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STABILITY TYPE RESULTS CONCERNING THE FUNDAMENTAL EQUATION OF INFORMATION OF MULTIPLICATIVE TYPE

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Abstract. The paper deals with the stability of the fundamental equation of information of multiplicative type. It is proved that the equation in question is stable in the sense of Hyers and Ulam under some assumptions. This result is applied to prove the stability of a system of functional equations that characterizes the recursive measures of information of multiplicative type.

1. Introduction. The stability theory of functional equations deals with the following question: When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? In case of a positive answer, we say that the equation in question is *stable*. This problem was raised by Ulam (see [Ula40]) and considered by Hyers who proved that the Cauchy equation is stable ([Hye41]). Since then, this result has been extended and generalized in several ways (see e.g. [For95], [Ger94] and [HIR98]). The investigation of the stability of the exponential Cauchy equation highlighted a new phenomenon which is now usually called *superstability* (see e.g. [HIR98]). The question of superstability is also dealt with in this paper. Solving a stability problem raised in [Mak07], we give an affirmative answer for the case of higher dimensional information functions.

Throughout this paper let k and n be fixed positive integers and define

$$\Gamma_n := \Big\{(p_1, \dots, p_n) \in \mathbb{R}^{kn} \;\Big|\; p_i \geq \mathbf{0},\, \sum_{i=1}^{n} p_i = \mathbf{1}\Big\}$$

and

$$D_k := \{(x, y) \in \mathbb{R}^{2k} \mid x, y \in [0, 1[^k,\, x + y \leq \mathbf{1}\}.$$

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Here $\mathbf{1}$ represents the k-vector $(1, \dots, 1) \in \mathbb{R}^k$ and all operations on vectors are to be done componentwise, e.g., $p_i \geq \mathbf{0}$ denotes that all coordinates of the vector $p_i \in \mathbb{R}^k$ are non-negative and we write $x + y \leq \mathbf{1}$ if $x_i + y_i \leq 1$ for all $i = 1, \dots, k$, where x_i and y_i denote the ith coordinates of the vectors x and y, respectively.

In what follows, we present some basic results from the theory of functional equations which we shall use throughout the paper; these results can be found for instance in [Kuc85].

A function $M : [0,1]^k \to \mathbb{R}$ is called *multiplicative* if

$$M(x \cdot y) = M(x) \cdot M(y) \quad \text{for all } x, y \in [0,1]^k.$$

We say that $A : [0,1]^k \to \mathbb{R}$ is *additive on D_k* if

$$A(x + y) = A(x) + A(y) \quad \text{for all } (x, y) \in D_k.$$

LEMMA 1.1. *If $M : [0,1]^k \to \mathbb{R}$ is both multiplicative on $[0,1]^k$ and additive on D_k, then M is either identically zero or a projection, i.e.,*

$$M(x) = M(x_1, \dots, x_k) = x_j, \quad x \in [0,1]^k,$$

for some $j \in \{0, \dots, k\}$.

In the proof of our theorem we shall use the following lemma.

LEMMA 1.2. *Let $M : [0,1]^k \to \mathbb{R}$ be multiplicative. Then the following statements are equivalent.*

(i) *M is additive on D_k;*
(ii) *$M(x) + M(\mathbf{1} - x) = 1$ for all $x \in [0,1]^k$.*

LEMMA 1.3. *Let $M : [0,1]^k \to \mathbb{R}$ be multiplicative. Then*

$$M(x) \geq 0 \quad \text{for all } x \in [0,1]^k.$$

LEMMA 1.4. *Let $M : [0,1]^k \to \mathbb{R}$ be multiplicative. Then*

$$M(x) = M(x_1, \dots, x_k) = \prod_{i=1}^{k} m_i(x_i)$$

for all $x \in [0,1]^k$, where each $m_i : [0,1] \to \mathbb{R}$ is multiplicative $(i = 1, \dots, k)$.

Now we turn to information measures (see [AD75], [ESS98]).

