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REAL HYPERSURFACES WITH AN INDUCED ALMOST CONTACT STRUCTURE

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Abstract. We study real affine hypersurfaces $f\colon M \to \mathbb{C}^{n+1}$ with an almost contact structure (φ, ξ, η) induced by any J-tangent transversal vector field. The main purpose of this paper is to show that if (φ, ξ, η) is metric relative to the second fundamental form then it is Sasakian and moreover $f(M)$ is a piece of a hyperquadric in $\mathbb{R}^{2n+2}$.

1. Introduction. In [2], V. Cruceanu studied centro-affine real hypersurfaces in complex affine spaces. He proved that such hypersurfaces are hyperquadrics if and only if the induced almost contact structure is metric relative to the affine fundamental form induced by a centro-affine transversal vector field.

In this paper we consider hypersurfaces with an arbitrary J-tangent transversal vector field. Such a vector field induces in a natural way an almost contact structure (φ, ξ, η) and the second fundamental form h. We prove that if (φ, ξ, η, h) is an almost contact metric structure then it is a Sasakian structure and the hypersurface is a piece of a hyperquadric, while the transversal vector field is centro-affine.

2. Preliminaries. We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [3]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X,Y)C$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C,$$

where X, Y are tangent vector fields. Here ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the second fundamental form, S is a tensor of type $(1,1)$, called the shape operator, and τ is a 1-form.

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We assume that h is non-degenerate so that h defines a semi-Riemannian metric on M. If h is non-degenerate, then we say that the hypersurface or the hypersurface immersion is *non-degenerate*. We have the following

THEOREM 2.1 (Fundamental equations). *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations*:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.1}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.2}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.3}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.4}$$

The equations (2.1), (2.2), (2.3), and (2.4) are called the equation of Gauss, Codazzi for h, Codazzi for S, and Ricci, respectively.

For an affine immersion the cubic form Q is defined by the formula

$$Q(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z). \tag{2.5}$$

It follows from the Codazzi equation (2.2) that Q is symmetric in all three arguments.

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$).

Let $\dim M = 2n+1$ and $f\colon (M,g) \to (\mathbb{R}^{2n+2}, \tilde{g})$ be a non-degenerate (relative to the second fundamental form) isometric immersion, where $\tilde{g}$ is the standard inner product on $\mathbb{R}^{2n+2}$. We assume that $\mathbb{R}^{2n+2}$ is endowed with the standard complex structure J,

$$J(x_1,\dots,x_{n+1},y_1,\dots,y_{n+1}) = (-y_1,\dots,-y_{n+1},x_1,\dots,x_{n+1}).$$

Let C be a transversal vector field on M. We say that C is *J-tangent* if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M to be the biggest J-invariant distribution on M, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap J(f_*(T_xM)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field X is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. Additionally we define two 1-dimensional distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ as follows:

$$\mathcal{D}_{1x} := \{X \in T_xM \colon g(X,Y) = 0 \ \forall Y \in \mathcal{D}_x\},$$
$$\mathcal{D}_{2x} := \{X \in T_xM \colon h(X,Y) = 0 \ \forall Y \in \mathcal{D}_x\},$$

where h is the second fundamental form on M relative to any transversal vector field. It follows from [3, Prop. 2.5] that the definition of $\mathcal{D}_2$ is independent of the choice of the transversal vector field. We say that the distribution $\mathcal{D}$ is *non-degenerate* if h is non-degenerate on $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is non-degenerate if and only if $\mathcal{D} \oplus \mathcal{D}_2 = TM$. To simplify notation, we will omit f_* in front of vector fields.

Denote by N^0 the metric normal for f (relative to $\tilde{g}$). The metric normal induces objects ∇^0, h^0 and S^0. Recall that the induced connection ∇^0 is the Levi-Civita connection of the metric g, and the objects h^0, S^0 and the metric g are related by $h^0(X,Y) = g(S^0X, Y)$ for every $X, Y \in T_xM$. We have the following

LEMMA 2.2. *The distributions* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *coincide if and only if* $\nabla^0_N N = 0$, *where* N *is a* g*-normal vector field to* $\mathcal{D}$ (*that is*, $g(N,N) = 1$ *and* $g(N,X) = 0$ *for every* $X \in \mathcal{D}$).

