^*W) = \phi^*(\nabla_V W)$ for all vector fields V, W on N, where the vector field ϕ^*V on M is defined by $(\phi^*V)_x = (T_x\phi)^{-1}(V_{\phi(x)})$ for every $x \in M$; consequently, if U is an open subset of $\mathbb{R}^n$, $\phi : U \to \mathbb{R}^n$ is an embedding and $\nabla \in \operatorname{Co}\mathbb{R}^n$, then for all $i, j, k \in \{1, \ldots, n\}$,

$$\frac{\partial \phi^i}{\partial x^p}\, \Gamma'^p_{jk} = (\Gamma^i_{qr} \circ \phi)\, \frac{\partial \phi^q}{\partial x^j}\, \frac{\partial \phi^r}{\partial x^k} + \frac{\partial^2 \phi^i}{\partial x^j \partial x^k}, \tag{2}$$

where Γ^i_{jk} denote the Christoffel symbols of ∇ with respect to the standard

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chart on $\mathbb{R}^n$ and $\mathit{\Gamma}'^i_{jk}$ denote the Christoffel symbols of $\phi^*\nabla$ with respect to the standard chart on U for all $i, j, k \in \{1, \ldots, n\}$).

By the *standard connection* on a finite-dimensional real vector space we mean the linear connection with zero Christoffel symbols with respect to a linear chart. By (2), this connection is independent of which linear chart we choose to define it. Let us denote by ∇_0 and ∇_0^A the standard connections on $\mathbb{R}^n$ and A^n respectively.

LEMMA 1. *If L is a lifting of torsion-free linear connections to linear connections on T^A, then $L_{\mathbb{R}^n}(\nabla_0) = \nabla_0^A$.*

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $\phi_\lambda : \mathbb{R}^n \ni x \mapsto \lambda x \in \mathbb{R}^n$. From (2) with $\phi = \phi_\lambda$ and $\nabla = \nabla_0$ it follows that $\phi_\lambda^*\nabla_0 = \nabla_0$. Therefore from (1) with $\phi = \phi_\lambda$ and $\nabla = \nabla_0$ it follows that $L_{\mathbb{R}^n}(\nabla_0) = (T^A\phi_\lambda)^*(L_{\mathbb{R}^n}(\nabla_0))$. Letting $\lambda \to 0$ yields the desired conclusion, by the analogue of (2) with $T^A\phi_\lambda : A^n \ni X \mapsto \lambda X \in A^n$ instead of $\phi : U \to \mathbb{R}^n$ and $L_{\mathbb{R}^n}(\nabla_0)$ instead of ∇.

Of course, $\mathrm{Co}\, M$ is an affine space modelled on the vector space of all tensor fields of type $(1,2)$ on M, which we will denote by $\mathrm{Te}\, M$, and $\mathrm{ToFrCo}\, M$ is an affine space modelled on the vector space of all symmetric tensor fields of type $(1,2)$ on M, which we will denote by $\mathrm{SyTe}\, M$ (obviously, $t \in \mathrm{Te}\, M$ is called symmetric if $t_x(z, y) = t_x(y, z)$ for every $x \in M$ and all $y, z \in T_xM$).

A *lifting of symmetric tensor fields of type* $(1,2)$ *to tensor fields of type* $(1,2)$ *on* T^A is, by definition, a family of maps $L'_M : \mathrm{SyTe}\, M \to \mathrm{Te}\, T^AM$ indexed by all n-dimensional manifolds and satisfying

$$L'_M(\phi^*t) = (T^A\phi)^*(L'_N(t))$$

for all n-dimensional manifolds M, N, every embedding $\phi : M \to N$ and every $t \in \mathrm{SyTe}\, N$ (obviously, the tensor field ϕ^*t on M is defined by $(\phi^*t)_x = (T_x\phi)^{-1} \circ t_{\phi(x)} \circ (T_x\phi, T_x\phi)$ for every $x \in M$). Such a lifting L' is said to be *linear* if L'_M is linear for each n-dimensional manifold M.

Similarly, a lifting L of torsion-free linear connections to linear connections on T^A is said to be *affine* if L_M is affine for each n-dimensional manifold M. This means that for each n-dimensional manifold M there is a unique linear map $\vec{L}_M : \mathrm{SyTe}\, M \to \mathrm{Te}\, T^AM$ (called the linear part of L_M) such that $L_M(\nabla + t) = L_M(\nabla) + \vec{L}_M(t)$ for every $\nabla \in \mathrm{ToFrCo}\, M$ and every $t \in \mathrm{SyTe}\, M$. We will call the family $\vec{L}$ the *linear part* of L.