DEFINITION 1.1. A sequence of functions $I_n : \Gamma_n \to \mathbb{R}$ $(n = 2, 3, \dots)$ is called an *information measure.*

The usual information-theoretical interpretation is that $I_n(p_1, \dots, p_n)$ is a measure of uncertainty as to the outcome of an experiment having n possible outcomes with probabilities $p_1, \dots, p_n$.

Some desiderata for information measures can be found in [AD75] as well as in [ESS98]. In this paper we will use only the following properties.

Definition 1.2. The sequence of functions $I_n : \mathit{\Gamma}_n \to \mathbb{R}$ $(n = 2, 3, \dots)$ is:

(i) *M-recursive* if with some multiplicative function $M : [0,1]^k \to \mathbb{R}$,

$$I_n(p_1, \dots, p_n) = I_{n-1}(p_1 + p_2, p_3, \dots, p_n) + M(p_1 + p_2) I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right)$$

for all $n = 3, 4, \dots$ and $(p_1, \dots, p_n) \in \mathit{\Gamma}_n$, with the convention $\frac{0}{0+0} = 0$;

(ii) 3-*semisymmetric* if

$$I_3(p_1, p_2, p_3) = I_3(p_1, p_3, p_2) \quad \text{for all } (p_1, p_2, p_3) \in \mathit{\Gamma}_3.$$

The following theorem enables transforming the characterization of information measures into solving functional equations (see, e.g., [ESS98]).

Theorem 1.1. *If the sequence of functions $I_n : \mathit{\Gamma}_n \to \mathbb{R}$ $(n = 2, 3, \dots)$ is M-recursive and* 3*-semisymmetric then the function $f : [0,1]^k \to \mathbb{R}$ defined by*

$$f(x) := I_2(\mathbf{1} - x, x)$$

satisfies the so-called fundamental equation of information of multiplicative type M, i.e.,

$$f(x) + M(\mathbf{1} - x) f\left(\frac{y}{\mathbf{1} - x}\right) = f(y) + M(\mathbf{1} - y) f\left(\frac{x}{\mathbf{1} - y}\right) \tag{1}$$

for all $(x, y) \in D_k$.

2. Known results. In [Mak07] it is proved that (1) is stable and superstable if $k = 1$ and $M : [0,1] \to \mathbb{R}$ is a power function, i.e., the stability of the following equation is investigated:

$$f(x) + (1 - x)^\alpha f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)^\alpha f\left(\frac{x}{1 - y}\right),$$

where $0 < \alpha \neq 1$.

In [Mor01] a stability type result is proved for $k = 1$ and $\alpha = 1$, i.e., for the Shannon entropy. However, Morando's theorem states the stability only on the rationals.

3. Main result. In this section we will show stability type results for the fundamental equation of information of multiplicative type. Our main result is

Theorem 3.1. *Let $\varepsilon \geq 0$ be arbitrary, $M : [0,1]^k \to \mathbb{R}$ be multiplicative but not additive, and $f : [0,1]^k \to \mathbb{R}$ be a function. Assume that*

$$\left| f(x) + M(\mathbf{1} - x) f\left(\frac{y}{\mathbf{1} - x}\right) - f(y) - M(\mathbf{1} - y) f\left(\frac{x}{\mathbf{1} - y}\right) \right| \leq \varepsilon \tag{2}$$

for all $(x,y)\in D_k$. *Then there exist* $a,b\in\mathbb{R}$ *and* $q^*\in\,]0,1[^k$ *such that*

$$
\begin{aligned}
(3)\qquad &|f(x)-(aM(x)+b(M(\mathbf{1}-x)-1))|\\
&\qquad\leq |M(q^*)+M(\mathbf{1}-q^*)-1|^{-1}\cdot(4\varepsilon+3\varepsilon M(\mathbf{1}-xq^*))
\end{aligned}
$$

for all $x\in[0,1]^k$. *Here* $q^*\in\,]0,1[^k$ *is any element such that* $M(q^*)+M(\mathbf{1}-q^*)\neq 1$.