Proof. Since N^0 is the metric normal, $N := JN^0$ is a tangent g-normal vector field to $\mathcal{D}$. We have

$$\begin{aligned} h^0(N,X) &= g(S^0N,X) = -g(D_N N^0, X) = g(D_N JN, X) \\ &= g(JD_N N, X) = -g(D_N N, JX) \\ &= -g(\nabla^0_N N + h^0(N,N)N^0, JX), \end{aligned}$$

where X is any tangent vector field. Now for every $X \in \mathcal{D}$ we have

$$h^0(N,X) = -g(\nabla^0_N N, JX).$$

Since ∇^0 is the Levi-Civita connection for g, we also have $g(\nabla^0_N N, N) = 0$. Thus $\nabla^0_N N \in \mathcal{D}$. It remains to observe that $\mathcal{D}_1 = \mathcal{D}_2$ if and only if

$$h^0(N,X) = -g(\nabla^0_N N, JX) = 0$$

for every $X \in \mathcal{D}$, that is, if and only if $\nabla^0_N N = 0$. ∎

3. Almost contact structures. A $(2n+1)$-dimensional manifold M is said to have an *almost contact structure* if there exist on M a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \tag{3.1}$$

$$\eta(\xi) = 1 \tag{3.2}$$

for every $X \in TM$. If additionally there is a semi-Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.3}$$

for every $X, Y \in TM$ then (φ, ξ, η, g) is called an *almost contact metric structure*. An almost contact metric structure is called *Sasakian* if

$$(\widehat{\nabla}_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{3.4}$$

where $\widehat{\nabla}$ is the Levi-Civita connection for g. An almost contact metric structure (φ, ξ, η, g) is called a *contact metric structure* if

$$g(X, \varphi Y) = d\eta(X, Y) \tag{3.5}$$

for every $X, Y \in TM$. We say that an almost contact structure (φ, ξ, η) is *normal* if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis tensor for φ. We have

THEOREM 3.1 ([1]). *A contact metric structure* (φ, ξ, η, g) *is Sasakian if and only if* (φ, ξ, η) *is normal.*

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a non-degenerate hypersurface with a J-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η and a tensor field φ of type $(1, 1)$ as follows:

$$\xi := JC, \tag{3.6}$$

$$\eta|_{\mathcal{D}} = 0 \quad \text{and} \quad \eta(\xi) = 1, \tag{3.7}$$

$$\varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \quad \text{and} \quad \varphi(\xi) = 0. \tag{3.8}$$

It is easy to see that (φ, ξ, η) is an almost contact structure on M. This structure will be called the *almost contact structure on M induced by C*. An induced almost contact structure (φ, ξ, η) is called *compatible* with the second fundamental form h if

$$\eta(X) = h(X, \xi) \quad \text{for every } X \in TM.$$

It is not difficult to see that if the distribution $\mathcal{D}$ is non-degenerate then there exists exactly one J-tangent transversal vector field such that the induced structure (φ, ξ, η) is compatible with h. Clearly, if (φ, ξ, η, h) is an almost contact metric structure then (φ, ξ, η) is compatible with h.

We shall now prove

THEOREM 3.2. *If* (φ, ξ, η) *is an induced almost contact structure on M then the following equations hold*:

$$\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.9}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y)SX - h(X, Y)\xi, \tag{3.10}$$

$$\begin{aligned}\eta([X, Y]) = &-h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X))\\ &+ \eta(Y)\tau(X) - \eta(X)\tau(Y),\end{aligned} \tag{3.11}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX, \tag{3.12}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.13}$$

$$\eta(SX) = h(X, \xi) \tag{3.14}$$

for every $X, Y \in \mathcal{X}(M)$.

