COLLOQUIUM MATHEMATICUM

VOL. 114 2009 NO. 1

IRREDUCIBLE TENSOR REPRESENTATIONS OF GENERAL LINEAR LIE SUPERALGEBRAS

BY

TADEUSZ JÓZEFIAK (Ann Arbor, MI)

Dedicated to the memory of Professor Stanisław Balcerzyk

Abstract. We present a description of irreducible tensor representations of general linear Lie superalgebras in terms of generalized determinants in the symmetric and exterior superalgebras of a superspace over a field of characteristic zero.

1. Introduction. Tensor representations of the general linear Lie superalgebras play a special role in the theory of representations of these superalgebras because of their relationship with the representation theory of the symmetric groups and use of combinatorial methods. The first construction of irreducible tensor representations was provided independently by Sergeev [10] and by Berele and Regev [3] using the Schur–Weyl duality in a space of tensors of a superspace over a field of characteristic 0. Implicitly, a different construction was contained earlier in a paper by Akin, Buchsbaum and Weyman [1] without mentioning Lie superalgebras. See also a more recent paper by Sergeev [11] for still another approach.

In the present paper we provide an approach to these representations using generalized determinants in the symmetric superalgebra of a superspace generalizing in this way the classical constructions in terms of products of minors for general linear groups, abundant in the literature (see, e.g., a paper by de Concini, Eisenbud and Procesi [4] and the bibliography cited there). The generalized determinants we use were first considered by Doubilet and Rota in [5] and by Grosshans, Rota and Stein in [7] for applications to invariant theory.

The dual version that places representations in the exterior superalgebra instead of the symmetric one is only stated for record without proofs. This generalizes earlier expositions for general linear groups by Barnabei [2] and Józefiak [9]. In an effort to make this paper relatively self-contained we provide some background material in Section 2; in Section 4C we rely on

2000 *Mathematics Subject Classification*: Primary 17B10; Secondary 05E15.

Key words and phrases: Lie superalgebra, tableau, irreducible representation.

classical results in the representation theory of symmetric groups that can be found in the books by James [8] and Fulton [6].

The results of this paper were obtained when the author was on leave from the Mathematical Reviews during academic years 1999/2000 and 2006/2007.

2. Preliminaries. Let $E = \{E_0, E_1\}$ be a vector superspace over a field K of characteristic 0. For a homogeneous element u of E we write $\overline{u}$ for its degree or parity (0 or 1). The symmetric superalgebra $S(E)$ of E is defined as a factor superalgebra of the tensor superalgebra $\bigotimes E$ by the ideal generated by the elements of the form

$$u \otimes v - (-1)^{\overline{u}\,\overline{v}} v \otimes u$$

for homogeneous $u, v \in E$. Similarly, the exterior superalgebra $\bigwedge(E)$ is the factor superalgebra of $\bigotimes E$ by the ideal generated by the elements of the form

$$u \otimes v + (-1)^{\overline{u}\,\overline{v}} v \otimes u$$

for homogeneous $u, v \in E$. Both $S(E)$ and $\bigwedge(E)$ are superalgebras graded by nonnegative integers, i.e.,

$$S(E) = \bigoplus S_k E, \qquad \bigwedge(E) = \bigoplus \bigwedge^k E$$

as superspaces over K, and multiplications preserve these gradings.

The space $\mathrm{End}_K(E)$ has a natural structure of a vector superspace over K. This superspace with a superbracket $[F, G] = FG - (-1)^{\overline{F}\,\overline{G}} GF$ for homogeneous $F, G \in \mathrm{End}_K(E)$ is known as the *general linear Lie superalgebra* of E and is denoted $\mathfrak{gl}(E)$. It acts on E in a natural way; the induced actions on $S(E)$ and $\bigwedge(E)$ are given by the formula

$$F.(u_1 \cdots u_k) = \sum_{i=1}^{k} (-1)^{\overline{F}(\sum_{j<i} \overline{u}_j)} u_1 \cdots F(u_i) \cdots u_k. \tag{1}$$

Let E^i be a copy of E indexed by $i \in \mathbb{N} = \{1, 2, \ldots\}$ and let u^i be the element of E^i corresponding to u. Let us consider the superalgebra $R(E) = S(\bigoplus E^i)$. For a sequence $U = (u_1, \ldots, u_k)$ of homogeneous elements of E we define

$$D(U) = D(u_1, \ldots, u_k) := \sum_{\sigma \in \Sigma_k} (\operatorname{sgn} \sigma) u_1^{\sigma(1)} \cdots u_k^{\sigma(k)} \in R(E),$$

where Σ_k is the symmetric group on $\{1, \ldots, k\}$. Note that

$$D(u_1, \ldots, u_j, u_{j+1}, \ldots, u_k) + (-1)^{\overline{u}_j \overline{u}_{j+1}} D(u_1, \ldots, u_{j+1}, u_j, \ldots, u_k) = 0,$$

which means that the map

$$\bigwedge^k E \to R(E), \quad u_1 \wedge \cdots \wedge u_k \mapsto D(U),$$

is well defined; it is also a map of $\mathfrak{gl}(E)$-modules and is functorial in E, as is easy to check.

Let λ be a partition of k and let $\mu = \lambda' = (\mu_1, \dots, \mu_l)$ be the conjugate partition of λ. We very often think of λ as a diagram of k boxes in the plane consisting of columns of lengths $\mu_1, \dots, \mu_l$ (in the English notation).

A sequence of homogeneous elements $u_1, \dots, u_k$ of E defines an element of $\bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_l} E$ as follows. We can put elements $u_1, \dots, u_k$ in the boxes of λ in such a way that u_i is in the jth column and pth row of λ if

$$i = \mu_1 + \dots + \mu_{j-1} + p \quad \text{for } 1 \le j \le l,\ 1 \le p \le \mu_j.$$

We call this a *filling* U of λ with elements $u_1, \dots, u_k$ (by abuse of notation).

Let $U_1, \dots, U_l$ be the columns of U. Then we write $e(U_j)$ for the product (down the column) of elements from U_j in $\bigwedge^{\mu_j} E$, and

$$e(U) = e(U_1) \otimes \dots \otimes e(U_l) \in \bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_l} E.$$

Example 2.1. If $\lambda = (3, 2)$ and

$$U = \begin{matrix} u_2 & u_3 & u_4 \\ u_1 & u_5 & \end{matrix}$$

is the filling associated with the sequence u_2, u_1, u_3, u_5, u_4 then $e(U_1) = u_2 \wedge u_1$, $e(U_2) = u_3 \wedge u_5$, $e(U_3) = u_4$ and $e(U) \in \bigwedge^2 E \otimes \bigwedge^2 E \otimes E$.

Using this notation we can define a map

$$\Phi_\lambda(E)\colon \bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_l} E \to R(E)$$

by

$$\Phi_\lambda(E)(e(E)) = D(U_1) \cdots D(U_l) =: D(U).$$

$\Phi_\lambda(E)$ is a $\mathfrak{gl}(E)$-homomorphism and so $E(\lambda) := \operatorname{im} \Phi_\lambda(E)$ is a $\mathfrak{gl}(E)$-module. We will prove that $E(\lambda)$ is an irreducible $\mathfrak{gl}(E)$-module and provide a basis and a presentation for $E(\lambda)$. We also write Φ_λ instead of $\Phi_\lambda(E)$ for short.