LEMMA 2. *If L is an affine lifting of torsion-free linear connections to linear connections on T^A, then the linear part $\vec{L}$ of L is a linear lifting of symmetric tensor fields of type* $(1,2)$ *to tensor fields of type* $(1,2)$ *on* T^A.

Proof. Let M, N be n-dimensional manifolds, $\phi : M \to N$ an embedding and $t \in \mathrm{SyTe}\, N$. Choose $\nabla \in \mathrm{ToFrCo}\, N$. Then by (1),

$$\begin{aligned}\vec{L}_M(\phi^* t) &= L_M(\phi^*\nabla + \phi^* t) - L_M(\phi^*\nabla) = L_M(\phi^*(\nabla + t)) - L_M(\phi^* t)\\ &= (T^A\phi)^*(L_N(\nabla + t)) - (T^A\phi)^*(L_N(\nabla))\\ &= (T^A\phi)^*(L_N(\nabla + t) - L_N(\nabla)) = (T^A\phi)^*(\vec{L}_N(t)),\end{aligned}$$

and the lemma is proved.

Lemma 2 enables us to use the classification of all linear liftings of symmetric tensor fields of type $(1,2)$ to Weil bundles established in [2]. We now sketch out this result. If $E \in A$, $F : A \to A$ is $\mathbb{R}$-linear and $G, H : A \times A \to A$ are $\mathbb{R}$-bilinear and such that

$$G(a, b \cdot c) = G(a, b) \cdot c + G(a, c) \cdot b, \tag{3}$$

$$H(a, b \cdot c) = H(a \cdot c, b) + H(a \cdot b, c) \tag{4}$$

for all $a, b, c \in A$, then there are unique linear liftings $\overline{E}$, $\overline{F}^{\mathrm{L}}$, $\overline{F}^{\mathrm{R}}$, $\overline{G}$, $\overline{H}^{\mathrm{L}}$, $\overline{H}^{\mathrm{R}}$ of symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A such that

$$\overline{E}_U(t)^i_X(Y, Z) = E \cdot (T^A t^i_{pq})(X) \cdot Y^p \cdot Z^q, \tag{5}$$

$$\overline{F}^{\mathrm{L}}_U(t)^i_X(Y, Z) = F((T^A t^p_{qp})(X) \cdot Y^q) \cdot Z^i, \tag{6}$$

$$\overline{F}^{\mathrm{R}}_U(t)^i_X(Y, Z) = F((T^A t^p_{qp})(X) \cdot Z^q) \cdot Y^i, \tag{7}$$

$$\overline{G}_U(t)^i_X(Y, Z) = \frac{1}{2}\, G\bigg(\bigg(T^A\bigg(\frac{\partial t^p_{rp}}{\partial x^q} - \frac{\partial t^p_{qp}}{\partial x^r}\bigg)\bigg)(X) \cdot Y^q \cdot Z^r, X^i\bigg), \tag{8}$$

$$\overline{H}^{\mathrm{L}}_U(t)^i_X(Y, Z) = \frac{1}{2}\, H\bigg(\bigg(T^A\bigg(\frac{\partial t^p_{rp}}{\partial x^q} - \frac{\partial t^p_{qp}}{\partial x^r}\bigg)\bigg)(X) \cdot Y^q, X^r\bigg) \cdot Z^i, \tag{9}$$

$$\overline{H}^{\mathrm{R}}_U(t)^i_X(Y, Z) = \frac{1}{2}\, H\bigg(\bigg(T^A\bigg(\frac{\partial t^p_{rp}}{\partial x^q} - \frac{\partial t^p_{qp}}{\partial x^r}\bigg)\bigg)(X) \cdot Z^q, X^r\bigg) \cdot Y^i \tag{10}$$

for every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $i \in \{1, \dots, n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$. On the other hand, if $n \geq 3$, then for each linear lifting L' of symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A there are uniquely determined $E \in A$, $\mathbb{R}$-linear $F, F' : A \to A$ and $\mathbb{R}$-bilinear $G : A \times A \to A$ with property (3) and $H, H' : A \times A \to A$ with property (4) such that

$$L' = \overline{E} + \overline{F}^{\mathrm{L}} + \overline{F'}^{\mathrm{R}} + \overline{G} + \overline{H}^{\mathrm{L}} + \overline{H'}^{\mathrm{R}}. \tag{11}$$

But not all such linear liftings of tensor fields are the linear parts of affine liftings of connections.