Proof. For every $X \in TM$ we have

$$JX = \varphi X - \eta(X)C.$$

Furthermore,

$$\begin{aligned} J(\mathrm{D}_X Y) &= J(\nabla_X Y + h(X,Y)C) = J(\nabla_X Y) + h(X,Y)JC \\ &= \varphi(\nabla_X Y) - \eta(\nabla_X Y)C + h(X,Y)\xi \end{aligned}$$

and

$$\begin{aligned} \mathrm{D}_X JY &= \mathrm{D}_X(\varphi Y - \eta(Y)C) = \mathrm{D}_X \varphi Y - X(\eta(Y))C - \eta(Y)\mathrm{D}_X C \\ &= \nabla_X \varphi Y + h(X,\varphi Y)C - X(\eta(Y))C - \eta(Y)(-SX + \tau(X)C) \\ &= \nabla_X \varphi Y + \eta(Y)SX + (h(X,\varphi Y) - X(\eta(Y)) - \eta(Y)\tau(X))C. \end{aligned}$$

Since $\mathrm{D}_X JY = J(\mathrm{D}_X Y)$, comparing these two equations, we obtain (3.9) and (3.10). Equations (3.11)—(3.14) follow directly from (3.9) and (3.10). (For (3.14), set $Y = \xi$ in (3.10).) ∎

From the above theorem we immediately get

COROLLARY 3.3. *For every $Z, W \in \mathcal{D}$ we have*

$$\eta(\nabla_Z W) = -h(Z, \varphi W), \tag{3.15}$$

$$\eta(\nabla_\xi Z) = -h(\xi, \varphi Z), \tag{3.16}$$

$$\varphi(\nabla_Z W) = \nabla_Z \varphi W - h(Z,W)\xi, \tag{3.17}$$

$$\eta([Z,W]) = -h(Z,\varphi W) + h(W,\varphi Z), \tag{3.18}$$

$$\eta([Z,\xi]) = h(\xi,\varphi Z) + \tau(Z). \tag{3.19}$$

Moreover,

$$S(\mathcal{D}) \subset \mathcal{D} \quad \textit{if and only if} \quad \xi \in \mathcal{D}_2. \tag{3.20}$$

Almost contact normal structures can be characterized as follows:

PROPOSITION 3.4 ([4, Th. 3.3]). *The induced almost contact structure (φ, ξ, η) is normal if and only if*

$$S\varphi Z - \varphi S Z + \tau(Z)\xi = 0 \quad \textit{for every } Z \in \mathcal{D}.$$

4. Main results. In this section we always assume that (φ, ξ, η) is an induced almost contact structure. Let us denote by $\widehat{\nabla}$ the Levi-Civita connection for the second fundamental form h. We have

Proposition 4.1. *If* (φ, ξ, η) *is an almost contact structure compatible with* h *then*

$$S(\mathcal{D}) \subset \mathcal{D}, \tag{4.1}$$

$$\xi \text{ is } h\text{-orthogonal to } \mathcal{D}, \tag{4.2}$$

$$S\xi = \xi + Z_0 \quad \text{where } Z_0 \in \mathcal{D}, \tag{4.3}$$

$$\tau(X) = -h(X, \varphi Z_0) \quad \text{for every } X \in \mathcal{D}, \tag{4.4}$$

$$\nabla_\xi \xi = -\varphi Z_0 + \tau(\xi)\xi, \tag{4.5}$$

$$\widehat{\nabla}_\xi \xi = -\varphi Z_0. \tag{4.6}$$

Proof. Properties (4.1) and (4.2) are obvious from (3.20), while (4.3) is an immediate consequence of the definition of an almost contact structure compatible with h and Theorem 3.2 (equation (3.14)). The Codazzi equation for S implies that

$$\nabla_X S\xi - S(\nabla_X \xi) - \tau(X)S\xi = \nabla_\xi SX - S(\nabla_\xi X) - \tau(\xi)SX.$$