Let $U = (u_i)$ be a filling of λ with entries $u_1, \dots, u_k$. If $\sigma \in \Sigma_k$ then $U_\sigma = (u_{\sigma(i)})$, i.e., the box where there is entry u_i in U is occupied by $u_{\sigma(i)}$ in U_σ. Let now $U = (u_1, \dots, u_k)$ be a sequence of homogeneous elements of E such that $u_1 \cdots u_k \ne 0$ in $S_k(E)$ and let $\sigma \in \Sigma_k$. Then $u_{\sigma(1)} \cdots u_{\sigma(k)} = \mp u_1 \cdots u_k$ where the sign depends on σ and the parity of the elements $u_1, \dots, u_k$. Call this sign $\Delta(\overline{U}; \sigma)$ where $\overline{U} = (\overline{u}_1, \dots, \overline{u}_k)$. In this way, Δ is a well-defined function $\Delta\colon Z_2^k \times \Sigma_k \to \{1, -1\}$; in particular, $\Delta(\overline{U}; (i, i+1)) = (-1)^{\overline{u}_j \overline{u}_{j+1}}$.

Analoguosly, one can define a function $\delta\colon Z_2^k \times \Sigma_k \to \{1, -1\}$ such that $u_{\sigma(1)} \wedge \dots \wedge u_{\sigma(k)} = \delta(\overline{U}; \sigma) u_1 \wedge \dots \wedge u_k$ in $\bigwedge^k E$; in particular, $\delta(\overline{U}; (i, i+1)) = (-1)^{\overline{u}_j \overline{u}_{j+1} + 1}$.

Proposition 2.2. *For any $\sigma \in \Sigma_k$ and $z \in Z_2^k$ we have*

$$\delta(z; \sigma) = (\operatorname{sgn} \sigma) \Delta(z; \sigma).$$

For the proof we need another fact which follows directly from the definitions.

LEMMA 2.3. *For $\sigma, \tau \in \Sigma_k$ we have*

$$\Delta(\overline{U};\sigma\tau) = \Delta(\overline{U};\sigma)\Delta(\overline{U}_\sigma;\tau), \qquad \delta(\overline{U};\sigma\tau) = \delta(\overline{U};\sigma)\delta(\overline{U}_\sigma;\tau).$$

In order to prove Proposition 2.2 we notice that the formula is true for a simple transposition. Assume that it is true for σ and τ. Then $\delta(\overline{U};\sigma\tau) = \delta(\overline{U};\sigma)\delta(\overline{U}_\sigma;\tau) = (\operatorname{sgn}\sigma)\Delta(\overline{U};\sigma)(\operatorname{sgn}\tau)\Delta(\overline{U}_\sigma;\tau) = (\operatorname{sgn}\sigma\tau)\Delta(\overline{U};\sigma\tau)$ by the lemma. Hence, by induction on the length of permutations we are done.

3. Determinants in $R(E)$. In order to get a better insight into some properties of the functions D introduced in Section 2 we need to introduce bases.

Let $\{X_1,\ldots,X_m\}$ be a K-basis of E_0 and let $\{X_{1'},\ldots,X_{n'}\}$ be a K-basis of E_1. We set $A = \{1,\ldots,m\}, A' = \{1',\ldots,n'\}$ and $\mathbb{A} = A \cup A'$. If U is a filling of λ with elements $u_1,\ldots,u_k$ (see Section 2) and $u_i = X_{a_i}, a_i \in \mathbb{A}$, then we write $e(T)$ instead of $e(U)$, where T is the filling of λ with entries in $\mathbb{A}$ such that T has entry a_i where U has u_i. Similarly, we write $D(T)$ instead of $D(U)$. If we set, in Example 2.1, $u_1 = X_{2'}, u_2 = X_2, u_3 = X_{3'}, u_4 = X_4$ and $u_5 = X_1$ then

$$T = \begin{matrix} 2 & 3' & 4 \\ 2' & 1 & \end{matrix} \qquad \text{and} \qquad D(T) = D(2,2')D(3',1)D(4).$$

We write $\overline{T} = (\overline{a}_i)$ for the filling of parities of $T = (a_i)$. The symmetric group Σ_k acts on the set of fillings T of λ with entries in $\mathbb{A}$; we have $T_\sigma = (a_{\sigma(i)})$ if $T = (a_i)$.

The basis $\{X_a \mid a \in \mathbb{A}\}$ of E leads to a basis $\{X_a^i \mid a \in \mathbb{A},\, i \in \mathbb{N}\}$ for $\bigoplus E^i$. For a sequence $(a_1,\ldots,a_k) \in \mathbb{A}^k$ we have, according to a previous definition and the above notation,

$$D(a_1,\ldots,a_k) := \sum_{\sigma\in\Sigma_k} (\operatorname{sgn}\sigma)\, X_{a_1}^{\sigma(1)}\cdots X_{a_k}^{\sigma(k)} \in R(E).$$

Note that $X_a^i X_b^j = (-1)^{\overline{a}\overline{b}} X_b^j X_a^i$.

Using the basis $\{X_a \mid a \in \mathbb{A}\}$, the Lie superalgebra $\mathfrak{gl}(E)$ can be identified with the Lie superalgebra $\mathfrak{gl}(m,n)$ of $(m+n)$-matrices with entries in K.

We record here the explicit formula for the action of $\mathfrak{gl}(m,n)$ on $\bigwedge^k E$ which follows from the formula (1) of Section 2:

$$(2) \qquad G.e(a_1,\ldots,a_k) = \sum_{j=1}^{k} (-1)^{\overline{G}(\sum_{i<j}\overline{a_i})} \sum_{c\in\mathbb{A}} g_{ca_j} e(a_1,\ldots,a_{j-1},c,a_{j+1},\ldots,a_k)$$

for a matrix $G = (g_{ca}) \in \mathfrak{gl}(m,n)$.

We will need more general determinants in $R(E)$. For k-tuples $R = (r_1, \dots, r_k) \in \mathbb{N}^k$, $P = (a_1, \dots, a_k) \in \mathbb{A}^k$ and $\sigma \in \Sigma_k$ we write $R_\sigma = (r_{\sigma(1)}, \dots, r_{\sigma(k)})$ and $P_\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(k)})$. We define

$$X(R;P) := X_{a_1}^{r_1} X_{a_2}^{r_2} \cdots X_{a_k}^{r_k}, \qquad D(R\|P) := \sum_{\tau \in \Sigma_k} (\operatorname{sgn} \tau) X(R_\tau; P).$$

Note that $D((1,\dots,k)\|P) = D(P)$ as defined previously.

Lemma 3.1. *For $\sigma, \tau \in \Sigma_k, R \in \mathbb{N}^k$ and $P \in \mathbb{A}^k$ we have:*

(i) $X(R_\sigma; P_\sigma) = \Delta(\overline{P}; \sigma)\, X(R;P)$.
(ii) *If $T = R_\sigma$ then $R = T_{\sigma^{-1}}$.*
(iii) $R_{\sigma\tau} = (R_\sigma)_\tau$.
(iv) $D(R\|P_\sigma) = \delta(\overline{P};\sigma) D(R\|P)$.
(v) $D(R_\tau\|P) = (\operatorname{sgn} \tau) D(R\|P)$.