Lemma 3. *If L is an affine lifting of torsion-free linear connections to linear connections on T^A, then $\vec{L}_U(\phi^*\nabla_0 - \nabla_0|_U) = (T^A\phi)^*\nabla^A_0 - \nabla^A_0|_{T^A U}$ for every open subset U of $\mathbb{R}^n$ and every embedding $\phi : U \to \mathbb{R}^n$.*

Proof. From (1) and Lemma 1 we have

$$\begin{aligned}\vec{L}_U(\phi^*\nabla_0 - \nabla_0|_U) &= L_U(\phi^*\nabla_0) - L_U(\nabla_0|_U)\\ &= (T^A\phi)^*(L_{\mathbb{R}^n}(\nabla_0)) - L_{\mathbb{R}^n}(\nabla_0)|_{T^AU}\\ &= (T^A\phi)^*\nabla_0^A - \nabla_0^A|_{T^AU},\end{aligned}$$

which completes the proof.

We now recall the complete lifting of connections to Weil bundles (see [3]). It transforms each $\nabla \in \operatorname{ToFrCo} M$, where M is an n-dimensional manifold, into $\nabla^A \in \operatorname{Co} T^AM$ in such a way that

$$(\nabla_Y^A Z)_X^i = (T^A\Gamma_{pq}^i)(X)\cdot Y^p \cdot Z^q \tag{12}$$

for every open subset U of $\mathbb{R}^n$, every $\nabla \in \operatorname{ToFrCo} U$, every $i \in \{1,\dots,n\}$, every $X \in T^AU$ and all $Y, Z \in A^n$, where Y and Z are interpreted as constant vector fields on T^AU, and Γ_{jk}^i for all $i,j,k \in \{1,\dots,n\}$ denote the Christoffel symbols of ∇ with respect to the standard chart on U. Note that our notation will cause no confusion, because the complete lift of the standard connection ∇_0 on $\mathbb{R}^n$ is nothing but the standard connection ∇_0^A on A^n.

Our purpose is to generalize the complete lifting. Let ∇ be a connection on a manifold M and let R denote the curvature tensor of ∇. We have the 2-form $\operatorname{tr} R$ on M such that, for every $x \in M$ and all $y, z \in T_xM$, $(\operatorname{tr} R)_x(y,z)$ is the trace of the endomorphism $T_xM \ni w \mapsto R_x(y,z)w \in T_xM$. Since

$$R_{jkl}^i = \frac{\partial\Gamma_{kl}^i}{\partial x^j} - \frac{\partial\Gamma_{jl}^i}{\partial x^k} + \Gamma_{jp}^i\Gamma_{kl}^p - \Gamma_{kp}^i\Gamma_{jl}^p$$

for all $i,j,k,l \in \{1,\dots,n\}$, we have

$$(\operatorname{tr} R)_{ij} = \frac{\partial\Gamma_{jp}^p}{\partial x^i} - \frac{\partial\Gamma_{ip}^p}{\partial x^j} \tag{13}$$

for all $i, j \in \{1,\dots,n\}$.

If $\alpha : A \to \mathbb{R}$ is linear, then there is a unique linear lifting $\widetilde{\alpha}$ of 2-forms to 2-forms on T^A such that $\widetilde{\alpha}_U(\omega)_X(Y,Z) = \alpha((T^A\omega_{pq})(X)\cdot Y^p \cdot Z^q)$ for every open subset U of $\mathbb{R}^n$, every 2-form ω on U, every $X \in T^AU$ and all $Y, Z \in A^n$ (see [3]). An $\mathbb{R}$-linear map $D : A \to A$ is said to be a derivation of A if $D(a\cdot b) = D(a)\cdot b + D(b)\cdot a$ for all $a, b \in A$. If D is a derivation of A, then there is a unique natural vector field $\widetilde{D}$ on T^A such that $(\widetilde{D}_U)_X^i = D(X^i)$ for every open subset U of $\mathbb{R}^n$, every $i \in \{1,\dots,n\}$ and every $X \in T^AU$ (see [4]). Thus $\widetilde{\alpha}_M\big(\frac{1}{2}\operatorname{tr} R\big) \otimes \widetilde{D}_M$ is a tensor field of type $(1,2)$ on T^AM. We can take a sum of tensor fields of this form. Consequently, for every $\mathbb{R}$-bilinear $G : A \times A \to A$ with property (3) we have constructed a tensor field of type $(1,2)$ on T^AM. We will denote it by $\widetilde{G}_M(\nabla)$. Of course, it depends naturally on ∇ and, by (13),