Since (φ, ξ, η) is compatible with h, formula (3.16) implies $\nabla_\xi Z \in \mathcal{D}$ for every $Z \in \mathcal{D}$. We also have (4.3). Thus, by (4.1),

$$\eta(\nabla_Z \xi) + \eta(\nabla_Z Z_0) - \eta(S(\nabla_Z \xi)) = \tau(Z) \tag{4.7}$$

for every $Z \in \mathcal{D}$. Now, using (3.9), (3.14) and compatibility of (φ, ξ, η) we get

$$\eta(\nabla_Z Z_0) = -h(Z, \varphi Z_0), \quad \eta(S(\nabla_Z \xi)) = \eta(\nabla_Z \xi)$$

for every $Z \in \mathcal{D}$. Hence equation (4.7) can be rewritten as

$$-h(Z, \varphi Z_0) = \tau(Z),$$

which proves (4.4). (4.5) can be easily deduced from (3.10), (3.13) and (4.3). To prove (4.6), note that

$$2h(\widehat{\nabla}_\xi \xi, X) = 2\xi(h(\xi, X)) + 2h([X, \xi], \xi).$$

Setting $X = \xi$ we obtain $h(\widehat{\nabla}_\xi \xi, \xi) = 0$, that is, $\widehat{\nabla}_\xi \xi \in \mathcal{D}$. On the other hand, if we take $X = Z \in \mathcal{D}$ then by Corollary 3.3 (equation (3.19)) we obtain

$$h(\widehat{\nabla}_\xi \xi, Z) = \tau(Z) = h(-\varphi Z_0, Z)$$

for every $Z \in \mathcal{D}$. Now, the non-degeneracy of h on $\mathcal{D}$ implies (4.6). ■

As an immediate consequence of Proposition 4.1 we get

Corollary 4.2. *If* (φ, ξ, η) *is an almost contact structure compatible with* h *then the following conditions are equivalent*:

$$\widehat{\nabla}_\xi \xi = 0, \tag{4.8}$$

$$S\xi = \xi, \tag{4.9}$$

$$\tau|_{\mathcal{D}} = 0. \tag{4.10}$$

Let us recall that the cubic form Q is given by the equation (2.5). We shall prove

LEMMA 4.3. *If* (φ, ξ, η, h) *is an almost contact metric structure then*

$$Q(X, W, Z) = Q(X, \varphi W, \varphi Z), \tag{4.11}$$

$$Q(W_1, W_2, W_3) = 0, \tag{4.12}$$

$$Q(\xi, W, W) = -h(SW, \varphi W) = h(S\varphi W, W) \tag{4.13}$$

for every $X \in \mathcal{X}(M)$ *and* $W, W_1, W_2, W_3, Z \in \mathcal{D}$.

Proof. Let $X \in \mathcal{X}(M)$ and $W, Z \in \mathcal{D}$. Then

$$\begin{aligned} Q(X, \varphi W, \varphi Z) &= X(h(\varphi W, \varphi Z)) - h(\nabla_X \varphi W, \varphi Z) - h(\varphi W, \nabla_X \varphi Z) \\ &\quad + \tau(X) h(\varphi W, \varphi Z) \\ &= X(h(W, Z)) - h(\nabla_X \varphi W, \varphi Z) - h(\varphi W, \nabla_X \varphi Z) \\ &\quad + \tau(X) h(W, Z). \end{aligned}$$

By Theorem 3.2 we see that

$$\nabla_X \varphi W = \varphi(\nabla_X W) + h(X, W)\xi$$

for every $X \in \mathcal{X}(M)$ and $W \in \mathcal{D}$. Thus

$$\begin{aligned} Q(X, \varphi W, \varphi Z) &= X(h(W, Z)) - h(\varphi(\nabla_X W), \varphi Z) - h(\varphi W, \varphi(\nabla_X Z)) \\ &\quad + \tau(X) h(W, Z) \\ &= X(h(W, Z)) - h(\nabla_X W, Z) - h(W, \nabla_X Z) + \tau(X) h(W, Z) \\ &= Q(X, W, Z), \end{aligned}$$