Proof. Properties (i)–(iii) follow directly from the definitions. Based on these we have

$$\begin{aligned} D(R\|P_\sigma) &= \sum_{\tau \in \Sigma_k} (\operatorname{sgn} \tau\sigma) X((R_\tau)_\sigma; P_\sigma) \\ &= (\operatorname{sgn} \sigma) \sum_{\tau \in \Sigma_k} (\operatorname{sgn} \tau) \Delta(\overline{P};\sigma) X(R_\tau; P) \\ &= \delta(\overline{P};\sigma) \sum_{\tau \in \Sigma_k} (\operatorname{sgn} \tau) X(R_\tau; P) = \delta(\overline{P};\sigma) D(R\|P), \end{aligned}$$

where the next to the last equality follows from Proposition 2.2. This proves (iv); the proof of (v) is similar.

If a partition $k = p + q$ is fixed and $P = (a_1, \dots, a_k) \in \mathbb{A}^k$ then we write $P' = (a_1, \dots, a_p)$, $P'' = (a_{p+1}, \dots, a_{p+q})$ and similarly for $R \in \mathbb{N}^k$. This means, in particular, that $P'_\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(p)})$ and $P''_\sigma = (a_{\sigma(p+1)}, \dots, a_{\sigma(k)})$ for $\sigma \in \Sigma_k$.

Let $\Sigma_k = \Sigma(\{1,\dots,k\})$, $\Sigma_p = \Sigma(\{1,\dots,p\})$ and $\Sigma_{p,q} = \Sigma(\{p+1, \dots, p+q\})$ be the symmetric groups on the indicated sets.

Lemma 3.2.

(i) *For any $\sigma \in \Sigma_k$,*

$$X(R'; P'_\sigma) X(R''; P''_\sigma) = \Delta(\overline{P};\sigma) X(R'_{\sigma^{-1}}; P') X(R''_{\sigma^{-1}}; P'').$$

(ii) *If $R_1 \in \mathbb{N}^p$, $R_2 \in \mathbb{N}^q$ and $\sigma \in \Sigma_k$, $\tau \in \Sigma_p \times \Sigma_{p,q}$ then*

$$\delta(\overline{P};\sigma\tau) D(R_1\|P'_{\sigma\tau}) D(R_2\|P''_{\sigma\tau}) = \delta(\overline{P};\sigma) D(R_1\|P'_\sigma) D(R_2\|P''_\sigma).$$

Proof. (i) is a generalization of Lemma 3.1(i) with a similar proof. For (ii) notice that $\tau = \tau' \times \tau'' = \tau'\tau'' = \tau''\tau'$ for some $\tau' \in \Sigma_p$ and $\tau'' \in \Sigma_{p,q}$.

Hence by Lemma 3.1(iv) we obtain

$$\delta(\overline{P'_\sigma};\tau')D(R_1\|P'_\sigma) = D(R_1\|P'_{\sigma\tau'}),$$

and similarly

$$\delta(\overline{P''_\sigma};\tau'')D(R_2\|P''_\sigma) = D(R_2\|P''_{\sigma\tau''}).$$

Multiplying both identities and taking into account that

$$\delta(\overline{P'_\sigma};\tau')\delta(\overline{P''_\sigma};\tau'') = \delta(\overline{P}_\sigma;\tau)$$

(because $\tau \in \Sigma_p \times \Sigma_{p,q}$) we obtain

$$\delta(\overline{P}_\sigma;\tau)D(R_1\|P'_\sigma)D(R_2\|P''_\sigma) = D(R_1\|P'_{\sigma\tau})D(R_2\|P''_{\sigma\tau}).$$

Now Lemma 2.3 for the function δ implies the required formula.

PROPOSITION 3.3. *Let λ be a partition of k and let T be a filling of λ with entries in the set $\mathbb{A}$ with columns $T_1,\ldots,T_l$; moreover, let $\mu = \lambda' = (\mu_1,\ldots,\mu_l)$ and $1 \le j < l$. Let V be a subset of the index set of the entries in T_j and let W be a subset of the index set of the entries in T_{j+1} such that $\#(V) + \#(W) > \mu_j$. Then*

$$\mathcal{G}_j(T;V,W) := \sum_\sigma \delta(\overline{T};\sigma)D(T_\sigma) = 0,$$

where the summation runs over a transversal $\{\sigma\}$ of $\Sigma(V\cup W)$ for $\Sigma(V)\times\Sigma(W)$; here $\Sigma(X)$ is the symmetric group on the set X, and a transversal means a complete set of left coset representatives.

REMARK 3.4. $\mathcal{G}_j(T;V,W)$ is well defined, i.e., does not depend on the choice of the transversal, as is clear from arguments given in the proof below.

Proof of Proposition 3.3. For ease of notation we can assume that $T_j = (a_1,\ldots,a_p)$, $T_{j+1} = (a_{p+1},\ldots,a_{p+q})$ for $p = \mu_j \ge q = \mu_{j+1}$, $a_i \in \mathbb{A}$; then $V \subset \{1,\ldots,p\}$ and $W \subset \{p+1,\ldots,p+q\}$. Let $G = \Sigma(V)\times\Sigma(W) \subset H = \Sigma(V\cup W) \subset \Sigma_{p+q}$; moreover, let $P' = T_j$, $P'' = T_{j+1}$ and $P = P'\cup P''$ for short. Since $D(T) = D(T_1)\cdots D(T_l)$, in order to show that $\mathcal{G}_j(T;V,W) = 0$ it is enough to prove that

$$\sum_{\sigma\in H/G} \delta(\overline{P};\sigma)D(P'_\sigma)D(P''_\sigma) = 0. \tag{3}$$

Note that for $\tau \in G$, $\sigma \in H$ we have

$$\delta(\overline{P};\sigma)D(P'_\sigma)D(P''_\sigma) = \delta(\overline{P};\sigma\tau)D(P'_{\sigma\tau})D(P''_{\sigma\tau})$$

by Lemma 3.2(ii). Therefore (since $\operatorname{char} K = 0$) it is sufficient to prove that

$$\sum_{\sigma\in H} \delta(\overline{P};\sigma)D(P'_\sigma)D(P''_\sigma) = 0. \tag{4}$$

Now the left side of (4) can be written as

$$\sum_{\sigma\in H}\sum_{\tau}\delta(\overline{P};\sigma)\,\delta(\overline{P};\tau)X(R'_\tau;P'_\sigma)X(R''_\tau;P''_\sigma)$$

where $R'=(r_1,\ldots,r_p)$, $R''=(r_{p+1},\ldots,r_{p+q})$ for

$$r_i=\begin{cases} i & \text{if } 1\le i\le p,\\ i-p & \text{if } p+1\le i\le p+q,\end{cases}$$

and τ ranges over $\Sigma_p\times\Sigma_{p,q}$. Therefore it is enough to prove the identity

$$\sum_{\sigma\in H}\delta(\overline{P};\sigma)X(R'_\tau;P'_\sigma)X(R''_\tau;P''_\sigma)=0 \tag{5}$$

for each τ as above.