$$\widetilde{G}_U(\nabla)^i_X(Y,Z) = \frac{1}{2}\, G\bigg(\bigg(T^A\bigg(\frac{\partial \Gamma^p_{rp}}{\partial x^q} - \frac{\partial \Gamma^p_{qp}}{\partial x^r}\bigg)\bigg)(X)\cdot Y^q \cdot Z^r, X^i\bigg) \tag{14}$$

for every open subset U of $\mathbb{R}^n$, every $\nabla \in \operatorname{ToFrCo} U$, every $i \in \{1,\dots,n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$, where Γ^i_{jk} for all $i,j,k \in \{1,\dots,n\}$ denote the Christoffel symbols of ∇ with respect to the standard chart on U.

If $\beta : A \times A \to \mathbb{R}$ is bilinear and such that $\beta(a, b\cdot c) = \beta(a\cdot c, b) + \beta(a\cdot b, c)$ for all $a,b,c \in A$, then there is a unique linear lifting $\widetilde{\beta}$ of 2-forms to 1-forms on T^A such that $\widetilde{\beta}_U(\omega)_X(Y) = \beta((T^A\omega_{pq})(X)\cdot Y^p, X^q)$ for every open subset U of $\mathbb{R}^n$, every 2-form ω on U, every $X \in T^A U$ and every $Y \in A^n$ (see [1]). If $a \in A$, then there is a unique natural tensor field $\widetilde{a}$ of type $(1,1)$ on T^A such that $(\widetilde{a}_U)^i_X(Y) = a \cdot Y^i$ for every open subset U of $\mathbb{R}^n$, every $i \in \{1,\dots,n\}$, every $X \in T^A U$ and every $Y \in A^n$ (see [4]). Thus $\widetilde{\beta}_M\big(\frac{1}{2}\operatorname{tr} R\big) \otimes \widetilde{a}_M$ and $\widetilde{a}_M \otimes \widetilde{\beta}_M\big(\frac{1}{2}\operatorname{tr} R\big)$ are tensor fields of type $(1,2)$ on $T^A M$. We can take sums of tensor fields of these forms. Consequently, for every $\mathbb{R}$-bilinear $H : A \times A \to A$ with property (4) we have constructed two tensor fields of type $(1,2)$ on $T^A M$. We will denote them by $\widetilde{H}^{\mathrm{L}}_M(\nabla)$ and $\widetilde{H}^{\mathrm{R}}_M(\nabla)$. Of course, they depend naturally on ∇ and, by (13),

$$\widetilde{H}^{\mathrm{L}}_U(\nabla)^i_X(Y,Z) = \frac{1}{2}\, H\bigg(\bigg(T^A\bigg(\frac{\partial \Gamma^p_{rp}}{\partial x^q} - \frac{\partial \Gamma^p_{qp}}{\partial x^r}\bigg)\bigg)(X)\cdot Y^q, X^r\bigg)\cdot Z^i, \tag{15}$$

$$\widetilde{H}^{\mathrm{R}}_U(\nabla)^i_X(Y,Z) = \frac{1}{2}\, H\bigg(\bigg(T^A\bigg(\frac{\partial \Gamma^p_{rp}}{\partial x^q} - \frac{\partial \Gamma^p_{qp}}{\partial x^r}\bigg)\bigg)(X)\cdot Z^q, X^r\bigg)\cdot Y^i \tag{16}$$

for every open subset U of $\mathbb{R}^n$, every $\nabla \in \operatorname{ToFrCo} U$, every $i \in \{1,\dots,n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$, where Γ^i_{jk} for all $i,j,k \in \{1,\dots,n\}$ denote the Christoffel symbols of ∇ with respect to the standard chart on U.