which proves (4.11). To prove (4.12) observe that from (4.11) we have

$$Q(W, W, W) = Q(W, \varphi W, \varphi W) = Q(\varphi W, W, \varphi W) = 0$$

for every $W \in \mathcal{D}$. Since Q is symmetric in all three arguments, the last equation implies that $Q(W_1, W_2, W_3) = 0$ for every $W_1, W_2, W_3 \in \mathcal{D}$. It is easy to see that

$$Q(\xi, W, W) = Q(W, \xi, W) = -h(\nabla_W \xi, W) - h(\xi, \nabla_W W)$$

for every $W \in \mathcal{D}$. Formulas (3.10) and (3.15) imply that for every $W \in \mathcal{D}$,

$$\varphi(\nabla_W \xi) = SW \quad \text{and} \quad \nabla_W W \in \mathcal{D}.$$

We now have

$$Q(\xi, W, W) = -h(SW, \varphi W) \quad \text{for every } W \in \mathcal{D}.$$

From (4.11) we obtain

$$Q(\xi, W, W) = Q(\xi, \varphi W, \varphi W)$$

and consequently

$$-h(SW,\varphi W)=h(S\varphi W,W),$$

which completes the proof. ∎

We shall now prove

THEOREM 4.4. *Let $f\colon M\to\mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a J-tangent transversal vector field and let (φ,ξ,η) be the induced almost contact structure on M. If (φ,ξ,η,h) is an almost contact metric structure then*

$$S=\mathrm{id}\quad \textit{and}\quad \tau=0.$$

Proof. Let $W,Z\in\mathcal{D}$. Formulas (2.3) and (4.1) imply that

$$\eta(\nabla_W SZ)-\eta(S(\nabla_W Z))=\eta(\nabla_Z SW)-\eta(S(\nabla_Z W)).$$

Thus, by (3.14),

$$\eta(\nabla_W SZ)-\eta(\nabla_Z SW)=\eta(S([W,Z]))=\eta([W,Z]).$$

By Corollary 3.3 (formulas (3.15) and (3.18)) we get

$$-h(W,\varphi SZ)+h(Z,\varphi SW)=-h(W,\varphi Z)+h(Z,\varphi W).$$

Replacing Z with φZ, and using the fact that (φ,ξ,η,h) is a metric structure we have

$$h(\varphi W,S\varphi Z)+h(Z,SW)=2h(W,Z)\quad\text{for every } W,Z\in\mathcal{D}. \tag{4.14}$$

Using the Gauss equation we get

$$\begin{aligned}(R(W,\varphi W)\cdot h)(\varphi W,\varphi W)&=-2h(R(W,\varphi W)\varphi W,\varphi W)\\&=-2h(W,W)h(SW,\varphi W)\end{aligned} \tag{4.15}$$

for every $W\in\mathcal{D}$. On the other hand,

$$\begin{aligned}(R(W,\varphi W)\cdot h)(\varphi W,\varphi W)&=(\nabla_W\nabla_{\varphi W}h)(\varphi W,\varphi W)\\&\quad-(\nabla_{\varphi W}\nabla_W h)(\varphi W,\varphi W)-(\nabla_{[W,\varphi W]}h)(\varphi W,\varphi W).\end{aligned}$$

The following formulas are obvious:

$$\begin{aligned}(\nabla_W\nabla_{\varphi W}h)(\varphi W,\varphi W)&=W(\nabla_{\varphi W}h(\varphi W,\varphi W))-2\nabla_{\varphi W}h(\nabla_W\varphi W,\varphi W),\\(\nabla_{\varphi W}\nabla_W h)(\varphi W,\varphi W)&=\varphi W(\nabla_W h(\varphi W,\varphi W))-2\nabla_W h(\nabla_{\varphi W}\varphi W,\varphi W).\end{aligned}$$