Let τ be fixed and consider the set $\{\tau(i)\,|\,i\in V\cup W\}$. Since $\#(V)+\#(W)>p$ there exist $s,i\in V\cup W$ such that $r_{\tau(s)}=r_{\tau(i)}$. We define $\gamma=(s,i)\in H$ and note that $R_{\tau\gamma}=R_\tau$ by the choice of γ. In fact, $\tau\gamma(i)=\tau(s)$ and $\tau\gamma(s)=\tau(i)$; hence $r_{\tau\gamma(i)}=r_{\tau(s)}=r_{\tau(i)}$ and $r_{\tau\gamma(s)}=r_{\tau(i)}=r_{\tau(s)}$. Obviously, $r_{\tau\gamma(x)}=r_{\tau(x)}$ for $x\ne s,i$. We claim that

$$\delta(\overline{P};\sigma)X(R'_\tau;P'_\sigma)X(R''_\tau;P''_\sigma)+\delta(\overline{P};\sigma\gamma)X(R'_\tau;P'_{\sigma\gamma})X(R''_\tau;P''_{\sigma\gamma})=0$$

for each $\sigma\in H$, which obviously implies (5).

Indeed, by Lemma 3.2(i) and Proposition 2.2 we have

$$\begin{aligned}\delta(\overline{P};\sigma\gamma)&X(R'_\tau;P'_{\sigma\gamma})X(R''_\tau;P''_{\sigma\gamma})\\ &=(\operatorname{sgn}\sigma\gamma)X(R'_{\tau(\sigma\gamma)^{-1}};P')X(R''_{\tau(\sigma\gamma)^{-1}};P'')\\ &=-(\operatorname{sgn}\sigma)\,X(R'_{\tau\sigma^{-1}};P')X(R''_{\tau\sigma^{-1}};P'')\\ &=-\delta(\overline{P};\sigma)X(R'_\tau;P'_\sigma)X(R''_\tau;P''_\sigma),\end{aligned}$$

which proves our claim. ∎

We provide an example in order to clarify the choice of γ.

EXAMPLE 3.5. Let $p=4$, $q=3$, $V=\{1,3,4\}$ and $W=\{6,7\}$ so that R can be expressed as

$$\begin{array}{cc} \boxed{r_1=1} & r_5=1\\ r_2=2 & \boxed{r_6=2}\\ \boxed{r_3=3} & \boxed{r_7=3}\\ \boxed{r_4=4} & \end{array}$$

where the boxed entries correspond to the sets V and W. If $\tau=(132)(56)$ then R_τ can be expressed by the filling

$$\begin{array}{cc} 3 & 2\\ 1 & 1\\ 2 & 3\\ 4 & \end{array}$$

We have here $r_{\tau(1)} = r_3 = 3 = r_7 = r_{\tau(7)}$ and $1,7 \in V \cup W$; therefore $\gamma = (1,7) \in \Sigma(V \cup W)$ and $R_{\tau\gamma} = R_\tau$, as is easy to see. Notice that also $r_{\tau(2)} = r_1 = 1 = r_5 = r_{\tau(6)}$; however, $2 \notin V \cup W$ so that the choice of $(2,6)$ instead of $(1,7)$ for γ would not satisfy the required conditions.

As an application of Proposition 3.3 we provide another quadratic identity among our determinants which generalizes Sylvester's classical identity for the usual determinants.

To this end we need to introduce a specific transversal of Σ_k for $\Sigma_p \times \Sigma_{p,q}$. With the notation as at the beginning of the proof of Proposition 3.3, if $B \subset V$ and $C \subset W$ with $\#(B) = \#(C)$ then we write $\tau_{B,C}$ for the element of order 2 in $\Sigma(V \cup W)$ which interchanges B and C, preserving the order of elements, and leaves all the remaining elements unchanged. Let

$$Z(V,W) = \{\tau_{B,C} \mid B \subset V,\, C \subset W,\, \#(B) = \#(C)\}.$$

Note that, in particular, $\tau_{\emptyset,\emptyset} = \mathrm{Id} \in Z(V,W)$. It is easy to see that $Z(V,W)$ is a transversal of $\Sigma(V \cup W)$ for $\Sigma(V) \times \Sigma(W)$. If $\#(V) = r$, $\#(W) = s$ and $s \le r$ then

$$\#Z(V,W) = \sum_{i=1}^{s} \binom{s}{i}\binom{r}{i} = \binom{s+r}{s} = \#(\Sigma(V \cup W)/\Sigma(V) \times \Sigma(W)).$$

Therefore it is enough to prove that distinct $\tau_{B,C}$'s determine distinct cosets, which is straightforward. We write $Z(W) = Z(\{1,\dots,p\}, W)$ in what follows. If $k = p+q$, $p \ge q$, $W \subset \{p+1,\dots,p+q\}$ and $P \in \mathbb{A}^k$, then by Proposition 3.3 we know that

$$\mathcal{G}(P;W) := \sum_{\sigma \in Z(W)} \delta(\overline{P};\sigma) D(P'_\sigma) D(P''_\sigma) = 0.$$

Now we define another set of permutations $Y(W) = \{\tau_{B,W}\}$ where B runs through all s-element subsets of $\{1,\dots,p\}$. Note that $\#Y(W) = \binom{p}{s}$ for $p \ge s > 0$. We adopt the convention that $Y(\emptyset) = \{\tau_{\emptyset,\emptyset} = \mathrm{Id}\}$; of course $\tau_{\emptyset,\emptyset} \notin Y(W)$ if $W \ne \emptyset$.

PROPOSITION 3.6. *Let $P \in \mathbb{A}^k$, $k = p+q$, $p \ge q$ and $\emptyset \ne W \subset \{p+1, \dots, p+q\}$. Then*

$$\mathcal{H}(P;W) := D(P')D(P'') - \sum_{\sigma \in Y(W)} \Delta(\overline{P};\sigma) D(P'_\sigma) D(P''_\sigma) = 0.$$

REMARK 3.7. Note that our conventions imply $\mathcal{G}(P;\emptyset) = D(P')D(P'')$ and $\mathcal{H}(P;\emptyset) = 0$.

REMARK 3.8. Since $\operatorname{sgn}\sigma = (-1)^s$ for $\sigma \in Y(W)$ if $\#W = s$, we can rewrite the formula of Proposition 3.6 in the form

$$D(P')D(P'') = (-1)^s \sum_{\sigma\in Y(W)} \delta(\overline{P};\sigma)D(P'_\sigma)D(P''_\sigma).$$

by Proposition 2.2.

Proof of Proposition 3.6. We first show that

$$\mathcal{G}(P;W) = \sum_{i=1}^{s}(-1)^{i+1} \sum_{C\subset W,\,\#C=i} \mathcal{H}(P;C). \tag{6}$$

Note that

$$Z(W) = \bigcup_{C\subset W} Y(C) = \bigcup_{i=0}^{s} \bigcup_{C\subset W,\,\#C=i} Y(C).$$

Hence and by Remark 3.8 we have

$$\begin{aligned}
\mathcal{G}(P;W) &= \sum_{i=0}^{s} \sum_{\sigma\in Y(C),\,\#C=i} \delta(\overline{P};\sigma)D(P'_\sigma)D(P''_\sigma)\\
&= \sum_{i=0}^{s} \sum_{\sigma\in Y(C),\#C=i} (-1)^i[(D(P')D(P'') - \mathcal{H}(P;C)]\\
&= \left(\sum_{i=0}^{s}(-1)^i\binom{s}{i}\right)D(P')D(P'') + \sum_{i=1}^{s}(-1)^{i+1} \sum_{C\subset W,\,\#C=i} \mathcal{H}(P;C),
\end{aligned}$$

and (6) follows since the sum in the first summand is zero and $\mathcal{H}(P;\emptyset) = 0$.

