SPHERICAL MEANS AND MEASURES WITH FINITE ENERGY

BY

THEMIS MITSIS (Heraklion)

Abstract. We prove a restricted weak type inequality for the spherical means operator with respect to measures with finite $\alpha$-energy, $\alpha \leq 1$. This complements recent results due to D. Oberlin.

Fix a small positive number $\delta$, and for $r > \delta$ denote by $S^\delta(\bar{x}, r)$ the $\delta$-neighborhood of the $(n - 1)$-dimensional sphere with center $\bar{x} \in \mathbb{R}^n$ and radius $r$:

$$S^\delta(\bar{x}, r) = \{y \in \mathbb{R}^n : r - \delta < |\bar{x} - y| < r + \delta\}.$$ (Here and for the rest of the paper we assume that $n \geq 3$.) Now, for suitable $f : \mathbb{R}^n \to \mathbb{R}$, consider the spherical means operator

$$T_\delta f : \mathbb{R}^n \times (\delta, \infty) \to \mathbb{R}$$

defined by

$$T_\delta f(\bar{x}, r) = \frac{1}{|S^\delta(\bar{x}, r)|} \int_{S^\delta(\bar{x}, r)} f,$$

where $| \cdot |$ denotes Lebesgue measure. The mapping properties of this operator, its variants, and the corresponding maximal operators have been studied extensively by several authors using Fourier analysis. Recently D. Oberlin [2] proved the following restricted weak type inequality for $T_\delta$ with respect to measures more general than the Lebesgue measure.

THEOREM 1. Let $1 < \alpha < n + 1$ and suppose $\mu$ is a compactly supported non-negative Borel measure in $\mathbb{R}^n \times (0, \infty)$ such that the $\alpha$-energy $I_\alpha(\mu)$ defined by

$$I_\alpha(\mu) = \iint d\mu(x) d\mu(y) \frac{1}{|x - y|^\alpha}$$

is finite. Let

$$r_0 = \inf\{r : \text{there exists } \bar{x} \in \mathbb{R}^n \text{ such that } (\bar{x}, r) \text{ is in the support of } \mu\}.$$

2000 Mathematics Subject Classification: Primary 42B25.

Key words and phrases: spherical means operator, energy of a measure.
Then for \( \lambda > 0 \) and \( 0 < \delta < r_0 \) one has the estimate

\[
\lambda^2 \mu(\{T_\delta \chi_E > \lambda\})^{2/\alpha} \leq C|E|
\]

for all Borel sets \( E \subset \mathbb{R}^n \) (\( \chi_E \) is the characteristic function). Here \( C \) is a positive constant independent of \( \delta \) and \( \lambda \) (it depends on \( \mu \) and \( n \)).

The case \( 0 < \alpha \leq 1 \) was left open in [2]. The example mentioned in [2] suggests that if \( 0 < \alpha \leq 1 \) then the right-hand side of (1) should be either corrected by a factor which tends to infinity as \( \delta \) tends to zero, or replaced with a larger norm. In that direction, one has the following result due to D. Oberlin, which is a special case of Theorem 4S in [3].

**Theorem 2.** Suppose \( 0 < \alpha \leq 1 \), and let \( B(x, \varrho) \) be the closed ball in \( \mathbb{R}^n \times (0, \infty) \) with center \( x \) and radius \( \varrho \). If

\[
\mu(B(x, \varrho)) \leq \varrho^\alpha
\]

for all \( x \) and \( \varrho \), then for every \( \varepsilon > 0 \) there exists a positive constant \( C_\varepsilon \) independent of \( \lambda \) and \( \delta \) such that

\[
\lambda^2 \mu(\{T_\delta \chi_E > \lambda\}) \leq C_\varepsilon \|\chi_E\|_{W^{2, (1-\alpha)/2+\varepsilon}}^2,
\]

where the norm on the right-hand side is the Sobolev space norm.

The proof of Theorem 2 is Fourier-analytic. In this paper we give an elementary proof of the following estimate which may be thought of as the “non-\( \delta \)-free counterpart” of (3) under a weaker energy-finiteness hypothesis ((2) implies that \( I_\beta(\mu) < \infty \) for all \( \beta < \alpha \)).