We have

$$(\nabla_X h)(Y,Z)=Q(X,Y,Z)-\tau(X)h(Y,Z)$$

for every $X, Y, Z \in \mathcal{X}(M)$. Thus Lemma 4.3 and the above formulas imply

$$\begin{aligned}
(\nabla_W \nabla_{\varphi W} h)(\varphi W, \varphi W) \\
&= W(Q(\varphi W, \varphi W, \varphi W) - \tau(\varphi W)h(\varphi W, \varphi W)) \\
&\quad - 2Q(\varphi W, \nabla_W \varphi W, \varphi W) + 2\tau(\varphi W)h(\nabla_W \varphi W, \varphi W) \\
&= -W(\tau(\varphi W))h(W, W) - \tau(\varphi W)W(h(\varphi W, \varphi W)) \\
&\quad - 2Q(\nabla_W \varphi W, W, W) + 2\tau(\varphi W)h(\nabla_W \varphi W, \varphi W) \\
&= -W(\tau(\varphi W))h(W, W) - \tau(\varphi W)(\nabla_W h)(\varphi W, \varphi W) \\
&\quad - 2\eta(\nabla_W \varphi W)Q(\xi, W, W) \\
&= -W(\tau(\varphi W))h(W, W) + \tau(\varphi W)\tau(W)h(W, W) \\
&\quad - 2\eta(\nabla_W \varphi W)Q(\xi, W, W) \\
&= -W(\tau(\varphi W))h(W, W) + \tau(\varphi W)\tau(W)h(W, W) \\
&\quad - 2h(W, W)Q(\xi, W, W),
\end{aligned}$$

where, in the last equality, we used (3.15), and

$$\begin{aligned}
(\nabla_{\varphi W} \nabla_W h)(\varphi W, \varphi W) \\
&= \varphi W(Q(W, \varphi W, \varphi W) - \tau(W)h(\varphi W, \varphi W)) \\
&\quad - 2Q(W, \nabla_{\varphi W} \varphi W, \varphi W) + 2\tau(W)h(\nabla_{\varphi W} \varphi W, \varphi W) \\
&= -\varphi W(\tau(W))h(W, W) - \tau(W)\varphi W(h(\varphi W, \varphi W)) \\
&\quad + 2\tau(W)h(\nabla_{\varphi W} \varphi W, \varphi W) \\
&= -\varphi W(\tau(W))h(W, W) - \tau(W)(\nabla_{\varphi W} h)(\varphi W, \varphi W) \\
&= -\varphi W(\tau(W))h(W, W) + \tau(W)\tau(\varphi W)h(W, W).
\end{aligned}$$

We also have, from (3.18),

$$\begin{aligned}
(\nabla_{[W,\varphi W]} h)(\varphi W, \varphi W) &= Q([W, \varphi W], \varphi W, \varphi W) - \tau([W, \varphi W])h(\varphi W, \varphi W) \\
&= \eta([W, \varphi W])Q(\xi, W, W) - \tau([W, \varphi W])h(W, W) \\
&= 2h(W, W)Q(\xi, W, W) - \tau([W, \varphi W])h(W, W).
\end{aligned}$$

Using (4.13) and the Ricci equation (2.4), we get

$$-2Q(\xi, W, W) = h(SW, \varphi W) - h(W, S\varphi W) = -2d\tau(W, \varphi W). \tag{4.16}$$

From (4.16) and the preceding formulas, we obtain

$$(R(W, \varphi W) \cdot h)(\varphi W, \varphi W) = -6d\tau(W, \varphi W)h(W, W)$$

and so, by (4.16) and (4.13),

$$(R(W, \varphi W) \cdot h)(\varphi W, \varphi W) = -6Q(\xi, W, W) = 6h(W, W)h(SW, \varphi W),$$

which, combined with (4.15), yields

$$h(SW, \varphi W) = 0 \tag{4.17}$$

for every $W \in \mathcal{D}$. (4.17) now implies

$$0 = h(S(W + 2\varphi Z), \varphi W - 2Z) = -2h(SW, Z) + 2h(S\varphi Z, \varphi W).$$