Now we prove that

$$\mathcal{H}(P;W) = \sum_{i=1}^{s}(-1)^{i+1} \sum_{C\subset W,\,\#C=i} \mathcal{G}(P;C). \tag{7}$$

Indeed, in view of (6) the right side of (7) is equal to

$$\begin{aligned}
\sum_{\emptyset\neq C\subset W} (-1)^{\#C+1}\mathcal{G}(P;C) &= \sum_{\emptyset\neq C\subset W} (-1)^{\#C+1}\Big(\sum_{\emptyset\neq B\subset C} (-1)^{\#B+1}\mathcal{H}(P;B)\Big)\\
&= \sum_{\emptyset\neq B\subset W} (-1)^{\#B+1}\Big(\sum_{\emptyset\neq C\subset W} (-1)^{\#C+1}\mathcal{H}(P;B)\Big).
\end{aligned}$$

The sum corresponding to $B = W$ is $\mathcal{H}(P;W)$, whereas if $B \neq W$ the sum is

$$\sum_{B\subset C\subset W} (-1)^{\#C+1} = -\sum_{i=0}^{s-t}(-1)^i\binom{s-t}{i} = 0$$

where $\#B = t$ and $s - t > 0$.

Now (7) implies that $\mathcal{H}(P;W) = 0$ since by Proposition 3.3, $\mathcal{G}(P;C) = 0$ for each $\emptyset \neq C \subset W$.

We end this section with a result which will not be used later (hence we do not provide a proof). We record it here because it can be used to give an alternative proof of Proposition 3.3 and because it is a generalization of a classical formula for the usual determinants.

PROPOSITION 3.9 (Laplace-type expansions). *Let* $k = p + q$, $R \in \mathbb{N}^k$, $P \in \mathbb{A}^k$.

(i) *If* $\tau \in \Sigma_k$ *then*

$$D(R\|P) = (\operatorname{sgn}\tau)\sum_{\sigma} \delta(\overline{P};\sigma) D(R'_\tau\|P'_\sigma) D(R''_\tau\|P''_\sigma)$$

where the summation runs over a transversal $\{\sigma\}$ *of* Σ_k *for* $\Sigma_p \times \Sigma_{p,q}$.

(ii) *If* $\sigma \in \Sigma_k$ *then*

$$D(R\|P) = \delta(\overline{P};\sigma)\sum_{\tau} (\operatorname{sgn}\sigma) D(R'_\tau\|P'_\sigma) D(R''_\tau\|P''_\sigma)$$

where the summation runs over a transversal $\{\tau\}$ *of* Σ_k *for* $\Sigma_p \times \Sigma_{p,q}$.

4. Main results

A. *Statement of results.* Let λ be a partition of k, and μ be the conjugate of λ. In Section 2 we defined, for λ and a superspace E over a field K of characteristic 0, a map of $\mathfrak{gl}(E)$-modules

$$\Phi_\lambda \colon \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E \to R(E)$$

such that $\Phi_\lambda(e(T)) = D(T)$ for any filling T of λ with entries in the set $\mathbb{A}$; here, as before, $\mathbb{A} = A \cup A'$, $A = \{1, \ldots, m\}$, $A' = \{1', \ldots, n'\}$ where $m = \dim E_0$ and $n = \dim E_1$.

The elements $\mathcal{G}_j(T; V, W) \in R(E)$ were defined in Section 3 for any filling T as above, for a number $1 \le j < l =$ length of μ and for a choice of a subset V of the index set of the entries in T_j and a subset W of the index set of the entries in T_{j+1}. We define the corresponding elements $G_j(T; V, W) \in \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E$ by the formula

$$G_j(T; V, W) := \sum_{\sigma} \delta(\overline{T};\sigma) e(T_\sigma)$$

where the summation runs over a transversal $\{\sigma\}$ of $\Sigma(V \cup W)$ for $\Sigma(V) \times \Sigma(W)$. Note that these elements do not depend on the choice of the transversal because $\delta(\overline{T};\sigma\tau)e(T_{\sigma\tau}) = \delta(\overline{T};\sigma)\, e(T_\sigma)$ for $\sigma \in \Sigma(V \cup W)$, $\tau \in \Sigma(V) \times \Sigma(W)$; this follows from formulas for the $e(T)$ analogous to those of Lemma 3.1(iv) and Lemma 3.2(ii) for the $D(T)$. From the very definitions we now have

$$\Phi_\lambda(G_j(T; V, W)) = \mathcal{G}_j(T; V, W),$$

hence in view of Proposition 3.3 we know that $G_j(T; V, W) \in \ker \Phi_\lambda$.

Let $C_\lambda(E)$ be the $\mathfrak{gl}(E)$-submodule of $\bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E$ generated by the $G_j(T; V, W)$ for all possible choices of j, T, V and W.

We order the set $\mathbb{A}$ in the following way: $1 < 2 < \cdots < m < 1' < \cdots < n'$. We call a filling T of λ with entries in $\mathbb{A}$ a *tableau* if the entries in each row of T are weakly increasing from left to right and are strictly increasing with respect to the set A', whereas the entries down each column are also weakly increasing but are strictly increasing with respect to the set A. An example of a tableau of shape $(3, 3, 2)$ is

$$\begin{matrix} 1 & 1 & 2' \\ 2 & 1' & 2' \\ 1' & 2' & \end{matrix}$$

Now we are ready to formulate the main result of this paper.

THEOREM 4.1. *Let K be a field of characteristic* 0 *and let E be a superspace over K with* $\dim E_0 = m$ *and* $\dim E_1 = n$. *With the previous notation for any partition λ with $\lambda_{m+1} \le n$ (called an (m, n)-hook partition) we have*:

(i) $E(\lambda) := \operatorname{im} \Phi_\lambda$ *is an irreducible* $\mathfrak{gl}(E)$*-module.*
(ii) *The set $\{D(T)\}$ where T runs over the set of tableaux of shape λ is a basis of $E(\lambda)$ over K.*
(iii) $C_\lambda(E) = \ker \Phi_\lambda$.

B. *Generators and independence.* As was mentioned above, we have $C_\lambda(E) \subset \ker \Phi_\lambda$. It is clear that in order to prove (ii) and (iii) of Theorem 4.1 it is enough to show (I) and (II) below:

(I) The set $\{D(T)\}$ where T runs over the set of tableaux of shape λ is linearly independent over K.
(II) The set $\{\overline{e}(T) := e(T) \bmod C_\lambda(E)\}$ where T runs over the set of tableaux of shape λ linearly spans the quotient

$$\Big(\bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E\Big)/C_\lambda(E).$$

As a consequence, for every filling T with entries in $\mathbb{A}$, $\overline{e}(T)$ and $D(T)$ can be identified by means of Φ_λ.