**Theorem 3.** If \( 0 < \alpha \leq 1 \) and \( I_\alpha(\mu) < \infty \) then

\[
\lambda^2 \mu(\{T_\delta \chi_E > \lambda\})^2 \leq C_\varepsilon |E|\delta^{\alpha-1-\varepsilon}.
\]

Note that (4) is not entirely satisfactory. A natural conjecture (corresponding to an \( L^2 \) bound) would be

\[
\lambda^2 \mu(\{T_\delta \chi_E > \lambda\}) \leq C_\varepsilon |E|\delta^{\alpha-1-\varepsilon}.
\]

We do not, however, know how to prove (or disprove) this.

**Proof of Theorem 3.** To simplify the presentation we will be using the standard notation \( x \lesssim y \) to denote \( x \leq Cy \) for some positive constant \( C \). Similarly, \( x \simeq y \) means that \( x \) and \( y \) are comparable.

Let

\[
F = \{T_\delta \chi_E > \lambda\} \subset \mathbb{R}^n \times (0, \infty).
\]
We will discretize the problem at scale $\delta$. First we show that $F$ can be decomposed into roughly $|\log \delta|$ sets on which $\mu$ behaves as if it were $\alpha$-dimensional. So, put

$$F_0 = \left\{ x \in F : \sup_{\varrho \geq \delta} \frac{\mu(B(x, \varrho))}{\varrho^\alpha} \leq 1 \right\},$$

$$F_i = \left\{ x \in F : 2^{i-1} < \sup_{\varrho \geq \delta} \frac{\mu(B(x, \varrho))}{\varrho^\alpha} \leq 2^i \right\}, \quad i = 1, 2, \ldots,$$

$$I = \{ i \in \mathbb{N} \cup \{0\} : \mu(F_i) \neq 0 \}.$$

Then $\mu(F) = \sum_{i \in I} \mu(F_i)$, and since $\mu$ is a finite measure, we have $|I| \lesssim |\log \delta|$ for $\delta$ small enough. Moreover,

$$\mu(B(x, \varrho)) \leq 2^i \varrho^\alpha \quad \text{for } x \in F_i, \ \varrho \geq \delta. \quad (5)$$

This means that, modulo the factor $2^i$, the measure $\mu$ is $\alpha$-dimensional on $F_i$. To estimate this factor, fix $i \in I$ with $i \geq 1$. Then, by the Besicovitch covering lemma, there exists a countable family of closed balls $B_j$ with radius $\varrho_j \geq \delta$ such that

- $\{B_j\}_j$ has bounded overlap.
- $\{B_j\}_j$ covers $F_i$.
- For all $j$ we have

$$\mu(B_j) > 2^{i-1} \varrho_j^\alpha. \quad (6)$$

Notice that

$$\frac{\mu(B_j)^2}{\varrho_j^\alpha} \lesssim \iint_{B_j \times B_j} \frac{d\mu(x) \ d\mu(y)}{|x - y|^\alpha}. \quad (7)$$

So, using (6) and (7), we get

$$2^i \mu(F_i) \leq \sum_j 2^i \mu(B_j) \lesssim \sum_j \varrho_j^{-\alpha} \mu(B_j)^2 \lesssim \sum_j \iint_{B_j \times B_j} \frac{d\mu(x) \ d\mu(y)}{|x - y|^\alpha} \lesssim I_\alpha(\mu), \quad (8)$$

where the last inequality follows from the fact that $\{B_j\}_j$ has bounded overlap. Therefore, (5) and (8) imply that

$$\mu(B(x, \varrho)) \lesssim \mu(F_i)^{-1} \varrho^\alpha \quad \text{for } x \in F_i, \ \varrho \geq \delta, \ i \in I, \ i \neq 0. \quad (9)$$

If $i \in I$ and $i = 0$ then (9) follows trivially from (5) because $\mu$ is finite.
Now, we use Córdoba’s orthogonality argument [1] to estimate the measure of each $F_i$, $i \in I$. (9) will be important here. We decompose $\mathbb{R}^{n+1}$ into a family $\mathcal{Q}$ of disjoint cubes of side length $\delta$:

$$\mathcal{Q} = \left\{ \prod_{l=1}^{n+1} [m_l \delta, (m_l + 1)\delta) : m_1, \ldots, m_{n+1} \in \mathbb{Z} \right\}.$$ 