Therefore

$$h(S\varphi Z, \varphi W) = h(SW, Z).$$

By (4.14) we also have

$$h(S\varphi Z, \varphi W) = 2h(W, Z) - h(SW, Z).$$

The above formulas imply that

$$h(SW, Z) = h(W, Z)$$

for every $Z \in \mathcal{D}$. Thus, since $\mathcal{D}$ is non-degenerate and $SW - W$ is h-orthogonal to $\mathcal{D}$ whenever $W \in \mathcal{D}$, it follows that $SW = W$ for every $W \in \mathcal{D}$. From Proposition 4.1 we easily get

$$SX = X + \eta(X)Z_0 \tag{4.18}$$

for every $X \in \mathcal{X}(M)$. We shall show that $Z_0 = 0$. Suppose $Z_0 \neq 0$; then using the Codazzi equation for S we have

$$\nabla_W SZ_0 - S(\nabla_W Z_0) - \tau(W)SZ_0 = \nabla_{Z_0} SW - S(\nabla_{Z_0} W) - \tau(Z_0)SW.$$

Since $\tau(Z_0) = 0$ (Prop. 4.1), using (4.18) we can rewrite the above equality as

$$-\eta(\nabla_W Z_0)Z_0 - \tau(W)Z_0 = -\eta(\nabla_{Z_0} W)Z_0,$$

that is, by (3.18) and (4.4),

$$\tau(W)Z_0 = \eta([Z_0, W])Z_0 = 2h(W, \varphi Z_0)Z_0 = -2\tau(W)Z_0.$$

The last equality implies that $\tau|_{\mathcal{D}} = 0$. Now, (4.4) implies $Z_0 = 0$, which contradicts our assumption. The property $\tau = 0$ easily follows from the fact $S = \mathrm{id}$ and the Codazzi equation for S. ∎

THEOREM 4.5. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a non-degenerate hypersurface with a J-tangent transversal vector field and let (φ, ξ, η) be the induced almost contact structure on M. The following conditions are equivalent*:

(4.19) (φ, ξ, η, h) *is an almost contact metric structure*,

(4.20) (φ, ξ, η, h) *is a contact metric structure*,

(4.21) (φ, ξ, η, h) *is a Sasakian structure.*

Proof. If (φ, ξ, η, h) is an almost contact metric structure then by Theorem 4.4 we obtain $\tau = 0$. Theorem 3.2 (eq. (3.11)) implies that (φ, ξ, η, h) is a contact metric structure. Again by Theorem 4.4 we get $S = \mathrm{id}$. Hence (φ, ξ, η) is normal (Prop. 3.4). Now Theorem 3.1 completes the proof. ∎

In [2] Cruceanu introduced a notion of *special hypersurfaces*, that is, centro-affine hypersurfaces with J-tangent centro-affine transversal vector field. He proved that if the induced almost contact structure is metric, then it is a hyperquadric. Now, using the Pick–Berwald theorem we will give an alternative proof of this theorem.

THEOREM 4.6. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a non-degenerate hypersurface with a J-tangent transversal vector field and let (φ,ξ,η) be the induced almost contact structure on M. If (φ,ξ,η,h) is an almost contact metric structure, then $f(M)$ is a piece of a hyperquadric.*

Proof. We shall show that $Q \equiv 0$. By Lemma 4.3 we have

$$Q(W_1,W_2,W_3)=0 \quad \text{for } W_1,W_2,W_3\in\mathcal{D},$$
$$Q(\xi,W_1,W_2)=0 \quad \text{for } W_1,W_2\in\mathcal{D}.$$

Since $\tau=0$ by Theorem 4.4, using (3.9) we obtain

$$Q(X,\xi,\xi)=-2h(\nabla_X\xi,\xi)=-2\eta(\nabla_X\xi)=-2\tau(X)=0$$

for every $X\in\mathcal{X}(M)$. The above equalities imply that

$$Q(X_1,X_2,X_3)=0 \quad \text{for all } X_1,X_2,X_3\in\mathcal{X}(M). \ \blacksquare$$

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