Proof of (I). We order the variables $\{X_a^i \mid 1 \le i \le \mu_1,\, a \in \mathbb{A}\}$ by declaring $X_a^i < X_b^j$ if $i < j$ or if $i = j$ and $a < b$. We order monomials in the X_a^i by the lexicographic ordering compatible with this ordering on the X_a^i.

Let us consider a one-column tableau T with entries $(a_1, \ldots, a_p) = (c_1^{k_1}, \ldots, c_s^{k_s})$, $c_i \ne c_j$ for $i \ne j$; note that for $c_i \in A$ one has $k_i \le 1$. Because of skew-commutativity of the X_a^i for $a \in A'$ we have

$$\begin{aligned} D(T) &= k_1! \cdots k_s! \sum_\sigma (\operatorname{sgn} \sigma) X_{a_1}^{\sigma(1)} \cdots X_{a_p}^{\sigma(p)} \\ &= k_1! \cdots k_s!\, X_{a_1}^1 \cdots X_{a_p}^p + \text{higher order terms} \end{aligned}$$

where the sum runs over a transversal $\{\sigma\}$ of Σ_p for $\Sigma_{k_1} \times \cdots \times \Sigma_{k_s}$. This extends to any tableau T. To describe the leading term of $D(T)$ we define $m_T(i,a)$ to be the number of a's in the ith row of T. Then we have

$$D(T) = s \prod_{i,a} (X_a^i)^{m_T(i,a)} + \text{higher order terms}, \quad 0 \neq s \in \mathbb{Z}.$$

The leading term is nonzero because $\operatorname{char} K = 0$ and $m_T(i,a) \le 1$ for $a \in A'$ by the definition of a tableau.

Now we order all tableaux of shape λ with entries in $\mathbb{A}$ by declaring $T \prec T'$ iff the first X_a^i for which $m_T(i,a) \neq m_{T'}(i,a)$ has $m_T(i,a) > m_{T'}(i,a)$ (take the first row i where T and T' differ; then for the first a where they differ in this row one should have $m_T(i,a) > m_{T'}(i,a)$). From this definition it follows immediately that if T and T' are tableaux of the same shape and $T \prec T'$ then the leading term of $D(T)$ is smaller than any term of $D(T')$.

Let now $\sum x_T\, D(T) = 0$ be a nontrivial linear combination and take minimal $\widetilde{T}$ with respect to $\prec$ such that $x_{\widetilde{T}} \neq 0$. By the above-mentioned fact the coefficient of the leading term of $D(\widetilde{T})$ of the left side of this identity is a nonzero multiple of $x_{\widetilde{T}}$, i.e., it is nonzero since $\operatorname{char} K = 0$, a contradiction.

Proof of (II). Let $\mathcal{F}_\lambda$ be the set of all fillings of λ with entries in $\mathbb{A}$ such that in each column any element of A occurs at most once. Notice that if, when performing operations on elements of $\mathcal{F}_\lambda$, we obtain a filling T with more than one occurrence of an element from A in the same column then $e(T) = 0$.

We define an equivalence relation in $\mathcal{F}_\lambda$ by $T \sim T'$ iff T' can be obtained from T by a permutation of entries in columns. Any equivalence class contains exactly one representative for which the columns are ordered, i.e., the entries in each colum are weakly increasing down the column and are strictly increasing with respect to the entries from A. Let $\widetilde{\mathcal{F}}_\lambda$ be the set of equivalence classes of elements in $\mathcal{F}_\lambda$ with respect to $\sim$. We identify an element in $\widetilde{\mathcal{F}}_\lambda$ with its unique representative that has ordered columns.

We define a well-ordering $\lessdot$ on $\widetilde{\mathcal{F}}_\lambda$ as follows: for $T, T' \in \widetilde{\mathcal{F}}_\lambda$ consider the rightmost column which is different in T and T'; we set $T \lessdot T'$ if the lowest box in that column has a larger entry in T'. We now prove that if $T \in \widetilde{\mathcal{F}}_\lambda$ is not a tableau then there exists a relation of the form

$$\overline{e}(T) = \sum_{T' \gtrdot T} c_{T,T'}\, \overline{e}(T'), \qquad T' \in \widetilde{\mathcal{F}}_\lambda,\ c_{T,T'} \in K. \tag{8}$$

Since $\widetilde{\mathcal{F}}_\lambda$ is finite, repeating this process (i.e. applying identities of the form (8) to summands $\overline{e}(T')$ where T' is not a tableau) finally leads to a required presentation of $\overline{e}(T)$ in terms of the $\overline{e}(T')$ for tableaux T'.

If $T \in \widetilde{\mathcal{F}}_\lambda$ is not a tableau then there are two consecutive columns T_j of length p and T_{j+1} of length q in T, $p \geq q$, and two entries in these columns and in the same row, say the sth, where the tableau condition is violated. The columns are of the form:

$$\begin{array}{lcl} T_j & & T_{j+1} \\ \hline a_1 & & a_{p+1} \\ a_2 & & a_{p+2} \\ \vdots & & \vdots \\ a_s & \geq & a_{p+s} \\ a_{s+1} & & a_{p+s+1} \\ \vdots & & \vdots \end{array}$$

(a) Let $a_s > a_{p+s}$ and let $V = \{s, \ldots, p\}$, $W = \{p+1, \ldots, p+s\}$. Since by definition $G_j(T; V, W) \in C_\lambda(E)$, we have the identity

$$0 = \overline{G}_j(T; V, W) = \sum_{\sigma \in F} \delta(\overline{T}; \sigma)\overline{e}(T_\sigma) \tag{9}$$

where F is a transversal of $\Sigma(\cup W)$ for $\Sigma(V) \times \Sigma(W)$ (one can take, e.g., $F = Z(V, W)$ introduced in Section 3). We can assume that one of the σ is the identity so that one of the summands in (9) is $\overline{e}(T)$. Using (9) we can express $\overline{e}(T)$ as a linear combination of the $\overline{e}(T')$ each corresponding to a nontrivial permutation from F. Each such permutation moves at least one entry from $\{a_s, \ldots, a_p\}$ to $\{a_{p+1}, \ldots, a_{p+s}\}$ and since $a_i < a_j$ for $i = s, \ldots, p, j = p+1, \ldots, p+s$, this means that we have $T' \gtrdot T$ for the resulting filling $T' \in \widetilde{\mathcal{F}}_\lambda$.

(b) Let $a_s = a_{p+s} = c$; then $c \in A'$ (indeed, if $c \in A$ then $a_s = a_{p+s} = c$ is not a violation of the tableau definition). We use again the identity (9) to express $\overline{e}(T)$ in the form (8). But first we cover the set F with three subsets F_0, F_1 and F_2: F_0 is the set of all $\sigma \in F$ that permute only entries equal to c; F_1 is the set of all $\sigma \in F$ that move at least one entry from column T_j that is different from c to column T_{j+1}; and finally, F_2 is the set of all $\sigma \in F$ that move at least one entry from column T_{j+1} that is different from c to column T_j. Let x be the largest integer such that $a_{s+x-1} = c$, $0 < x \leq p - s + 1$, and let y be the largest integer such that $a_{p+s-y+1} = c$, $0 < y < s$.