Proof. Define $F:\,]0,1[^k\times[0,1]^k\to\mathbb{R}$ by

$$F(p,q)=f(\mathbf{1}-p)+M(p)f(q)-f(pq)-M(\mathbf{1}-pq)f\left(\frac{\mathbf{1}-p}{\mathbf{1}-pq}\right).$$

Then the substitution of $x=\mathbf{1}-p$ and $y=pq$ in (2) implies that

$$|F(p,q)|\leq\varepsilon \tag{4}$$

for all $p,q\in\,]0,1[^k$. On the other hand,

$$
\begin{aligned}
&|[M(q)+M(\mathbf{1}-q)-1]\cdot[f(p)-f(\mathbf{1})M(p)]\\
&\qquad-[M(p)+M(\mathbf{1}-p)-1]\cdot[f(q)-f(\mathbf{1})M(q)]|\\
&=F(q,p)+F(p,q)-F(q,\mathbf{1})+F(p,\mathbf{1})\\
&\qquad+M(\mathbf{1}-pq)\left[F\left(\frac{\mathbf{1}-p}{\mathbf{1}-pq},\mathbf{1}\right)+F\left(\frac{\mathbf{1}-p}{\mathbf{1}-pq},\mathbf{1}\right)-F\left(\frac{\mathbf{1}-p}{\mathbf{1}-pq},q\right)\right]
\end{aligned}
$$

for all $p,q\in\,]0,1[^k$. Now using (4) we get

$$
\begin{aligned}
(5)\qquad &|[M(q)+M(\mathbf{1}-q)-1]\cdot[f(p)-f(\mathbf{1})M(p)]\\
&\qquad\qquad-[M(p)+M(\mathbf{1}-p)-1]\cdot[f(q)-f(\mathbf{1})M(q)]|\\
&\qquad\leq 4\varepsilon+3\varepsilon M(\mathbf{1}-pq).
\end{aligned}
$$

Since M is not additive there exists a $q^*\in\,]0,1[^k$ such that

$$M(q^*)+M(\mathbf{1}-q^*)\neq 1. \tag{6}$$

Substituting $q=q^*$ in (5) we obtain

$$
\begin{aligned}
&|[M(q^*)+M(\mathbf{1}-q^*)-1]\cdot[f(p)-f(\mathbf{1})M(p)]\\
&\qquad\qquad-[M(p)+M(\mathbf{1}-p)-1]\cdot[f(q^*)-f(\mathbf{1})M(q^*)]|\\
&\qquad\leq 4\varepsilon+3\varepsilon M(\mathbf{1}-pq^*).
\end{aligned}
$$

Due to (6) we find that

$$
\begin{aligned}
&\left|[f(p)-f(\mathbf{1})M(p)]-\frac{f(q^*)-f(\mathbf{1})M(q^*)}{M(q^*)+M(\mathbf{1}-q^*)-1}\cdot[M(p)+M(\mathbf{1}-p)-1]\right|\\
&\qquad\leq|M(q^*)+M(\mathbf{1}-q^*)-1|^{-1}\cdot(4\varepsilon+3\varepsilon M(\mathbf{1}-pq^*)),
\end{aligned}
$$

which is (3) for $x = p \in]0,1[^k$ with

$$a = f(\mathbf{1})M(p) + \frac{f(q^*) - f(\mathbf{1})M(q^*)}{M(q^*) + M(\mathbf{1} - q^*) - 1},$$

$$b = \frac{f(q^*) - f(\mathbf{1})M(q^*)}{M(q^*) + M(\mathbf{1} - q^*) - 1}.$$

A direct calculation shows that (3) also holds in case $x \in [0,1]^k \setminus]0,1[^k$. ∎

Define $K : [0,1]^k \to \mathbb{R}$ by

$$K(x) = \frac{4\varepsilon + 3\varepsilon M(\mathbf{1} - xq^*)}{|M(q^*) + M(\mathbf{1} - q^*) - 1|}, \tag{7}$$

where $M : [0,1]^k \to \mathbb{R}$ is multiplicative but not additive, $\varepsilon \geq 0$ arbitrary but fixed and $q^* \in [0,1]^k$ is such that $M(q^*) + M(\mathbf{1} - q^*) \neq 1$.

Using the previous theorem we can deduce the following.