Let $\{Q_j\}_j = \{Q \in \mathcal{Q} : Q \cap F_i \neq \emptyset\}$ and pick $(\bar{x}_j, r_j) \in Q_j$ ($\bar{x}_j \in \mathbb{R}^n$, $r_j > 0$) such that

$$\frac{1}{|S^\delta(\bar{x}_j, r_j)|} \int_{S^\delta(\bar{x}_j, r_j)} \chi_E > \lambda.$$ 

Since $\mu$ is compactly supported, the $r_j$’s are bounded, therefore $|S^\delta(\bar{x}_j, r_j)| \simeq \delta$. Thus

$$\mu(F_i) = \sum_j \mu(Q_j \cap F_i) = \frac{1}{\lambda \delta} \sum_j \lambda \delta \mu(Q_j \cap F_i)$$

$$\lesssim \frac{1}{\lambda \delta} \sum_j \mu(Q_j \cap F_i) \int_E \chi_{S^\delta(\bar{x}_j, r_j)}$$

$$\leq \frac{|E|^{1/2}}{\lambda \delta} \left[ \int_E \left( \sum_j \mu(Q_j \cap F_i) \chi_{S^\delta(\bar{x}_j, r_j)} \right)^2 \right]^{1/2}$$

$$\leq \frac{|E|^{1/2}}{\lambda \delta} \left[ \sum_{j, k} \mu(Q_j \cap F_i) \mu(Q_k \cap F_i) \chi_{S^\delta(\bar{x}_j, r_j) \cap S^\delta(\bar{x}_k, r_k)} \right]^{1/2}$$

$$= \frac{|E|^{1/2}}{\lambda \delta} \left[ \sum_{j, k} \mu(Q_j \cap F_i) \mu(Q_k \cap F_i) |S^\delta(\bar{x}_j, r_j) \cap S^\delta(\bar{x}_k, r_k)| \right]^{1/2}.$$ 

By Lemma 1 in [2],

$$|S^\delta(\bar{x}_j, r_j) \cap S^\delta(\bar{x}_k, r_k)| \lesssim \frac{\delta^2}{\delta + |(\bar{x}_j, r_j) - (\bar{x}_k, r_k)|}.$$ 

Moreover, for all $x \in Q_j$ and $y \in Q_k$ we have

$$\delta + |x - y| \lesssim \delta + |(\bar{x}_j, r_j) - (\bar{x}_k, r_k)|.$$ 

Therefore

$$(10) \lesssim \frac{|E|^{1/2}}{\lambda} \left[ \sum_{j, k} \int_{(Q_j \times Q_k) \cap (F_i \times F_i)} \frac{d\mu(x) d\mu(y)}{\delta + |x - y|} \right]^{1/2}$$

$$= \frac{|E|^{1/2}}{\lambda} \left[ \int_{F_i \times F_i} \frac{d\mu(x) d\mu(y)}{\delta + |x - y|} \right]^{1/2}.$$
To estimate the integral in the square brackets, we use the distribution function. For each $x \in F_i$ we have

\[
\int_{F_i} \frac{d\mu(y)}{\delta + |x - y|} = \int_0^{1/\delta} \mu(\{y \in F_i : \delta + |x - y| < \varrho^{-1}\}) \, d\varrho \\
\leq \int_0^{1/\delta} \mu(B(x, \varrho^{-1})) \, d\varrho
\]

(12)

Since $\varrho^{-1} \geq \delta$, (9) implies that

\[
(12) \lesssim \frac{1}{\mu(F_i)} \int_0^{1/\delta} \frac{d\varrho}{\varrho^\alpha} \lesssim \frac{\delta^{\alpha - 1}}{\mu(F_i)}.
\]

Consequently, (11) yields

\[
\mu(F_i) \lesssim \frac{1}{\lambda} |E|^{1/2} \delta^{(\alpha - 1)/2}.
\]

Summing up these inequalities over $i \in I$ we obtain

\[
\mu(F) \lesssim \frac{1}{\lambda} |E|^{1/2} \log \delta \delta^{(\alpha - 1)/2} \leq C \varepsilon \frac{1}{\lambda} |E|^{1/2} \delta^{(\alpha - 1)/2 - \varepsilon}
\]

as claimed.

The same argument shows that if $\alpha = 1$ then

\[
\mu(F) \leq C \varepsilon \frac{1}{\lambda} |E|^{1/2} \delta^{-\varepsilon}.
\]

REFERENCES


Department of Mathematics
University of Crete
Knossos Ave., 71409 Heraklion, Greece
E-mail: themis.mitsis@gmail.com

Received 6 January 2008