If $F_1 = F_2 = \emptyset$ then $F_0 = F$ and consequently $x = p - s + 1, y = s - 1$. Hence all summands in (9) are of the form $\overline{e}(T)$ since, as is easy to see, $\delta(\overline{T}; \sigma) = 1$ in this case; therefore $\overline{e}(T) = 0$ since $\operatorname{char} K = 0$.

If $F_1 \neq \emptyset$ and $\sigma \in F_1$ then all entries moved by σ from T_j to T_{j+1} are greater than or equal to entries in T_{j+1} indexed by W and at least one entry is greater; hence for the resulting summand $\delta(\overline{T'};\sigma)\overline{e}(T'_\sigma)$ in (9), $T' \in \widetilde{\mathcal{F}}_\lambda$, we have $T' \gtrdot T$. We argue similarly for $F_2 \neq \emptyset$.

Now, from (9) and the fact that the cardinality of F_0 is $\binom{x+y}{x}$ it follows that

$$\binom{x+y}{x}\overline{e}(T) = \sum_{T' \gtrdot T} \mp\overline{e}(T'), \tag{10}$$

where the summands on the right correspond to the set $F_1 \cup F_2 \neq \emptyset$; hence (10) leads to the required expression of type (8) for $\overline{e}(T)$ by dividing both sides by the binomial coefficient.

C. *Irreducibility.* In our proof of the irreducibility of $E(\lambda)$ we will use a specific description of the Specht modules S^λ for Σ_k. Here we briefly recall what we need in the following (for details, see e.g. Fulton [6] or James [8]).

Let M^λ be the vector space over K of row tabloids of shape λ and let $\{S\}$ be the tabloid determined by a filling S of λ with distinct numbers $1,\dots,k$. For such an S we have an element $v_S \in M^\lambda$ defined by the formula

$$v_S = \sum_{\tau \in C(S)} (\operatorname{sgn}\tau)\{\tau S\}$$

where $C(S)$ is the subgroup of Σ_k of all column-preserving permutations of S. The *Specht module* S^λ is defined as the Σ_k-submodule of M^λ generated by the v_S. We also need the module dual to M^λ which is denoted $\widetilde{M}^\lambda$. It is the Σ_k-module spanned by the column tabloids $[S]$, where S, as above, is a filling of λ with distinct numbers $1,\dots,k$ and $[S]$ is defined up sign.

We have a well-defined Σ_k-epimorphism $f_\lambda\colon \widetilde{M}^\lambda \to S^\lambda$, $f_\lambda[S] = v_S$. For $V \subset S_j$ and $W \subset S_{j+1}$ we can define the so called *Garnir elements*

$$g_j(S;V,W) := \sum_{\sigma \in Z(V,W)} (\operatorname{sgn}\sigma)[\sigma S] \in \widetilde{M}^\lambda.$$

It is a classical result (see, e.g., James [8, p. 27]) that $g_j(S;V,W) \in \ker f_\lambda$ for any j, S, V and W.

In addition to the left $\mathfrak{gl}(E)$-action we consider the right Σ_k-action on $\bigotimes^k E$ defined by

$$(u_1 \otimes \dots \otimes u_k)\cdot\sigma = \Delta(\overline{U};\sigma)u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(k)}$$

for a sequence (u_i) of homogeneous elements of E. It follows from Lemma 2.3 that this action is well defined. The action induces the corresponding action on $\bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_l} E$ such that $e(U)\cdot\sigma = \Delta(\overline{U};\sigma)e(U_\sigma)$.

Now let S° be a filling of λ with distinct numbers $1,\dots,k$ such that i is in the jth column and sth row of λ iff

$$i = \mu_1 + \dots + \mu_{j-1} + s \quad \text{for } 1 \le j \le l,\ 1 \le s \le \mu_j.$$

For any Σ_k-module N we define $E(N) := \bigotimes^k E \otimes_{\Sigma_k} N$, which is in a natural way a left $\mathfrak{gl}(E)$-module. Finally, we can define a map

$$\beta_\lambda \colon \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E \to E(\widetilde{M}^\lambda)$$

by $\beta_\lambda(e(U)) = u_1 \otimes \cdots \otimes u_k \otimes [S^\circ]$ for any filling of λ corresponding to $U = (u_i)$, u_i homogeneous elements of E (see Section 2). Note first that β_λ is a well-defined map of $\mathfrak{gl}(E)$-modules. In order to prove this we take two entries from the jth column of U and denote by τ their transposition; then

$$\begin{aligned}\beta_\lambda(e(U) + \Delta(\overline{U};\tau)e(U_\tau)) &= e(U)\cdot(1+\tau)\otimes[S^\circ]\\ &= e(U)\otimes([S^\circ]+[\tau S^\circ]) = 0\end{aligned}$$

since $[\tau\, S^\circ] = -[S^\circ]$ in $\widetilde{M}^\lambda$. It is easy to see that β_λ is an isomorphism (note that $\widetilde{M}^\lambda$ is a cyclic Σ_k-module generated by $[S^\circ]$).

We now have the following diagram of $\mathfrak{gl}(E)$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \Phi_\lambda & \longrightarrow & \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E & \xrightarrow{\Phi_\lambda} & E(\lambda) & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \beta_\lambda} & & & & \\ 0 & \longrightarrow & E(\ker f_\lambda) & \longrightarrow & E(\widetilde{M}^\lambda) & \xrightarrow{1\otimes f_\lambda} & E(S^\lambda) & \longrightarrow & 0 \end{array}$$

with exact rows. The exactness of the bottom row follows from the fact that S^λ is a projective (even irreducible) Σ_k-module. We will show that $\beta_\lambda(\ker \Phi_\lambda) \subset E(\ker f_\lambda)$. Indeed, by Theorem 4.1(iii) we know that $\ker \Phi_\lambda$ is generated by the elements of the form $G_j(T; V, W)$. We have

$$\begin{aligned}(1\otimes f_\lambda)\beta_\lambda(G_j(T;V,W)) &= (1\otimes f_\lambda)\Big[\Big(\sum_{\sigma\in Z(V,W)} \delta(\overline{T};\sigma)e(T_\sigma)\Big)\otimes[S^\circ]\Big]\\ &= (1\otimes f_\lambda)\Big[e(T)\cdot\Big(\sum_{\sigma\in Z(V,W)}(\operatorname{sgn}\sigma)\sigma\Big)\Big]\otimes[S^\circ]\\ &= e(T)\otimes f_\lambda(g_j(S^\circ;V,W)) = 0,\end{aligned}$$

the last identity by the fact that $g_j(S^\circ; V, W) \in \ker f_\lambda$. Therefore the map β_λ induces a map $\widetilde{\beta}_\lambda \colon E(\lambda) \to E(S^\lambda)$ such that $\widetilde{\beta}_\lambda(D(U)) = e(U) \otimes v_{S^\circ}$ for any U as above. It is obvious that this map is an epimorphism (note that S^λ is a cyclic Σ_k-module generated by v_{S°). On the other hand, both $E(\lambda)$ and $E(S^\lambda)$ have the same dimension over K. Indeed, we proved earlier in part B of this section that the dimension of $E(\lambda)$ is equal to the number of tableaux of shape λ with entries in $\mathbb{A}$. Sergeev [10] and Berele and Regev [3] proved independently that $E(S^\lambda)$ has the same dimension (and a basis indexed by the same set of tableaux). They also proved that $E(S^\lambda)$ is irreducible (in both cases using the Schur–Weyl duality for the actions of Σ_k and $\mathfrak{gl}(E)$ on $\bigotimes^k E$). Hence $\widetilde{\beta}_\lambda$ is an isomorphism and $E(\lambda)$ is irreducible as a $\mathfrak{gl}(E)$-module.