Corollary 3.1. *In case $\varepsilon = 0$, Theorem 3.1 yields the general solution of equation (1).*

Corollary 3.2. *If the function $M : [0,1]^k \to \mathbb{R}$ is bounded above by a constant $B \in \mathbb{R}$ then inequality (2) on D_k implies*

$$|f(x) - (aM(x) + b(M(\mathbf{1} - x) - 1))| \leq |M(q^*) + M(\mathbf{1} - q^*) - 1|^{-1} \cdot (4\varepsilon + 3B\varepsilon)$$

on $[0,1]^k$, for any $q^* \in]0,1[^k$ is such that $M(q^*) + M(\mathbf{1} - q^*) \neq 1$.

Corollary 3.2 implies

Corollary 3.3. *The equation*

$$f(x) + M(\mathbf{1} - x) f\left(\frac{y}{\mathbf{1} - x}\right) = f(y) + M(\mathbf{1} - y) f\left(\frac{x}{\mathbf{1} - y}\right)$$

is superstable on D_k in case M is bounded above.

Remark 3.1. If $M(x) = x^\alpha$ ($x \in [0,1]$), where $0 < \alpha \neq 1$, then we get the result of Maksa (see [Mak07]).

Finally, the following theorem concerns the stability of a system of equations.

Theorem 3.2. *Let $I_n : \Gamma_n \to \mathbb{R}$ $(n \geq 2)$ be a sequence of functions, and let $M : [0,1]^k \to \mathbb{R}$ be a multiplicative function. Suppose that there exists a sequence (ε_n) of non-negative real numbers such that*

$$\left| I_n(p_1, \ldots, p_n) - I_{n-1}(p_1 + p_2, p_3, \ldots, p_n) - M(p_1 + p_2) I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \right| \leq \varepsilon_{n-1} \tag{8}$$

for all $n \geq 3$ and $(p_1, \dots, p_n) \in \Gamma_n$, and

$$|I_3(p_1, p_2, p_3) - I_3(p_1, p_3, p_2)| \leq \varepsilon_1$$

on Γ_3. Then there exist $c, d \in \mathbb{R}$ and a $q^* \in [0,1]^k$ such that

$$\Big|I_n(p_1, \dots, p_n) - \Big[c\Big(\sum_{i=1}^{n} M(p_i) - 1\Big) - d(M(p_1) - 1)\Big]\Big| \leq \sum_{k=2}^{n-1} \varepsilon_k + (1 + (n-2)M(p_1 + p_2))K(p_2) \tag{9}$$

for all $n \geq 2$ and $(p_1, \dots, p_n) \in \Gamma_n$, where the convention $\sum_{k=2}^{1} \varepsilon_k = 0$ is adopted and the function K is defined by (7).

Proof. The proof is by induction on n. Let $(x, y) \in D_k$, $n = 3$ and substitute

$$p_1 = 1 - x - y, \quad p_2 = y, \quad p_3 = x$$

into (8). Then

$$\left|I_3(\mathbf{1} - x - y, y, x) - I_2(\mathbf{1} - x, x) - M(\mathbf{1} - x)I_2\left(\mathbf{1} - \frac{y}{\mathbf{1} - x}, \frac{y}{\mathbf{1} - x}\right)\right| \leq \varepsilon_2.$$

Hence the function $f : [0,1]^k \to \mathbb{R}$ defined by

$$f(x) = I_2(\mathbf{1} - x, x) \quad (x \in [0,1]^k)$$

satisfies

$$\begin{aligned}
&\left|f(x) + M(\mathbf{1} - x)f\left(\frac{y}{\mathbf{1} - x}\right) - f(x) - M(\mathbf{1} - x)f\left(\frac{x}{\mathbf{1} - y}\right)\right| \\
&\quad \leq \left|f(x) + M(\mathbf{1} - x)f\left(\frac{y}{\mathbf{1} - x}\right) - I_3(\mathbf{1} - x - y, y, x)\right| \\
&\qquad + |I_3(\mathbf{1} - x - y, y, x) - I_3(\mathbf{1} - y - x, x, y)| \\
&\qquad + \left|I_3(\mathbf{1} - y - x, y, x) - f(y) - M(\mathbf{1} - y)f\left(\frac{y}{\mathbf{1} - y}\right)\right| \leq \varepsilon_1 + 2\varepsilon_2
\end{aligned}$$