Now, for any $\mathbb{R}$-bilinear $G : A \times A \to A$ with property (3) and $H, H' : A \times A \to A$ with property (4) the family of maps

$$\operatorname{ToFrCo} M \ni \nabla \mapsto \nabla^A + \widetilde{G}_M(\nabla) + \widetilde{H}^{\mathrm{L}}_M(\nabla) + \widetilde{H'}^{\mathrm{R}}_M(\nabla) \in \operatorname{Co} T^A M$$

indexed by all n-dimensional manifolds is a lifting of torsion-free linear connections to linear connections on T^A. It is affine, which is clear from (12) and (14)–(16).

We can now formulate our main result.

THEOREM. *If $n \geq 3$, then for each affine lifting L of torsion-free linear connections to linear connections on T^A there are uniquely determined $\mathbb{R}$-bilinear $G : A \times A \to A$ with property* (3) *and $H, H' : A \times A \to A$ with property* (4) *such that*

$$L_M(\nabla) = \nabla^A + \widetilde{G}_M(\nabla) + \widetilde{H}^{\mathrm{L}}_M(\nabla) + \widetilde{H'}^{\mathrm{R}}_M(\nabla) \tag{17}$$

for every n-dimensional manifold M and every $\nabla \in \operatorname{ToFrCo} M$.

Proof. Lemma 1 makes it obvious that two affine liftings of torsion-free linear connections to linear connections on T^A are equal if their linear parts are equal. Comparing (12) with (5), (14) with (8), (15) with (9) and (16) with (10) we see that the linear part of the lifting defined by the right hand side of (17) equals $\overline{1} + \overline{G} + \overline{H}^{\mathrm{L}} + \overline{H'}^{\mathrm{R}}$. Hence it suffices to show that $\vec{L}$ is of this form.

Let U be an open subset of $\mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ an embedding. From (2) with $\nabla = \nabla_0$ it follows that

$$\frac{\partial \phi^i}{\partial x^p}(x)(\phi^*\nabla_0 - \nabla_0|_U)^p_x(y, z) = \frac{\partial^2 \phi^i}{\partial x^q \partial x^r}(x) y^q z^r \tag{18}$$

for every $i \in \{1, \ldots, n\}$, every $x \in U$ and all $y, z \in \mathbb{R}^n$. But (18) can be rewritten as $\phi'_x((\phi^*\nabla_0 - \nabla_0|_U)_x(y, z)) = \phi''_x(y, z)$ for every $x \in U$ and all $y, z \in \mathbb{R}^n$. From the analogue of this formula with $T^A\phi : T^AU \to A^n$ instead of $\phi : U \to \mathbb{R}^n$ and ∇^A_0 instead of ∇_0 it follows that

$$\begin{aligned}\left(T^A \frac{\partial \phi^i}{\partial x^p}\right)(X) \cdot ((T^A\phi)^*\nabla^A_0 - \nabla^A_0|_{T^AU})^p_X(Y, Z) \\ = \left(T^A \frac{\partial^2 \phi^i}{\partial x^q \partial x^r}\right)(X) \cdot Y^q \cdot Z^r\end{aligned} \tag{19}$$

for every $i \in \{1, \ldots, n\}$, every $X \in T^AU$ and all $Y, Z \in A^n$.

Fix an affine lifting L of torsion-free linear connections to linear connections on T^A. We know already that $\vec{L}$ is of the form (11). Put $U = \{x \in \mathbb{R}^n : x^1 > 0\}$ and

$$\phi : U \ni x \mapsto \left(\frac{(x^1)^2}{2}, x^2, \ldots, x^n\right) \in \mathbb{R}^n.$$

Writing $t = \phi^*\nabla_0 - \nabla_0|_U$, from (18) we obtain

$$t^i_{jk}(x) = \begin{cases} 1/x^1 & \text{if } i, j, k = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j, k \in \{1, \ldots, n\}$ and every $x \in U$. Lemma 3, (5)–(10) and (19) with $i = 1$ lead to

$$\frac{E \cdot Y^1 \cdot Z^1}{X^1} + F\left(\frac{Y^1}{X^1}\right) \cdot Z^1 + F'\left(\frac{Z^1}{X^1}\right) \cdot Y^1 = \frac{Y^1 \cdot Z^1}{X^1} \tag{20}$$

and with $i = 2$ to

$$F\left(\frac{Y^1}{X^1}\right) \cdot Z^2 + F'\left(\frac{Z^1}{X^1}\right) \cdot Y^2 = 0 \tag{21}$$

for every $X \in T^AU$ and all $Y, Z \in A^n$. From (21) we see that $F = 0$ and $F' = 0$. Next, from (20) we see that $E = 1$. Thus $\vec{L} = \overline{1} + \overline{G} + \overline{H}^{\mathrm{L}} + \overline{H'}^{\mathrm{R}}$, which completes the proof.