REMARK 4.2. For T a filling of λ with entries in $\mathbb{A}, 1 \le j \le l$ and $W \subset S^{\circ}_{j+1}$ we can define an element

$$H_j(T;W) := e(T) - \sum_{\sigma \in Y(W)} \Delta(\overline{T};\sigma) e(T_\sigma)$$

in $\bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_l} E$. By Proposition 3.6 we find that $H_j(T;W) \in \ker \Phi_\lambda$. Fulton showed in [6] that the $H_j(T;W)$ generate $\ker \Phi_\lambda$ in case $E_1 = 0$, i.e., for the Lie algebra $\mathfrak{gl}(E_0)$, or equivalently, for a general linear group. I have been unable to prove Theorem 4.1(ii) using the elements $H_j(T;W)$ instead of $G_j(T;V,W)$.

REMARK 4.3. The proof of Theorem 4.1 shows that the relations $\mathcal{G}_j(T;V,W)$ of Proposition 3.3 correspond to the Garnir relations for symmetric groups.

REMARK 4.4. It would be interesting to find a proof of the irreducibility of the $E(\lambda)$ without using the Schur–Weyl duality for actions of Σ_k and $\mathfrak{gl}(E)$ on $\bigotimes^k E$.

5. Dual results. The usual duality between symmetric and exterior superalgebras allows us to present dual results. We formulate below the basic definitions and the counterpart of Theorem 4.1 only.

For a superspace E over a field K let $\widetilde{R}(E) := \bigwedge(\bigoplus E^i)$. If $(a_1, \ldots, a_k) \in \mathbb{A}^k$ then we define

$$\widetilde{D}(a_1, \ldots, a_k) := \sum_{\sigma \in \Sigma_k} (\operatorname{sgn} \sigma) Z^{\sigma(1)}_{a_1} \wedge \cdots \wedge Z^{\sigma(k)}_{a_k} \in \widetilde{R}(E),$$

where $\{Z^i_a \mid a \in \mathbb{A}\}$ is the K-basis of E^i corresponding to a basis $\{Z_a \mid a \in \mathbb{A}\}$ for E.

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition and let T be a filling of λ with entries in $\mathbb{A}$ and with rows $\widetilde{T}_1, \ldots, \widetilde{T}_p$. We define $\widetilde{D}(T) := \widetilde{D}(\widetilde{T}_1) \cdots \widetilde{D}(\widetilde{T}_p)$. There is a well-defined map of $\mathfrak{gl}(E)$-modules

$$\Psi_\lambda(E) \colon S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_p} E \to \widetilde{R}(E)$$

such that $\Psi_\lambda(E)(\widetilde{e}(T)) = \widetilde{D}(T)$ where

$$\widetilde{e}(T) := \widetilde{e}(\widetilde{T}_1) \otimes \cdots \otimes \widetilde{e}(\widetilde{T}_p) \in S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_p} E$$

and $\widetilde{e}(a_1, \ldots, a_k) := Z_{a_1} \cdots Z_{a_k} \in S_k E$. We define $\widetilde{E}(\lambda) := \operatorname{im} \Psi_\lambda(E)$.

For a partition λ and a filling T of λ as before let us fix j such that $1 \le j < p$; moreover, let V be a subset of the index set of the entries in $\widetilde{T}_j$ and let W be a subset of the index set of the entries in $\widetilde{T}_{j+1}$. Then we define

$$\widetilde{G}_j(T;V,W) := \sum_{\sigma} \Delta(\overline{T};\sigma) \widetilde{e}(T_\sigma)$$

where the summation runs over a transversal $\{\sigma\}$ of $\Sigma(V \cup W)$ for $\Sigma(V) \times \Sigma(W)$.

THEOREM 5.1. *Let K be a field of characteristic* 0 *and let E be a superspace over K with* $\dim E_0 = m$ *and* $\dim E_1 = n$. *With the above notations and definitions we have, for any partition λ with $\lambda_{m+1} \leq n$:*

(i) *$\widetilde{E}(\lambda)$ is an irreducible $\mathfrak{gl}(E)$-module.*
(ii) *The set $\{\widetilde{D}(T)\}$ where T runs over the set of tableaux of shape λ is a basis of $\widetilde{E}(\lambda)$ over K.*
(iii) $\ker \Psi_\lambda(E)$ *is generated by elements of the form $\widetilde{G}_j(T;V,W)$ with varying j, T, V and W.*

REMARK 5.2. The elements

$$\widetilde{H}_j(T;W) := \widetilde{e}(T) - \sum_{\sigma \in Y(W)} \delta(\overline{T};\sigma)\widetilde{e}(T_\sigma)$$

are counterparts of the elements $H_j(T;W)$ of Remark 4.2 and play a similar role in the dual theory.

REMARK 5.3. The elements $H_j(T;W)$ and $\widetilde{H}_j(T;W)$ correspond to relations discussed already by Young in [13] for symmetric groups and Towber in [12] for general linear groups. These relations formed a basis of Fulton's approach in [6].

REFERENCES

[1] K. Akin, D. A. Buchsbaum and J. Weyman, *Schur functors and Schur complexes*, Adv. Math. 44 (1982), 207–278.

[2] M. Barnabei, *Schur modules, Weyl modules, and Capelli operators*, ibid. 151 (2000), 1–35.

[3] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, ibid. 64 (1987), 118–175.

[4] C. de Concini, D. Eisenbud and C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. 56 (1980), 129–165.

[5] P. Doubilet and G.-C. Rota, *Skew-symmetric invariant theory*, Adv. Math. 21 (1976), 196–201.

[6] W. Fulton, *Young Tableaux. With Applications to Representation Theory and Geometry*, Cambridge Univ. Press, Cambridge, 1997.

[7] F. D. Grosshans, G.-C. Rota and J. A. Stein, *Invariant Theory and Superalgebras*, CBMS Reg. Conf. Ser. Math. 69, Amer. Math. Soc., Providence, RI, 1987.

[8] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. 682, Springer, Berlin, 1978.

[9] T. Józefiak, *A construction of irreducible* GL(m) *representations*, Michigan Math. J. 48 (2000), 349–358.

[10] A. N. Sergeev, *Tensor algebra of the identity representation as a module over the Lie superalgebras* Gl(n,m) *and* $Q(n)$, Mat. Sb. (N.S.) 123 (165) (1984), 422–430 (in Russian).

[11] A. N. Sergeev, *An analog of the classical invariant theory for Lie superalgebras*, *I*, *II*, Michigan Math. J. 49 (2001), 113–146, 147–168.
[12] J. Towber, *Two new functors from modules to algebras*, J. Algebra 47 (1977), 80–104.
[13] A. Young, *On quantitative substitutional analysis*, *II*, Proc. London Math. Soc. 34 (1902), 361–397.

Mathematical Reviews
416 Fourth Street
Ann Arbor, MI 48107, U.S.A.
E-mail: tsj@ams.org

Received 16 January 2008 (5005)