for all $(x, y) \in D_k$. Thus, by Theorem 3.1, there exist $a, b \in \mathbb{R}$ and a $q^* \in [0,1]$ such that

$$\begin{aligned}
&|f(x) - [aM(x) + b(M(\mathbf{1} - x) - 1)]| \\
&\quad \leq |M(q^*) + M(\mathbf{1} - q^*) - 1|^{-1} \cdot (4(\varepsilon_1 + 2\varepsilon_2) + 3(\varepsilon_1 + 2\varepsilon_2)M(\mathbf{1} - xq^*))
\end{aligned}$$

for all $x \in [0,1]^k$. Let now $(p_1, p_2) \in \Gamma_2$. Then

$$\begin{aligned}
&|I_2(p_1, p_2) - [aM(p_2) + b(M(p_1) - 1)]| \\
&\qquad \leq |M(q^*) + M(\mathbf{1} - q^*) - 1|^{-1} \cdot (4\varepsilon + 3\varepsilon M(\mathbf{1} - p_2 q^*))|.
\end{aligned}$$

Define $c = a$ and $d = b - a$. Then

$$\begin{aligned}\Big|I_2(p_1,p_2) - c\Big[\sum_{k=1}^{2} M(p_k) - d(M(p_1)-1)\Big]\Big| \\ &\leq |M(q^*) + M(\mathbf{1}-q^*) - 1|^{-1} \cdot (4\varepsilon + 3\varepsilon M(\mathbf{1}-p_2 q^*))| \\ &= \sum_{k=2}^{2-1} \varepsilon_k + (1-(1-1)M(p_1+p_2)) \cdot K(p_2),\end{aligned}$$

hence the statement holds for $n = 2$.

Assume now that (8) holds and set

$$J_n(p_1,\ldots,p_n) = c\Big(\sum_{k=1}^{n} M(p_k) - 1\Big) + d(M(p_1)-1)$$

for all $n \geq 2$, $(p_1,\ldots,p_n) \in \mathit{\Gamma}_n$. It can be easily seen that $J_n : \mathit{\Gamma}_n \to \mathbb{R}$ is an M-recursive and 3-semisymmetric information measure ($n \in \mathbb{N}$). Therefore

$$\begin{aligned}&|I_{n+1}(p_1,\ldots,p_{n+1}) - J_{n+1}(p_1,\ldots,p_{n+1})| \\ &= \Big|I_{n+1}(p_1,\ldots,p_{n+1}) - J_n(p_1+p_2,\ldots,p_{n+1}) \\ &\quad - M(p_1+p_2)I_2\Big(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\Big)\Big| \\ &\leq \Big|I_{n+1}(p_1,\ldots,p_{n+1}) - I_n(p_1+p_2,\ldots,p_{n+1}) \\ &\quad - M(p_1+p_2)I_2\Big(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\Big)\Big| \\ &\quad + |I_n(p_1+p_2,\ldots,p_{n+1}) - J_n(p_1+p_2,\ldots,p_n)| \\ &\quad + \Big|M(p_1+p_2)I_2\Big(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\Big) - M(p_1+p_2)J_2\Big(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\Big)\Big| \\ &\leq \varepsilon_n + \sum_{k=2}^{n-1} \varepsilon_n + (1+(n-2)M(p_1+p_2))K(p_2) + M(p_1+p_2)K(p_2) \\ &= \sum_{k=2}^{n} \varepsilon_k + (1+(n-1)M(p_1+p_2))K(p_2)\end{aligned}$$

for all $(p_1,\ldots,p_n) \in \mathit{\Gamma}_{n+1}$, that is, (9) holds for $n+1$ instead of n, which ends the proof. ∎

Remark 3.2. Our argument does not work in case M is a projection, i.e., we cannot prove stability concerning the fundamental equation of information in this case, neither on the closed nor on the open domain.

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