Suppose that L' is given by (11). From (5)–(10) it is evident that for every n-dimensional manifold M and every $t \in \mathrm{SyTe}\, M$,

$$\Big(\overline{E} + \overline{\tfrac{1}{2}(F+F')}^{\mathrm{L}} + \overline{\tfrac{1}{2}(F+F')}^{\mathrm{R}} + \overline{\tfrac{1}{2}(H+H')}^{\mathrm{L}} + \overline{\tfrac{1}{2}(H+H')}^{\mathrm{R}}\Big)_M(t) \tag{22}$$

is symmetric and

$$\Big(\overline{\tfrac{1}{2}(F-F')}^{\mathrm{L}} - \overline{\tfrac{1}{2}(F-F')}^{\mathrm{R}} + \overline{G} + \overline{\tfrac{1}{2}(H-H')}^{\mathrm{L}} - \overline{\tfrac{1}{2}(H-H')}^{\mathrm{R}}\Big)_M(t) \tag{23}$$

is skew-symmetric. Moreover, L' is the sum of the liftings defined by (22) and (23). Therefore from the above-mentioned result of [2] we get the following propositions.

PROPOSITION 1. *If $n \geq 3$, then for each linear lifting L' of symmetric tensor fields of type $(1,2)$ to symmetric tensor fields of type $(1,2)$ on T^A there are uniquely determined $E \in A$, $\mathbb{R}$-linear $F : A \to A$ and $\mathbb{R}$-bilinear $H : A \times A \to A$ with property (4) such that $L' = \overline{E} + \overline{F}^{\mathrm{L}} + \overline{F}^{\mathrm{R}} + \overline{H}^{\mathrm{L}} + \overline{H}^{\mathrm{R}}$.*

PROPOSITION 2. *If $n \geq 3$, then for each linear lifting L' of symmetric tensor fields of type $(1,2)$ to skew-symmetric tensor fields of type $(1,2)$ on T^A there are uniquely determined $\mathbb{R}$-linear $F : A \to A$ and $\mathbb{R}$-bilinear $G : A \times A \to A$ with property (3) and $H : A \times A \to A$ with property (4) such that $L' = \overline{F}^{\mathrm{L}} - \overline{F}^{\mathrm{R}} + \overline{G} + \overline{H}^{\mathrm{L}} - \overline{H}^{\mathrm{R}}$.*

By (12), $\nabla^A \in \mathrm{ToFrCo}\, T^AM$ for every n-dimensional manifold M and every $\nabla \in \mathrm{ToFrCo}\, M$. Therefore, on account of the above remarks, from our theorem we get the following corollary.

COROLLARY. *If $n \geq 3$, then for each affine lifting L of torsion-free linear connections to torsion-free linear connections on T^A there is a uniquely determined $\mathbb{R}$-bilinear $H : A \times A \to A$ with property (4) such that*

$$L_M(\nabla) = \nabla^A + \widetilde{H}^{\mathrm{L}}_M(\nabla) + \widetilde{H}^{\mathrm{R}}_M(\nabla)$$

for every n-dimensional manifold M and every $\nabla \in \mathrm{ToFrCo}\, M$.

REFERENCES

[1] J. Dębecki, *Linear liftings of p-forms to q-forms on Weil bundles*, Monatsh. Math. 148 (2006), 101–117.

[2] —, *Linear liftings of symmetric tensor fields of type* $(1,2)$ *to Weil bundles*, Ann. Polon. Math. 92 (2007), 13–27.

[3] J. Gancarzewicz, W. Mikulski and Z. Pogoda, *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. 135 (1994), 1–41.

[4] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

[5] A. Morimoto, *Prolongation of connections to bundles of infinitely near points*, J. Differential Geom. 11 (1976), 479–498.

Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4
30-059 Kraków, Poland
E-mail: debecki@im.uj.edu.pl

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